2 Spherical Astronomy

Spherical astronomy is a science studying astronomical coordinate frames, directions and apparent motions of celestial objects, determination of position from astronomical observations, observational errors, etc. We shall concentrate mainly on astronomical coordinates, apparent motions of stars and time reckoning. Also, some of the most important star catalogues will be introduced.

For simplicity we will assume that the observer is always on the northern hemisphere. Although all definitions and equations are easily generalised for both hemispheres, this might be unnecessarily confusing. In spherical astronomy all angles are usually expressed in degrees; we will also use degrees unless otherwise mentioned.

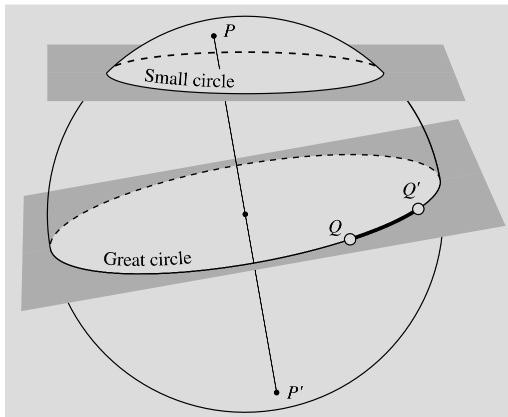

Fig. 2.1 A great circle is the intersection of a sphere and a plane passing through its centre. P and P' are the poles of the great circle. The shortest path from Q to Q' follows the great circle

2.1 Spherical Trigonometry

For the coordinate transformations of spherical astronomy, we need some mathematical tools, which we present now.

If a plane passes through the centre of a sphere, it will split the sphere into two identical hemispheres along a circle called a *great circle* (Fig. 2.1). A line perpendicular to the plane and passing through the centre of the sphere intersects the sphere at the *poles* P and P'. If a sphere is intersected by a plane not containing the centre, the intersection curve is a *small circle*. There is exactly one great circle passing through two given points Q and Q' on a sphere (unless these points are antipodal, in which case all circles passing through both of them are great circles). The arc QQ' of this great circle is the shortest path on the surface of the sphere between these points.

A *spherical triangle* is not just any three-cornered figure lying on a sphere; its sides must be arcs of great circles. The spherical triangle ABC in Fig. 2.2 has the arcs AB, BC and AC as its sides. If the radius of the sphere is r, the length of the arc AB is

$$|AB| = rc, \quad [c] = \text{rad},$$

where c is the angle subtended by the arc AB as seen from the centre. This angle is called the *central angle* of the side AB. Because lengths of sides and central angles correspond to each other in a unique way, it is customary to give the central angles instead of the sides. In this way, the radius

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of the sphere does not enter into the equations of spherical trigonometry. An angle of a spherical triangle can be defined as the angle between the tangents of the two sides meeting at a vertex, or as the dihedral angle between the planes intersecting the sphere along these two sides. We denote the angles of a spherical triangle by capital letters (A, B, C) and the opposing sides, or, more correctly, the corresponding central angles, by lowercase letters (a, b, c).

The sum of the angles of a spherical triangle is always greater than 180 degrees; the excess

$$E = A + B + C - 180^\circ \qquad (2.1)$$

is called the *spherical excess*. It is not a constant, but depends on the triangle. Unlike in plane geometry, it is not enough to know two of the angles to determine the third one. The area of a spherical triangle is related to the spherical excess in a very simple way:

$$\text{Area} = Er^2, \quad [E] = \text{rad}. \qquad (2.2)$$

This shows that the spherical excess equals the solid angle in steradians (see Appendix A.1), subtended by the triangle as seen from the centre.

To prove (2.2), we extend all sides of the triangle Δ to great circles (Fig. 2.3). These great circles will form another triangle Δ', congruent with Δ but antipodal to it. If the angle A is expressed in radians, the area of the slice $S(A)$ bounded by the two sides of A (the shaded area in Fig. 2.3) is obviously $2A/2\pi = A/\pi$ times the area of the sphere, $4\pi r^2$. Similarly, the slices $S(B)$ and $S(C)$ cover fractions B/π and C/π of the whole sphere.

Together, the three slices cover the whole surface of the sphere, the equal triangles Δ and Δ' belonging to every slice, and each point outside the triangles, to exactly one slice. Thus the area of the slices $S(A)$, $S(B)$ and $S(C)$ equals the area of the sphere plus four times the area of Δ, $\mathcal{A}(\Delta)$:

$$\frac{A + B + C}{\pi} 4\pi r^2 = 4\pi r^2 + 4\mathcal{A}(\Delta),$$

whence

$$\mathcal{A}(\Delta) = (A + B + C - \pi) r^2 = E r^2.$$

As in the case of plane triangles, we can derive relationships between the sides and angles of spherical triangles. The easiest way to do this is by inspecting certain coordinate transformations.

Suppose we have two rectangular coordinate frames $Oxyz$ and $Ox'y'z'$ (Fig. 2.4), such that the $x'y'z'$ frame is obtained from the xyz frame by rotating it around the x axis by an angle χ.

The position of a point P on a unit sphere is uniquely determined by giving two angles. The angle ψ is measured counterclockwise from the positive x axis along the xy plane; the other angle θ tells the angular distance from the xy plane.

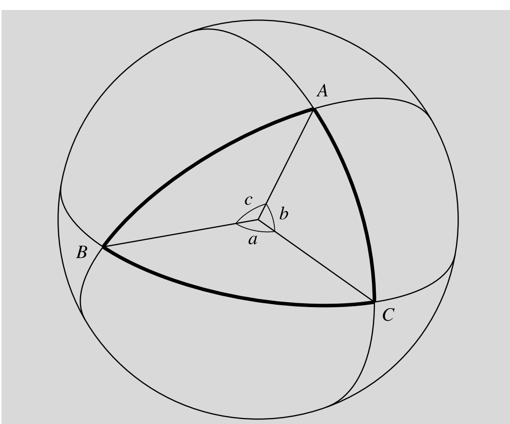


Fig. 2.2 A spherical triangle is bounded by three arcs of great circles, AB, BC and CA. The corresponding central angles are c, a, and b

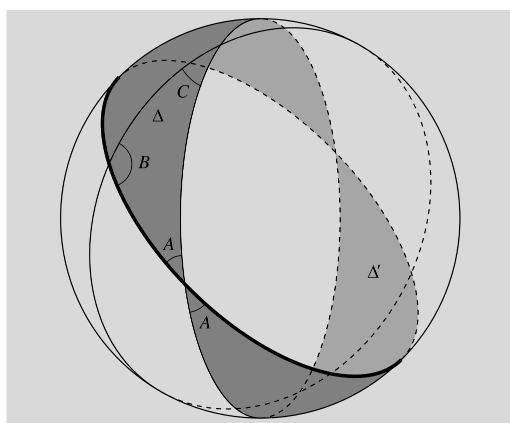


Fig. 2.3 If the sides of a spherical triangle are extended all the way around the sphere, they form another triangle Δ', antipodal and equal to the original triangle Δ. *The shaded area* is the slice $S(A)$

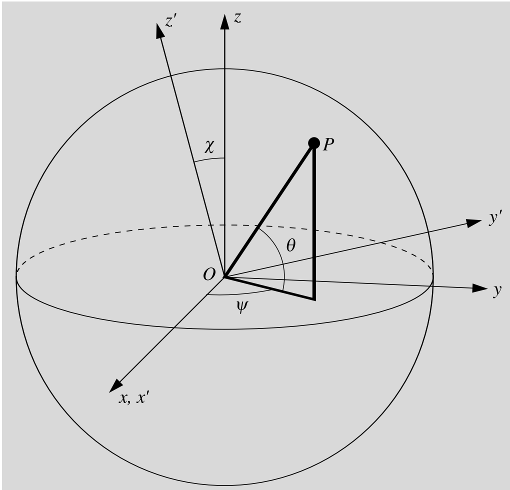


Fig. 2.4 The location of a point P on the surface of a unit sphere can be expressed by rectangular xyz coordinates or by two angles, ψ and θ. The $x'y'z'$ frame is obtained by rotating the xyz frame around its x axis by an angle χ

In an analogous way, we can define the angles ψ' and θ', which give the position of the point P in the $x'y'z'$ frame. The rectangular coordinates of the point P as functions of these angles are:

$$\begin{aligned} x &= \cos\psi\cos\theta, \qquad & x' &= \cos\psi'\cos\theta', \\ y &= \sin\psi\cos\theta, \qquad & y' &= \sin\psi'\cos\theta', \\ z &= \sin\theta, \qquad & z' &= \sin\theta'. \end{aligned} \tag{2.3}$$

We also know that the dashed coordinates are obtained from the undashed ones by a rotation in the yz plane (Fig. 2.5):

$$\begin{aligned} x' &= x, \\ y' &= y\cos\chi + z\sin\chi, \\ z' &= -y\sin\chi + z\cos\chi. \end{aligned} \tag{2.4}$$

By substituting the expressions of the rectangular coordinates (2.3) into (2.4), we have

$$\begin{aligned} \cos\psi'\cos\theta' &= \cos\psi\cos\theta, \\ \sin\psi'\cos\theta' &= \sin\psi\cos\theta\cos\chi + \sin\theta\sin\chi, \\ \sin\theta' &= -\sin\psi\cos\theta\sin\chi + \sin\theta\cos\chi. \end{aligned} \tag{2.5}$$

In fact, these equations are quite sufficient for all coordinate transformations we may encounter.

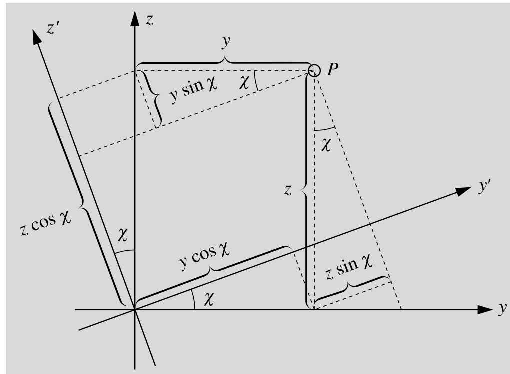


Fig. 2.5 The coordinates of the point P in the rotated frame are $x' = x$, $y' = y\cos\chi + z\sin\chi$, $z' = z\cos\chi - y\sin\chi$

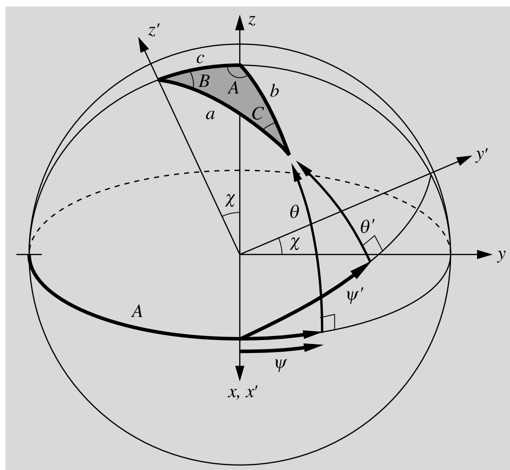


Fig. 2.6 To derive triangulation formulas for the spherical triangle ABC, the spherical coordinates ψ, θ, ψ' and θ' of the vertex C are expressed in terms of the sides and angles of the triangle

However, we shall also derive the usual equations for spherical triangles. To do this, we set up the coordinate frames in a suitable way (Fig. 2.6). The z axis points towards the vertex A and the z' axis, towards B. Now the vertex C corresponds to the point P in Fig. 2.4. The angles ψ, θ, ψ', θ' and χ can be expressed in terms of the angles and sides of the spherical triangle:

$$\begin{aligned} \psi &= A - 90^\circ, \qquad & \theta &= 90^\circ - b, \\ \psi' &= 90^\circ - B, \qquad & \theta' &= 90^\circ - a, \qquad \chi = c. \end{aligned} \tag{2.6}$$

Substitution into (2.5) gives

$$\begin{aligned}
&\cos(90^\circ - B)\cos(90^\circ - a)\\
&\quad = \cos(A - 90^\circ)\cos(90^\circ - b),\\
&\sin(90^\circ - B)\cos(90^\circ - a)\\
&\quad = \sin(A - 90^\circ)\cos(90^\circ - b)\cos c\\
&\qquad + \sin(90^\circ - b)\sin c,\\
&\sin(90^\circ - a)\\
&\quad = -\sin(A - 90^\circ)\cos(90^\circ - b)\sin c\\
&\qquad + \sin(90^\circ - b)\cos c,
\end{aligned}$$

or

$$\begin{aligned}
\sin B \sin a &= \sin A \sin b,\\
\cos B \sin a &= -\cos A \sin b \cos c + \cos b \sin c,\\
\cos a &= \cos A \sin b \sin c + \cos b \cos c.
\end{aligned} \tag{2.7}$$

Equations for other sides and angles are obtained by cyclic permutations of the sides a, b, c and the angles A, B, C. For instance, the first equation also yields

$$\begin{aligned}
\sin C \sin b &= \sin B \sin c,\\
\sin A \sin c &= \sin C \sin a.
\end{aligned}$$

All these variations of the *sine formula* can be written in an easily remembered form:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{2.8}$$

If we take the limit, letting the sides a, b and c shrink to zero, the spherical triangle becomes a plane triangle. If all angles are expressed in radians, we have approximately

$$\sin a \approx a, \qquad \cos a \approx 1 - \frac{1}{2}a^2.$$

Substituting these approximations into the sine formula, we get the familiar sine formula of plane geometry:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The second equation in (2.7) is the *sine-cosine formula*, and the corresponding plane formula is a trivial one:

$$c = b \cos A + a \cos B.$$

This is obtained by substituting the approximations of sine and cosine into the sine-cosine formula and ignoring all quadratic and higher-order terms. In the same way we can use the third equation in (2.7), the *cosine formula*, to derive the planar cosine formula:

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

2.2 The Earth

A position on the Earth is usually given by two spherical coordinates (although in some calculations rectangular or other coordinates may be more convenient). If necessary, also a third coordinate, e.g. the distance from the centre, can be used.

The reference plane is the *equatorial plane*, perpendicular to the rotation axis and intersecting the surface of the Earth along the *equator*. Small circles parallel to the equator are called *parallels of latitude*. Semicircles from pole to pole are *meridians*. The geographical *longitude* is the angle between the meridian and the zero meridian passing through Greenwich Observatory. We shall use positive values for longitudes east of Greenwich and negative values west of Greenwich. Sign convention, however, varies, and negative longitudes are not used in maps; so it is usually better to say explicitly whether the longitude is east or west of Greenwich.

The *latitude* is usually supposed to mean the *geographical latitude*, which is the angle between the plumb line and the equatorial plane. The latitude is positive in the northern hemisphere and negative in the southern one. The geographical latitude can be determined by astronomical observations (Fig. 2.7): the altitude of the celestial pole measured from the horizon equals the geographical latitude. (The celestial pole is the intersection of the rotation axis of the Earth and the infinitely distant celestial sphere; we shall return to these concepts a little later.)

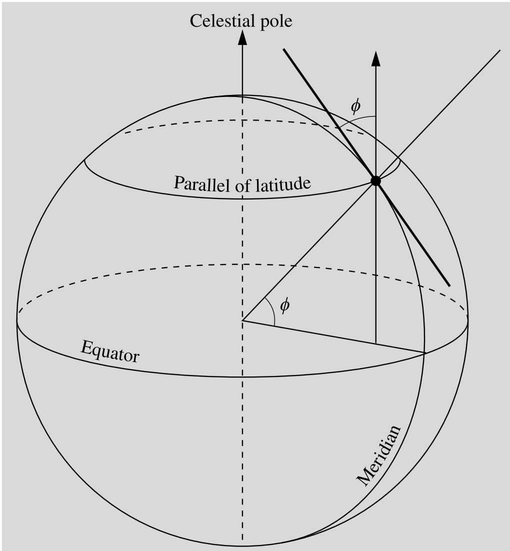


Fig. 2.7 The latitude ϕ is obtained by measuring the altitude of the celestial pole. The celestial pole can be imagined as a point at an infinite distance in the direction of the Earth's rotation axis

Because the Earth is rotating, it is slightly flattened. The exact shape is rather complicated, but for most purposes it can by approximated by an oblate spheroid, the short axis of which coincides with the rotation axis (Sect. 7.6). In 1979 the International Union of Geodesy and Geophysics (IUGG) adopted the Geodetic Reference System 1980 (GRS-80), which is used when global reference frames fixed to the Earth are defined. The GRS-80 reference ellipsoid has the following dimensions:

equatorial radius	$a = 6{,}378{,}137$ m,
polar radius	$b = 6{,}356{,}752$ m,
flattening	$f = (a-b)/a$
	$= 1/298.25722210.$

The shape defined by the surface of the oceans, called the *geoid*, differs from this spheroid at most by about 100 m.

The angle between the equator and the normal to the ellipsoid approximating the true Earth is called the *geodetic latitude*. Because the surface of a liquid (like an ocean) is perpendicular to the plumb line, the geodetic and geographical latitudes are practically the same.

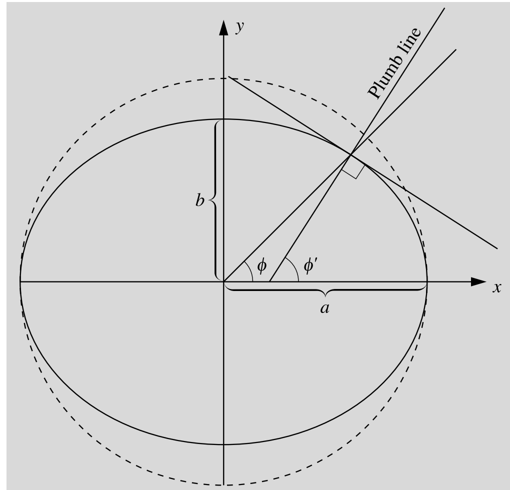


Fig. 2.8 Due to the flattening of the Earth, the geographic latitude ϕ and geocentric latitude ϕ' are different

Because of the flattening, the plumb line does not point to the centre of the Earth except at the poles and on the equator. An angle corresponding to the ordinary spherical coordinate (the angle between the equator and the line from the centre to a point on the surface), the *geocentric latitude* ϕ' is therefore a little smaller than the geographic latitude ϕ (Fig. 2.8).

We now derive an equation between the geographic latitude ϕ and geocentric latitude ϕ', assuming the Earth is an oblate spheroid and the geographic and geodesic latitudes are equal. The equation of the meridional ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The direction of the normal to the ellipse at a point (x, y) is given by

$$\tan\phi = -\frac{dx}{dy} = \frac{a^2}{b^2}\frac{y}{x}.$$

The geocentric latitude is obtained from

$$\tan\phi' = y/x.$$

Hence

$$\tan\phi' = \frac{b^2}{a^2}\tan\phi = \left(1 - e^2\right)\tan\phi, \qquad (2.9)$$

where

$$e = \sqrt{1 - b^2/a^2}$$

is the eccentricity of the ellipse. The difference $\Delta\phi = \phi - \phi'$ has a maximum 11.5′ at the latitude 45°.

Since the coordinates of celestial bodies in astronomical almanacs are given with respect to the centre of the Earth, the coordinates of nearby objects must be corrected for the difference in the position of the observer, if high accuracy is required. This means that one has to calculate the *topocentric* coordinates, centered at the observer. The easiest way to do this is to use rectangular coordinates of the object and the observer (Example 2.6).

One arc minute along a meridian is called a *nautical mile*. Since the radius of curvature varies with latitude, increasing towards the poles, the length of the nautical mile so defined would depend on the latitude (1843 m on the equator, 1862 m at the poles). Therefore one nautical mile has been defined to be equal to one minute of arc at $\phi = 45°$, whence 1 nautical mile = 1852 m.

2.3 The Celestial Sphere

The ancient universe was confined within a finite spherical shell. The stars were fixed to this shell and thus were all equidistant from the Earth, which was at the centre of the spherical universe. This simple model is still in many ways as useful as it was in antiquity: it helps us to easily understand the diurnal and annual motions of stars, and, more important, to predict these motions in a relatively simple way. Therefore we will assume for the time being that all the stars are located on the surface of an enormous sphere and that we are at its centre. Because the radius of this celestial sphere is practically infinite, we can neglect the effects due to the changing position of the observer, caused by the rotation and orbital motion of the Earth. These effects will be considered later in Sects. 2.9 and 2.10.

Since the distances of the stars are ignored, we need only two coordinates to specify their directions. Each coordinate frame has some fixed reference plane passing through the centre of the celestial sphere and dividing the sphere into two hemispheres along a great circle. One of the coordinates indicates the angular distance from this reference plane. There is exactly one great circle going through the object and intersecting this plane perpendicularly; the second coordinate gives the angle between that point of intersection and some fixed direction.

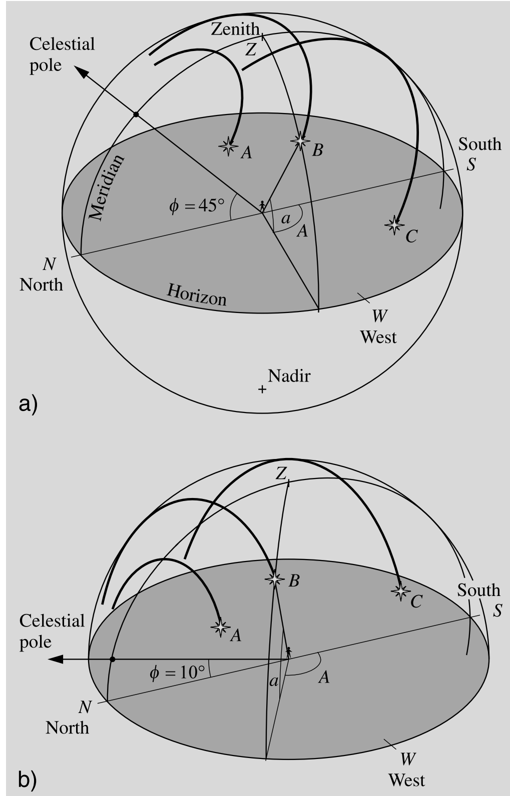


Fig. 2.9 (**a**) The apparent motions of stars during a night as seen from latitude $\phi = 45°$. (**b**) The same stars seen from latitude $\phi = 10°$

2.4 The Horizontal System

The most natural coordinate frame from the observer's point of view is the *horizontal frame* (Fig. 2.9). Its reference plane is the tangent plane of the Earth passing through the observer; this horizontal plane intersects the celestial sphere along the *horizon*. The point just above the observer is called the *zenith* and the antipodal point below the observer is the *nadir*. (These two points are the poles corresponding to the horizon.) Great circles through the zenith are called *verticals*. All verticals intersect the horizon perpendicularly.

By observing the motion of a star over the course of a night, an observer finds out that it follows a track like one of those in Fig. 2.9. Stars rise in the east, reach their highest point, or *culminate*, on the vertical NZS, and set in the west. The vertical NZS is called the *meridian*. North and south directions are defined as the intersections of the meridian and the horizon.

One of the horizontal coordinates is the *altitude* or *elevation*, a, which is measured from the horizon along the vertical passing through the object. The altitude lies in the range $[-90°, +90°]$; it is positive for objects above the horizon and negative for the objects below the horizon. The *zenith distance*, or the angle between the object and the zenith, is obviously

$$z = 90° - a. \tag{2.10}$$

The second coordinate is the *azimuth*, A; it is the angular distance of the vertical of the object from some fixed direction. Unfortunately, in different contexts, different fixed directions are used; thus it is always advisable to check which definition is employed. The azimuth is usually measured from the north or south, and though clockwise is the preferred direction, counterclockwise measurements are also occasionally used. In geography, navigation and many other fields, the azimuth is measured from the north. In this book we have adopted a fairly common astronomical convention, measuring the azimuth *clockwise* from the *south*. Its values are usually normalised between 0° and 360° (or $-180°$–$180°$). The reason for this definition is that some other important angles are also measured from the south.

In Fig. 2.9a we can see the altitude and azimuth of three stars at some instant. As a star moves along its daily track, both of its coordinates will change. Another difficulty with this coordinate frame is its local character. In Fig. 2.9b we have the same stars, but the observer is now further south. We can see that the coordinates of the same star at the same moment are different for different observers. Since the horizontal coordinates are time and position dependent, they cannot be used, for instance, in star catalogues.

2.5 The Equatorial System

The direction of the rotation axis of the Earth remains almost constant and so does the equatorial plane perpendicular to this axis. Therefore the equatorial plane is a suitable reference plane for a coordinate frame that has to be independent of time and the position of the observer.

The intersection of the celestial sphere and the equatorial plane is a great circle, which is called the *equator of the celestial sphere*. The north pole of the celestial sphere is one of the poles corresponding to this great circle. It is also the point in the northern sky where the extension of the Earth's rotational axis meets the celestial sphere. The celestial north pole is at a distance of about one degree (which is equivalent to two full moons) from the moderately bright star Polaris. The meridian always passes through the north pole; it is divided by the pole into north and south meridians.

The angular separation of a star from the equatorial plane is not affected by the rotation of the Earth. This angle is called the *declination* δ.

Stars seem to revolve around the pole once every day (Fig. 2.10). To define the second coordinate, we must again agree on a fixed direction, unaffected by the Earth's rotation. From a mathematical point of view, it does not matter which point on the equator is selected. However, for later purposes, it is more appropriate to employ a certain point with some valuable properties, which will be explained in the next section. This point is called the *vernal equinox*. Because it used to be in the constellation Aries (the Ram), it is also called the first point of Aries ant denoted by the sign of Aries, Υ. Now we can define the second coordinate as the angle from the vernal equinox measured along the equator. This angle is the *right ascension* α (or R.A.) of the object, measured counterclockwise from Υ.

Since declination and right ascension are independent of the position of the observer and the motions of the Earth, they can be used in star maps and catalogues. As will be explained later, in many telescopes one of the axes (the hour axis) is parallel to the rotation axis of the Earth. The other axis (declination axis) is perpendicular to the hour axis. Declinations can be read immediately on the declination dial of the telescope. But the zero point of the right ascension seems to move in the sky, due to the diurnal rotation of the Earth. So we cannot use the right ascension to find an object unless we know the direction of the vernal equinox.

Since the south meridian is a well-defined line in the sky, we use it to establish a local coordinate corresponding to the right ascension. The *hour angle h* is measured clockwise from the meridian. The hour angle of an object is not a constant, but grows at a steady rate, due to the Earth's rotation. The hour angle of the vernal equinox is called the *sidereal time* Θ. Figure 2.11 shows that for any object,

$$\Theta = h + \alpha, \qquad (2.11)$$

where h is the object's hour angle and α its right ascension.

Since hour angle and sidereal time change with time at a constant rate, it is practical to express them in units of time. Also the closely related right ascension is customarily given in time units. Thus 24 hours equals 360 degrees, 1 hour = 15 degrees, 1 minute of time = 15 minutes of arc, and so on. All these quantities are in the range [0 h, 24 h). This custom also explains the name *hour circle*; it is a great circle along which the hour angle (or right ascension) is constant.

In practice, the sidereal time can be readily determined by pointing the telescope to an easily recognisable star and reading its hour angle on the hour angle dial of the telescope. The right ascension found in a catalogue is then added to the hour angle, giving the sidereal time at the moment of observation. For any other time, the sidereal time can be evaluated by adding the time elapsed since the observation. If we want to be accurate, we have to use a sidereal clock to measure time intervals. A sidereal clock runs 3 min 56.56 s fast a day as compared with an ordinary solar time

Fig. 2.10 At night, stars seem to revolve around the celestial pole. The altitude of the pole from the horizon equals the latitude of the observer. (Photo Aimo Sillanpää and Pasi Nurmi)

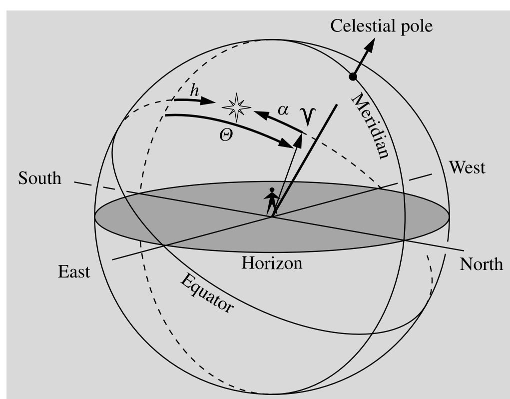


Fig. 2.11 The relation between the right ascension α, hour angle h and sidereal time Θ. The right ascension is measured counterclockwise from the vernal equinox, the hour angle and sidereal time clockwise from the South meridian. The sidereal time is the hour angle the vernal equinox

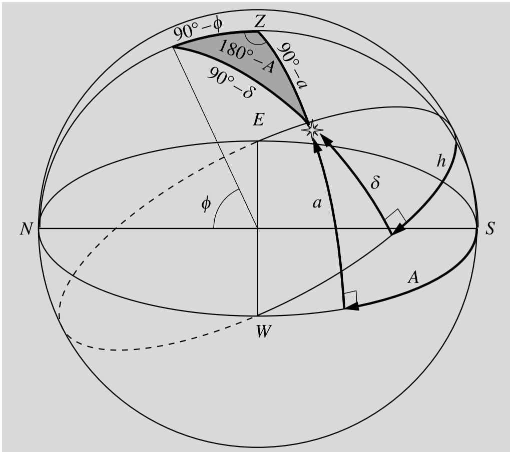


Fig. 2.12 The nautical triangle for deriving transformations between the horizontal and equatorial frames

clock:

$$\begin{aligned} &24\text{ h solar time} \\ &\quad = 24\text{ h } 3\text{ min } 56.56\text{ s sidereal time.} \end{aligned} \tag{2.12}$$

The reason for this is the orbital motion of the Earth: stars seem to move faster than the Sun across the sky; hence, a sidereal clock must run faster. (This is further discussed in Sect. 2.13.)

Transformations between the horizontal and equatorial frames are easily obtained from spherical trigonometry. Comparing Figs. 2.6 and 2.12, we find that we must make the following substitutions into (2.5):

$$\begin{aligned} &\psi = 90^\circ - A, \qquad \theta = a, \\ &\psi' = 90^\circ - h, \qquad \theta' = \delta, \qquad \chi = 90^\circ - \phi. \end{aligned} \tag{2.13}$$

The angle ϕ in the last equation is the altitude of the celestial pole, or the latitude of the observer. Making the substitutions, we get

$$\begin{aligned} \sin h \cos\delta &= \sin A \cos a, \\ \cos h \cos\delta &= \cos A \cos a \sin\phi + \sin a \cos\phi, \\ \sin\delta &= -\cos A \cos a \cos\phi + \sin a \sin\phi. \end{aligned} \tag{2.14}$$

The inverse transformation is obtained by substituting

$$\begin{aligned} &\psi = 90^\circ - h, \qquad \theta = \delta, \\ &\psi' = 90^\circ - A, \qquad \theta' = a, \\ &\chi = -(90^\circ - \phi), \end{aligned} \tag{2.15}$$

whence

$$\begin{aligned} \sin A \cos a &= \sin h \cos\delta, \\ \cos A \cos a &= \cos h \cos\delta \sin\phi - \sin\delta \cos\phi, \\ \sin a &= \cos h \cos\delta \cos\phi + \sin\delta \sin\phi. \end{aligned} \tag{2.16}$$

Since the altitude and declination are in the range $[-90^\circ, +90^\circ]$, it suffices to know the sine of one of these angles to determine the other angle unambiguously. Azimuth and right ascension, however, can have any value from 0° to 360° (or from 0 h to 24 h), and to solve for them, we have to know both the sine and cosine to choose the correct quadrant.

The altitude of an object is greatest when it is on the south meridian (the great circle arc between the celestial poles containing the zenith). At that moment (called *upper culmination*, or *transit*) its hour angle is 0 h. At the *lower culmination* the hour angle is $h = 12$ h. When $h = 0$ h, we get from the last equation in (2.16)

$$\begin{aligned} \sin a &= \cos\delta \cos\phi + \sin\delta \sin\phi \\ &= \cos(\phi - \delta) = \sin(90^\circ - \phi + \delta). \end{aligned}$$

Thus the altitude at the upper culmination is

$$a_{\max} = \begin{cases} 90^\circ - \phi + \delta, & \text{if the object culminates south of zenith,} \\ 90^\circ + \phi - \delta, & \text{if the object culminates north of zenith.} \end{cases} \tag{2.17}$$

The altitude is positive for objects with $\delta > \phi - 90^\circ$. Objects with declinations less than $\phi - 90^\circ$ can never be seen at the latitude ϕ. On the other hand, when $h = 12$ h we have

$$\begin{aligned} \sin a &= -\cos\delta \cos\phi + \sin\delta \sin\phi \\ &= -\cos(\delta + \phi) = \sin(\delta + \phi - 90^\circ), \end{aligned}$$

and the altitude at the lower culmination is

$$a_{\min} = \delta + \phi - 90^\circ . \tag{2.18}$$

Stars with $\delta > 90^\circ - \phi$ will never set. For example, in Helsinki ($\phi \approx 60^\circ$), all stars with a declination higher than 30° are such *circumpolar* stars. And stars with a declination less than −30° can never be observed there.

We shall now study briefly how the (α, δ) frame can be established by observations. Suppose we observe a circumpolar star at its upper and lower culmination (Fig. 2.13). At the upper transit, its altitude is $a_{\max} = 90^\circ - \phi + \delta$ and at the lower transit, $a_{\min} = \delta + \phi - 90^\circ$. Eliminating the latitude, we get

$$\delta = \frac{1}{2}(a_{\min} + a_{\max}). \tag{2.19}$$

Thus we get the same value for the declination, independent of the observer's location. Therefore we can use it as one of the absolute coordinates. From the same observations, we can also determine the direction of the celestial pole as well as the latitude of the observer. After these preparations, we can find the declination of any object by measuring its distance from the pole.

The equator can be now defined as the great circle all of whose points are at a distance of 90° from the pole. The zero point of the second coordinate (right ascension) can then be defined as the point where the Sun seems to cross the equator from south to north.

In practice the situation is more complicated, since the direction of Earth's rotation axis changes due to perturbations. Therefore the equatorial coordinate frame is nowadays defined using certain standard objects the positions of which are known very accurately. The best accuracy is achieved by using the most distant objects, *quasars* (Sect. 19.7), which remain in the same direction over very long intervals of time.

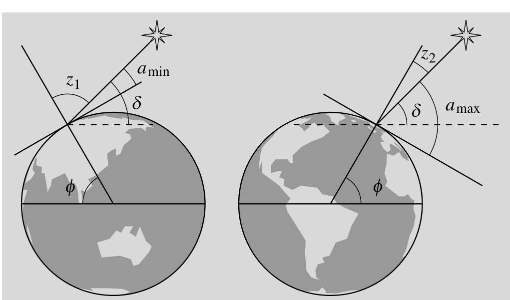


Fig. 2.13 The altitude of a circumpolar star at upper and lower culmination

2.6 Rising and Setting Times

From the last equation (2.16), we find the hour angle h of an object at the moment its altitude is a:

$$\cos h = -\tan\delta \tan\phi + \frac{\sin a}{\cos\delta \cos\phi} . \tag{2.20}$$

This equation can be used for computing rising and setting times. Then $a = 0$ and the hour angles corresponding to rising and setting times are obtained from

$$\cos h = -\tan\delta \tan\phi . \tag{2.21}$$

If the right ascension α is known, we can use (2.11) to compute the sidereal time Θ. (Later, in Sect. 2.14, we shall study how to transform the sidereal time to ordinary time.)

If higher accuracy is needed, we have to correct for the refraction of light caused by the atmosphere of the Earth (see Sect. 2.9). In that case, we must use a small negative value for a in (2.20). This value, the *horizontal refraction*, is about $-34'$.

The rising and setting times of the Sun given in almanacs refer to the time when the upper edge of the Solar disk just touches the horizon. To compute these times, we must set $a = -50'$ $(= -34' - 16')$.

Also for the Moon almanacs give rising and setting times of the upper edge of the disk. Since the distance of the Moon varies considerably, we cannot use any constant value for the radius of the Moon, but it has to be calculated separately each time. The Moon is also so close that its direction with respect to the background stars varies due to the rotation of the Earth. Thus the rising and setting times of the Moon are defined as the instants when the altitude of the Moon is $-34' - s + \pi$,

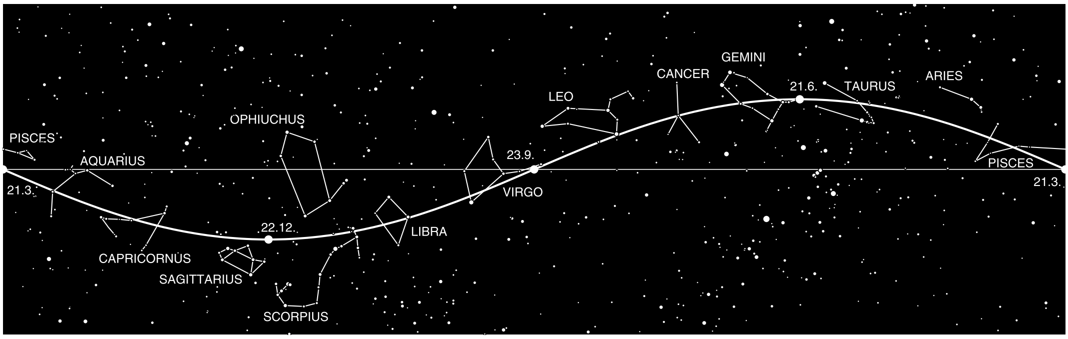


Fig. 2.14 The ecliptic or the apparent orbit of the Sun resembles somewhat a sine curve in the equatorial coordinate frame. The ecliptic passes through 13 constellations. Since many of them have been named after animals, the neighbourhood of the ecliptic is called the *zodiac*. Astrologers divide the ecliptic into 12 equal signs of 30° each. (Ophiuchus is unknown to them.) Due to the precession of the coordinate frame the astrological signs and the constellations with the same name are no more aligned. Also the sizes of the actual constellations are quite different

where s is the apparent radius ($15.5'$ on the average) and π the parallax ($57'$ on the average). The latter quantity is explained in Sect. 2.9.

Finding the rising and setting times of the Sun, planets and especially the Moon is complicated by their motion with respect to the stars. We can use, for example, the coordinates for the noon to calculate estimates for the rising and setting times, which can then be used to interpolate more accurate coordinates for the rising and setting times. When these coordinates are used to compute new times a pretty good accuracy can be obtained. The iteration can be repeated if even higher precision is required.

2.7 The Ecliptic System

The orbital plane of the Earth, the *ecliptic*, is the reference plane of another important coordinate frame. The ecliptic can also be defined as the great circle on the celestial sphere described by the Sun in the course of one year. This frame is used mainly for planets and other bodies of the solar system. The orientation of the Earth's equatorial plane remains invariant, unaffected by annual motion. In spring, the Sun appears to move from the southern hemisphere to the northern one (Fig. 2.14). The time of this remarkable event as well as the direction to the Sun at that moment are called the *vernal equinox*. At the vernal

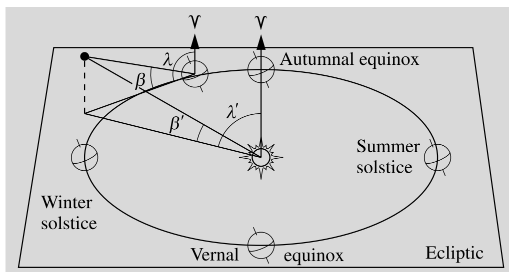


Fig. 2.15 The ecliptic geocentric (λ, β) and heliocentric (λ', β') coordinates are equal only if the object is very far away. The geocentric coordinates depend also on the Earth's position in its orbit

equinox, the Sun's right ascension and declination are zero. The equatorial and ecliptic planes intersect along a straight line directed towards the vernal equinox. Thus we can use this direction as the zero point for both the equatorial and ecliptic coordinate frames. The point opposite the vernal equinox is the *autumnal equinox*, it is the point at which the Sun crosses the equator from north to south.

The *ecliptic latitude* β is the angular distance from the ecliptic; it is in the range $[-90°, +90°]$. The other coordinate is the *ecliptic longitude* λ, measured counterclockwise from the vernal equinox (Fig. 2.15).

Transformation equations between the equatorial and ecliptic frames can be derived analogously to (2.14) and (2.16):

$$\begin{aligned}\sin\lambda\cos\beta &= \sin\delta\sin\varepsilon + \cos\delta\cos\varepsilon\sin\alpha,\\ \cos\lambda\cos\beta &= \cos\delta\cos\alpha, \qquad (2.22)\\ \sin\beta &= \sin\delta\cos\varepsilon - \cos\delta\sin\varepsilon\sin\alpha,\end{aligned}$$

$$\begin{aligned}\sin\alpha\cos\delta &= -\sin\beta\sin\varepsilon + \cos\beta\cos\varepsilon\sin\lambda,\\ \cos\alpha\cos\delta &= \cos\lambda\cos\beta, \qquad (2.23)\\ \sin\delta &= \sin\beta\cos\varepsilon + \cos\beta\sin\varepsilon\sin\lambda.\end{aligned}$$

The angle ε appearing in these equations is the *obliquity of the ecliptic*, or the angle between the equatorial and ecliptic planes. Its value is roughly 23°26′ (a more accurate value is given in Box 2.1).

Depending on the problem to be solved, we may encounter *heliocentric* (origin at the Sun), *geocentric* (origin at the centre of the Earth) or *topocentric* (origin at the observer) coordinates. For very distant objects the differences are negligible, but not for bodies of the solar system. To transform heliocentric coordinates to geocentric coordinates or vice versa, we must also know the distance of the object. This transformation is most easily accomplished by computing the rectangular coordinates of the object and the new origin, then changing the origin and finally evaluating the new latitude and longitude from the rectangular coordinates (see Examples 2.5 and 2.6).

2.8 The Galactic Coordinates

For studies of the Milky Way Galaxy, the most natural reference plane is the plane of the Milky Way (Fig. 2.16). Since the Sun lies very close to that plane, we can put the origin at the Sun. The *galactic longitude l* is measured counterclockwise (like right ascension) from the direction of the centre of the Milky Way (in Sagittarius, α = 17 h 45.7 min, $\delta = -29°00'$). The

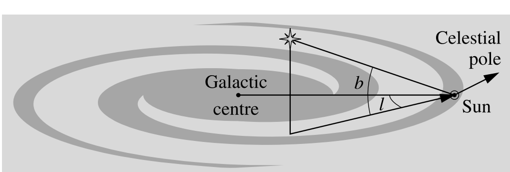


Fig. 2.16 The galactic coordinates l and b

galactic latitude b is measured from the galactic plane, positive northwards and negative southwards. This definition was officially adopted only in 1959, when the direction of the galactic centre was determined from radio observations accurately enough. The old galactic coordinates l^{I} and b^{I} had the intersection of the equator and the galactic plane as their zero point.

The galactic coordinates can be obtained from the equatorial ones with the transformation equations

$$\begin{aligned}\sin(l_{\mathrm{N}} - l)\cos b &= \cos\delta\sin(\alpha - \alpha_{\mathrm{P}}),\\ \cos(l_{\mathrm{N}} - l)\cos b &= -\cos\delta\sin\delta_{\mathrm{P}}\cos(\alpha - \alpha_{\mathrm{P}})\\ &\quad + \sin\delta\cos\delta_{\mathrm{P}},\\ \sin b &= \cos\delta\cos\delta_{\mathrm{P}}\cos(\alpha - \alpha_{\mathrm{P}})\\ &\quad + \sin\delta\sin\delta_{\mathrm{P}},\end{aligned} \qquad (2.24)$$

where the direction of the Galactic north pole is α_{P} = 12 h 51.4 min, $\delta_{\mathrm{P}} = 27°08'$, and the galactic longitude of the celestial pole, $l_{\mathrm{N}} = 123.0°$.

2.9 Perturbations of Coordinates

Even if a star remains fixed with respect to the Sun, its coordinates can change, due to several disturbing effects. Naturally its altitude and azimuth change constantly because of the rotation of the Earth, but even its right ascension and declination are not quite free from perturbations.

Precession The Earth is not quite spherical but slightly flattened. Since most of the members of the solar system orbit close to the ecliptic, they tend to pull the equatorial bulge of the Earth towards it. Most of this "flattening" torque is caused by the Moon and the Sun. But the Earth is rotating and therefore the torque cannot change the inclination of the equator relative to the ecliptic. Instead, the rotation axis turns in a direction perpendicular to the axis and the torque, thus describing a cone once in roughly 26,000 years. This slow turning of the rotation axis is called *precession* (Fig. 2.17). Because of precession, the

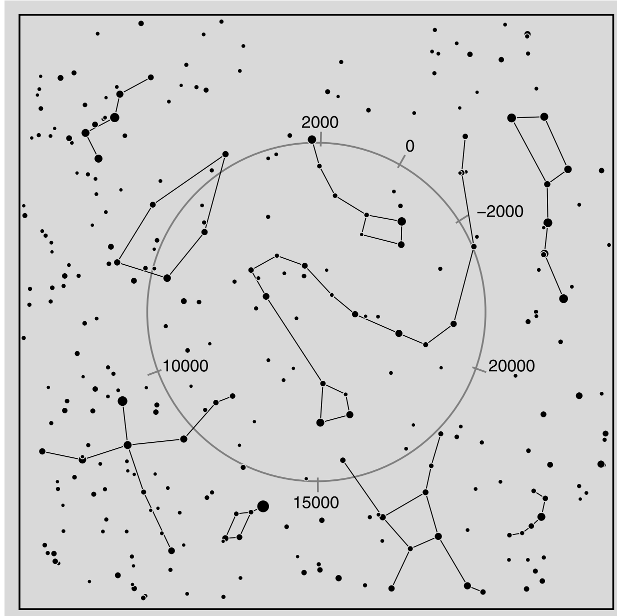


Fig. 2.17 Due to precession the celestial pole moves and makes a full revolution in about 26,000 years. For a while it is still approaching Polaris, but will then begin to recede. In about 14,000 it will be close to Vega. Somewhat over 2000 years BC the pole was close to the α star of Drace, Thuban

vernal equinox moves along the ecliptic clockwise about 50 seconds of arc every year, thus increasing the ecliptic longitudes of all objects at the same rate. At present the rotation axis points about one degree away from Polaris, but after 12,000 years, the celestial pole will be roughly in the direction of Vega. The changing ecliptic longitudes also affect the right ascension and declination. Thus we have to know the instant of time, or *epoch*, for which the coordinates are given.

Currently most maps and catalogues use the epoch J2000.0, which means the beginning of the year 2000, or, to be exact, the noon of January 1, 2000, or the Julian date 2,451,545.0 (see Sect. 2.15).

Let us now derive expressions for the changes in right ascension and declination. Taking the last transformation equation in (2.23),

$$\sin\delta = \cos\varepsilon\sin\beta + \sin\varepsilon\cos\beta\sin\lambda,$$

and differentiating, we get

$$\cos\delta\,\mathrm{d}\delta = \sin\varepsilon\cos\beta\cos\lambda\,\mathrm{d}\lambda.$$

Applying the second equation in (2.22) to the right-hand side, we have, for the change in declination,

$$\mathrm{d}\delta = \mathrm{d}\lambda\sin\varepsilon\cos\alpha. \tag{2.25}$$

By differentiating the equation

$$\cos\alpha\cos\delta = \cos\beta\cos\lambda,$$

we get

$$\begin{aligned}&-\sin\alpha\cos\delta\,\mathrm{d}\alpha - \cos\alpha\sin\delta\,\mathrm{d}\delta\\ &\quad = -\cos\beta\sin\lambda\,\mathrm{d}\lambda;\end{aligned}$$

and, by substituting the previously obtained expression for $\mathrm{d}\delta$ and applying the first equation (2.22), we have

$$\begin{aligned}&\sin\alpha\cos\delta\,\mathrm{d}\alpha\\ &\quad = \mathrm{d}\lambda\big(\cos\beta\sin\lambda - \sin\varepsilon\cos^2\alpha\sin\delta\big)\\ &\quad = \mathrm{d}\lambda\big(\sin\delta\sin\varepsilon + \cos\delta\cos\varepsilon\sin\alpha\\ &\qquad -\sin\varepsilon\cos^2\alpha\sin\delta\big).\end{aligned}$$

Simplifying this, we get

$$\mathrm{d}\alpha = \mathrm{d}\lambda(\sin\alpha\sin\varepsilon\tan\delta + \cos\varepsilon). \tag{2.26}$$

If $\mathrm{d}\lambda$ is the annual increment of the ecliptic longitude (about $50''$), the precessional changes in

Table 2.1 Precession constants m and n. Here, "a" means a tropical year

Epoch	m	n	
1800	3.07048 s/a	1.33703 s/a =	20.0554″/a
1850	3.07141	1.33674	20.0511
1900	3.07234	1.33646	20.0468
1950	3.07327	1.33617	20.0426
2000	3.07419	1.33589	20.0383

right ascension and declination in one year are thus

$$\begin{aligned} d\delta &= d\lambda \sin\varepsilon \cos\alpha, \\ d\alpha &= d\lambda(\sin\varepsilon \sin\alpha \tan\delta + \cos\varepsilon). \end{aligned} \tag{2.27}$$

These expressions are usually written in the form

$$\begin{aligned} d\delta &= n \cos\alpha, \\ d\alpha &= m + n \sin\alpha \tan\delta, \end{aligned} \tag{2.28}$$

where

$$\begin{aligned} m &= d\lambda \cos\varepsilon, \\ n &= d\lambda \sin\varepsilon \end{aligned} \tag{2.29}$$

are the *precession constants*. Since the obliquity of the ecliptic is not exactly a constant but changes with time, m and n also vary slowly with time. However, this variation is so slow that usually we can regard m and n as constants unless the time interval is very long. The values of these constants for some epochs are given in Table 2.1. For intervals longer than a few decades a more rigorous method should be used. Its derivation exceeds the level of this book, but the necessary formulas are given in Box 2.1.

Nutation The Moon's orbit is inclined with respect to the ecliptic, resulting in precession of its orbital plane. One revolution takes 18.6 years, producing perturbations with the same period in the precession of the Earth. This effect, *nutation*, changes ecliptic longitudes as well as the obliquity of the ecliptic (Fig. 2.18). Calculations are now much more complicated, but fortunately nutational perturbations are relatively small, only fractions of an arc minute. Thus they can often be calculated using the simple formulas of Box 2.1 or just omitted.

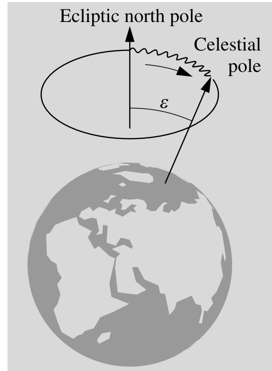


Fig. 2.18 Due to precession the rotation axis of the Earth turns around the ecliptic north pole. Nutation is the small wobble disturbing the smooth precessional motion. In this figure the magnitude of the nutation is highly exaggerated

Parallax If we observe an object from different points, we see it in different directions. The difference of the observed directions (as well as the whole phenomenon) is called the *parallax*. Since the amount of parallax depends on the distance of the observer from the object, we can utilise the parallax to measure distances. Human stereoscopic vision is based (at least to some extent) on this effect. For astronomical purposes we need much longer baselines than the distance between our eyes (about 7 cm). Appropriately large and convenient baselines are the radius of the Earth and the radius of its orbit.

Distances to the nearest stars can be determined from the *annual parallax*, which is the angle subtended by the radius of the Earth's orbit (called the *astronomical unit*, au) as seen from the star. (We shall discuss this further in Sect. 2.10.)

By *diurnal parallax* we mean the change of direction due to the daily rotation of the Earth. In addition to the distance of the object, the diurnal parallax also depends on the latitude of the observer. If we talk about the parallax of a body in our solar system, we always mean the angle subtended by the Earth's equatorial radius (6378 km) as seen from the object (Fig. 2.19). This equals the apparent shift of the object with respect to the background stars seen by an observer at the equator if (s)he observes the object moving from the

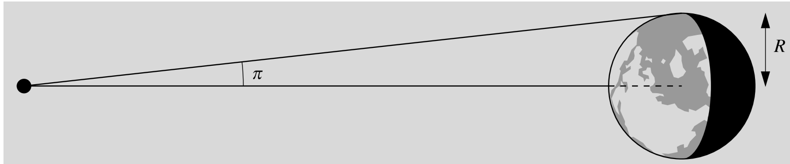


Fig. 2.19 The horizontal parallax π of an object is the angle subtended by the Earth's equatorial radius as seen from the object

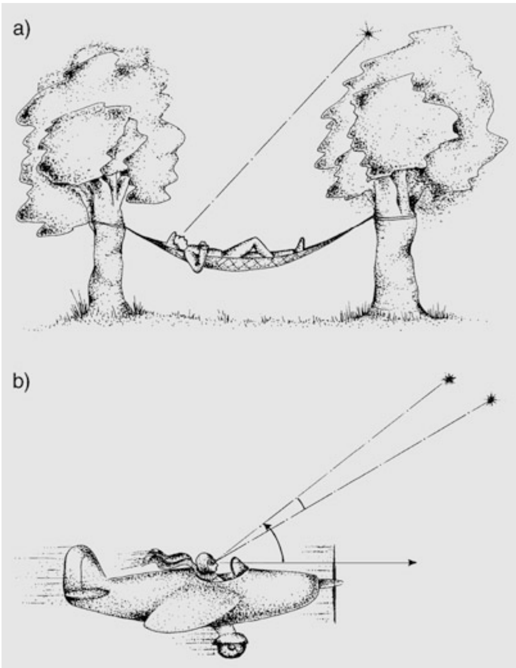


Fig. 2.20 The effect of aberration on the apparent direction of an object. (**a**) Observer at rest. (**b**) Observer in motion

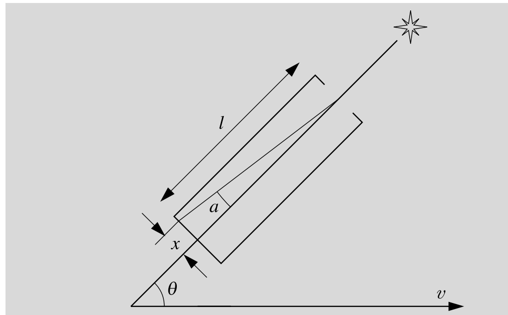


Fig. 2.21 A telescope is pointed in the true direction of a star. It takes a time $t = l/c$ for the light to travel the length of the telescope. The telescope is moving with velocity v, which has a component $v \sin\theta$, perpendicular to the direction of the light beam. The beam will hit the bottom of the telescope displaced from the optical axis by a distance $x = tv \sin\theta = l(v/c) \sin\theta$. Thus the change of direction in radians is $a = x/l = (v/c) \sin\theta$

horizon to the zenith. The parallax of the Moon, for example, is about $57'$, and that of the Sun $8.79''$.

In astronomy parallax may also refer to distance in general, even if it is not measured using the shift in the observed direction.

Aberration Because of the finite speed of light, an observer in motion sees an object shifted in the direction of her/his motion (Fig. 2.20). This change of apparent direction is called the *aberration*. To derive the exact value we have to use the special theory of relativity, but for practical purposes it suffices to use the approximate value

$$a = \frac{v}{c} \sin\theta, \quad [a] = \text{rad}, \qquad (2.30)$$

where v is the velocity of the observer, c is the speed of light and θ is the angle between the true direction of the object and the velocity vector of the observer (Fig 2.21). The greatest possible value of the aberration due to the orbital motion of the Earth, v/c, called the *aberration constant*, is $21''$. The maximal shift due to the Earth's rotation, the diurnal aberration constant, is much smaller, about $0.3''$.

Refraction Since light is refracted by the atmosphere, the direction of an object differs from the true direction by an amount depending on the atmospheric conditions along the line of sight. Since this refraction varies with atmospheric pressure and temperature, it is very difficult to predict it accurately. However, an approximation good enough for most practical purposes is easily derived. If the object is not too far from the zenith, the atmosphere between the object and the observer can be approximated by a stack of parallel planar layers, each of which has a certain index of refraction n_i (Fig. 2.22). Outside the atmosphere, we have $n = 1$.

Let the true zenith distance be z and the apparent one, ζ. Using the notations of Fig. 2.22, we obtain the following equations for the boundaries of the successive layers:

$$\begin{aligned}\sin z &= n_k \sin z_k,\\ &\vdots\\ n_2 \sin z_2 &= n_1 \sin z_1,\\ n_1 \sin z_1 &= n_0 \sin \zeta.\end{aligned}$$

Simplifying this set of equations we get

$$\sin z = n_0 \sin \zeta. \tag{2.31}$$

When the *refraction angle* $R = z - \zeta$ is small and is expressed in radians, we have

$$\begin{aligned}n_0 \sin \zeta &= \sin z = \sin(R + \zeta)\\ &= \sin R \cos \zeta + \cos R \sin \zeta\\ &\approx R \cos \zeta + \sin \zeta.\end{aligned}$$

Thus we get

$$R = (n_0 - 1) \tan \zeta, \quad [R] = \text{rad}. \tag{2.32}$$

The index of refraction depends on the density of the air, which further depends on the pressure and temperature. When the altitude is over 15°, we can use an approximate formula

$$R = \frac{P}{273 + T} 0.00452° \tan(90° - a), \tag{2.33}$$

where a is the altitude in degrees, T temperature in degrees Celsius, and P the atmospheric pressure in hectopascals (or, equivalently, in millibars). At lower altitudes the curvature of the atmosphere must be taken into account. An approximate formula for the refraction is then

$$R = \frac{P}{273 + T} \frac{0.1594 + 0.0196a + 0.00002a^2}{1 + 0.505a + 0.0845a^2}. \tag{2.34}$$

These formulas are widely used, although they are against the rules of dimensional analysis. To get correct values, all quantities must be expressed in correct units. Figure 2.23 shows the refraction under different conditions evaluated from these formulas.

Altitude is always (except very close to zenith) increased by refraction. On the horizon the change is about 34′, which is slightly more than the diameter of the Sun. When the lower limb of the Sun just touches the horizon, the Sun has in reality already set.

Light coming from the zenith is not refracted at all if the boundaries between the layers are horizontal. Under some climatic conditions, a boundary (e.g. between cold and warm layers) can be slanted, and in this case, there can be a small

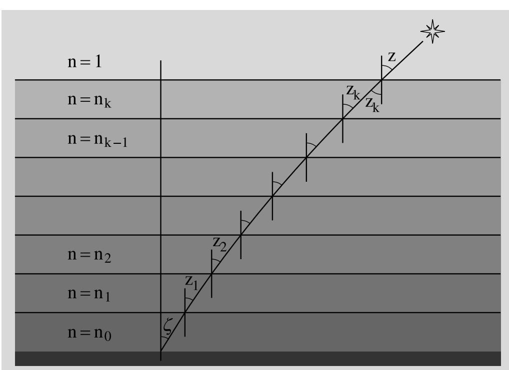


Fig. 2.22 Refraction of a light ray travelling through the atmosphere. If the altitude is not very low we can think that the atmosphere consists of planar layers each of which has its own index of refraction. Outside the atmosphere the index is $n = 1$ and increases towards the ground

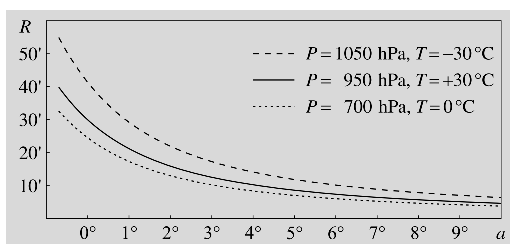


Fig. 2.23 Refraction at different altitudes. The refraction angle R tells how much higher the object seems to be compared with its true altitude a. Refraction depends on the density and thus on the pressure and temperature of the air. *The upper curves* give the refraction at sea level during rather extreme weather conditions. At the altitude of 2.5 kilometers the average pressure is only 700 hPa, and thus the effect of refraction smaller (*lowest curve*)

zenith refraction, which is of the order of a few arc seconds.

Stellar positions given in star catalogues are *mean places*, from which the effects of parallax, aberration and nutation have been removed. The mean place of the date (i.e. at the observing time) is obtained by correcting the mean place for the proper motion of the star (Sect. 2.10) and precession. The *apparent place* is obtained by correcting this place further for nutation, parallax and aberration. There is a catalogue published annually that gives the apparent places of certain references stars at intervals of a few days. These positions have been corrected for precession, nutation, parallax and annual aberration. The effects of diurnal aberration and refraction are not included because they depend on the location of the observer.

2.10 Positional Astronomy

The position of a star can be measured either with respect to some reference stars (relative astrometry) or with respect to a fixed coordinate frame (absolute astrometry).

Absolute coordinates are usually determined using a *meridian circle*, which is a telescope that can be turned only in the meridional plane (Fig. 2.27). It has only one axis, which is aligned exactly in the east-west direction. Since all stars cross the meridian in the course of a day, they all come to the field of the meridian circle at some time or other. When a star culminates, its altitude and the time of the transit are recorded. If the time is determined with a sidereal clock, the sidereal time immediately gives the right ascension of the star, since the hour angle is $h = 0$ h. The other coordinate, the declination δ, is obtained from the altitude:

$$\delta = a - (90^\circ - \phi),$$

where a is the observed altitude and ϕ is the geographic latitude of the observatory.

Relative coordinates are measured on photographic plates (Fig. 2.25) or CCD images containing some known reference stars. The scale of the plate as well as the orientation of the coordinate frame can be determined from the reference stars. After this has been done, the right ascension and declination of any object in the image can be calculated if its coordinates in the image are measured.

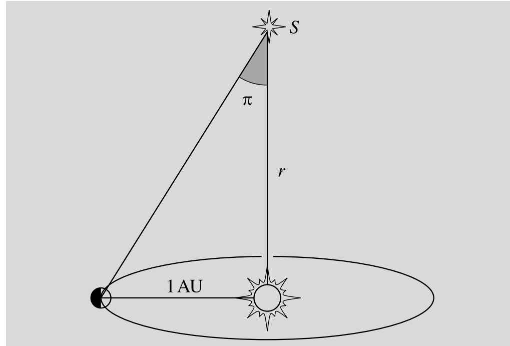


Fig. 2.24 The trigonometric parallax π of a star S is the angle subtended by the radius of the orbit of the Earth, or one astronomical unit, as seen from the star

All stars in a small field are almost equally affected by the dominant perturbations, precession, nutation, and aberration. The much smaller effect of parallax, on the other hand, changes the relative positions of the stars.

The shift in the direction of a star with respect to distant background stars due to the annual motion of the Earth is called the *trigonometric parallax* of the star. It gives the distance of the star: the smaller the parallax, the farther away the star is. Trigonometric parallax is, in fact, the only direct method we currently have of measuring distances to stars. Later we shall be introduced to some other, indirect methods, which require certain assumptions on the motions or structure of stars. The same method of triangulation is employed to measure distances of earthly objects. To measure distances to stars, we have to use the longest baseline available, the diameter of the orbit of the Earth.

During the course of one year, a star will appear to describe a circle if it is at the pole of the ecliptic, a segment of line if it is in the ecliptic, or an ellipse otherwise. The semimajor axis of this ellipse is called the parallax of the star. It is usually denoted by π. It equals the angle subtended by the radius of the Earth's orbit as seen from the star (Fig. 2.24).

The unit of distance used in astronomy is *parsec* (pc). At a distance of one parsec, one astronomical unit subtends an angle of one arc second. Since one radian is about 206,265″, 1 pc equals 206,265 au. Furthermore, because 1 au = 1.496×10^{11} m, 1 pc $\approx 3.086 \times 10^{16}$ m. If the parallax is given in arc seconds, the distance is simply

$$r = 1/\pi, \quad [r] = \text{pc}, \quad [\pi] = {}''. \tag{2.35}$$

In popular astronomical texts, distances are usually given in *light-years*, one light-year being the distance light travels in one year, or 9.5×10^{15} m. Thus one parsec is about 3.26 light-years.

The first parallax measurement was accomplished by *Friedrich Wilhelm Bessel* (1784–1846) in 1838. He found the parallax of 61 Cygni to be 0.3″. The nearest star Proxima Centauri has a parallax of 0.762″ and thus a distance of 1.31 pc.

In addition to the motion due to the annual parallax, many stars seem to move slowly in a direction that does not change with time. This effect is caused by the relative motion of the Sun and the stars through space; it is called the *proper motion*. The appearance of the sky and the shapes of the constellations are constantly, although extremely slowly, changed by the proper motions of the stars (Fig. 2.26).

The velocity of a star with respect to the Sun can be divided into two components (Fig. 2.28), one of which is directed along the line of sight (the radial component or the *radial velocity*), and the other perpendicular to it (the tangential component). The tangential velocity results in the proper motion, which can be measured by taking pictures at intervals of several years or decades. The proper motion μ has two components, one giving the change in declination μ_δ and the other, in right ascension, $\mu_\alpha \cos\delta$. The coefficient $\cos\delta$ is used to correct the scale of right ascension: hour circles (the great circles with $\alpha =$ constant) approach each other towards the poles, so the coordinate difference must be multiplied by $\cos\delta$ to obtain the true angular separation. The total proper motion is

$$\mu = \sqrt{\mu_\alpha^2 \cos^2\delta + \mu_\delta^2}. \tag{2.36}$$

The greatest known proper motion belongs to Barnard's Star, which moves across the sky at the enormous speed of 10.3 arc seconds per year. It needs less than 200 years to travel the diameter of a full moon.

In order to measure proper motions, we must observe stars for a long time. The radial component, on the other hand, is readily obtained from a single observation, thanks to the *Doppler effect*. By the Doppler effect we mean the change in frequency and wavelength of radiation due to the radial velocity of the radiation source. The same effect can be observed, for example, in the sound of an ambulance, the pitch being higher when the ambulance is approaching and lower when it is receding.

The formula for the Doppler effect for small velocities can be derived as in Fig. 2.29. The source of radiation transmits electromagnetic waves, the period of one cycle being T. In time T, the radiation approaches the observer by a distance $s = cT$, where c is the speed of propagation. During the same time, the source moves with respect to the observer a distance $s' = vT$, where v is the speed of the source, positive for a receding source and negative for an approaching one. We find that the length of one cycle, the wavelength λ, equals

$$\lambda = s + s' = cT + vT.$$

If the source were at rest, the wavelength of its radiation would be $\lambda_0 = cT$. The motion of the source changes the wavelength by an amount

$$\Delta\lambda = \lambda - \lambda_0 = cT + vT - cT = vT,$$

and the relative change $\Delta\lambda$ of the wavelength is

$$\frac{\Delta\lambda}{\lambda_0} = \frac{v}{c}. \tag{2.37}$$

This is valid only when $v \ll c$. For very high velocities, we must use the relativistic formula

$$\frac{\Delta\lambda}{\lambda_0} = \sqrt{\frac{1 + v/c}{1 - v/c}} - 1. \tag{2.38}$$

These formulas are valid only for electromagnetic radiation. The Doppler shift of e.g. sound

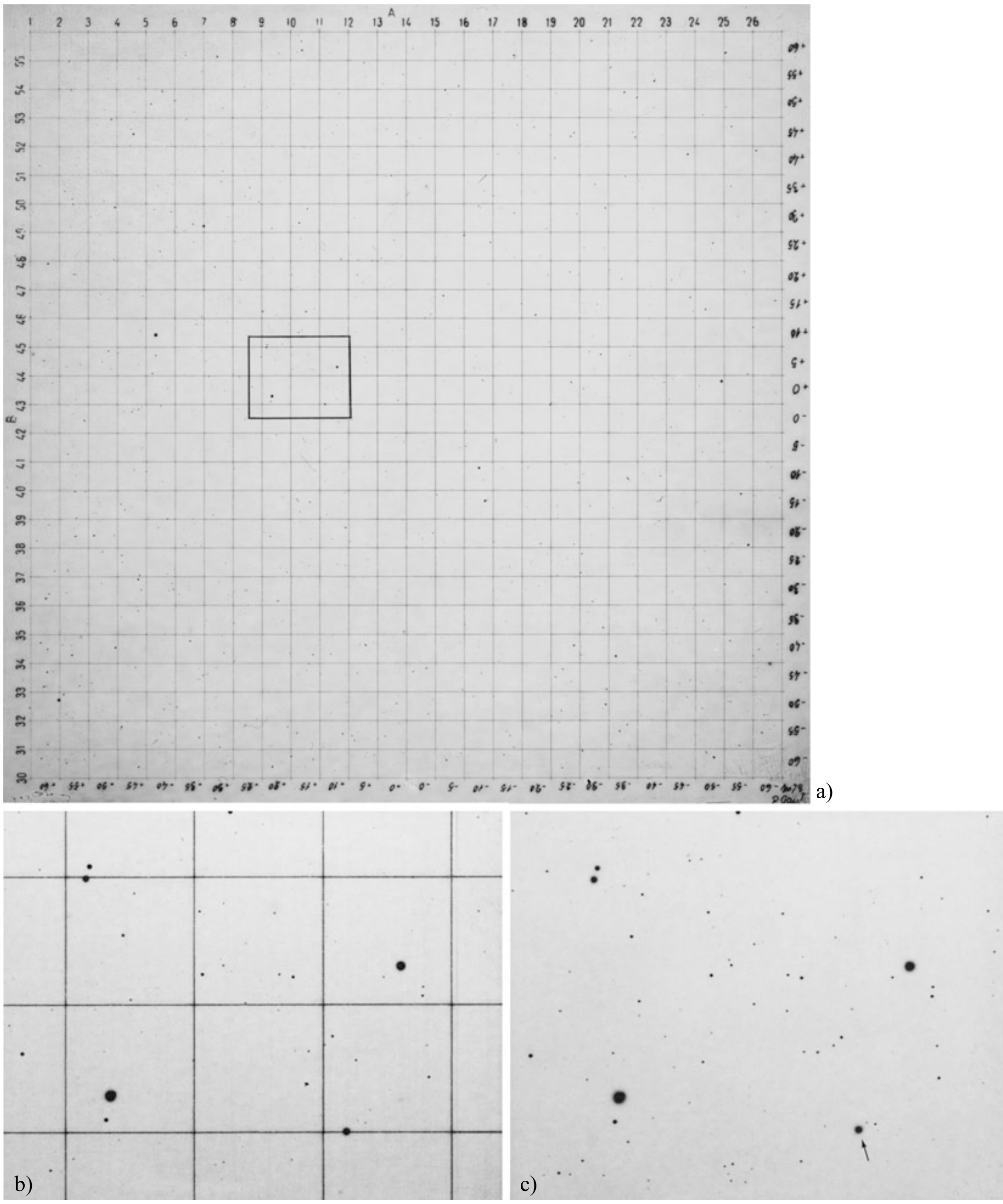


Fig. 2.25 (**a**) A plate photographed for the Carte du Ciel project in Helsinki on November 21, 1902. The centre of the field is at $\alpha = 18$ h 40 min, $\delta = 46°$, and the area is $2° \times 2°$. Distance between coordinate lines (exposed separately on the plate) is 5 minutes of arc. (**b**) The framed region on the same plate. (**c**) The same area on a plate taken on November 7, 1948. The bright star in the lower right corner (SAO 47747) has moved about 12 seconds of arc. The brighter, slightly drop-shaped star to the left is a binary star (SAO 47767); the separation between its components is $8''$

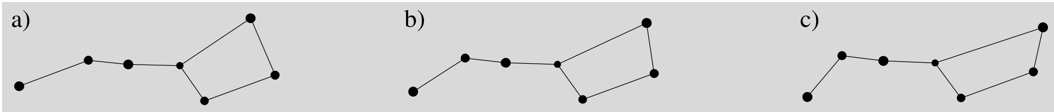


Fig. 2.26 Proper motions of stars slowly change the appearance of constellations. (**a**) The Big Dipper during the last ice age 30,000 years ago, (**b**) nowadays, and (**c**) after 30,000 years

Fig. 2.27 Astronomers discussing observations with the transit circle of Helsinki Observatory in 1904

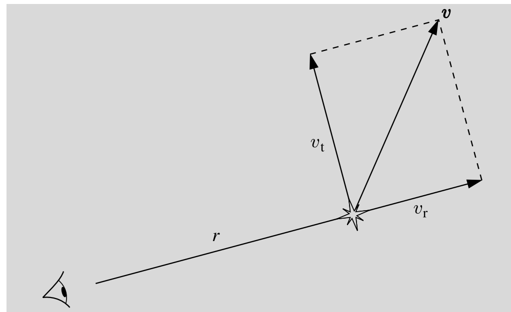


Fig. 2.28 The radial and tangential components, v_r and v_t of the velocity $\boldsymbol{v}$ of a star. The latter component is observed as proper motion

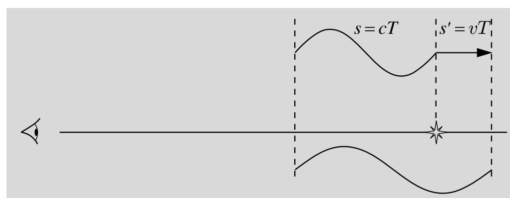


Fig. 2.29 The wavelength of radiation increases if the source is receding

waves is different depending on the motion of the source or the observer.

In astronomy the Doppler effect can be seen in stellar spectra, in which the spectral lines are often displaced towards the blue (shorter wavelengths) or red (longer wavelengths) end of the spectrum. A *blueshift* means that the star is approaching, while a *redshift* indicates that it is receding.

The displacements due to the Doppler effect are usually very small. In order to measure them, a reference spectrum is exposed on the plate next to the stellar spectrum. Now that CCD-cameras have replaced photographic plates, separate calibration exposures of reference spectra are taken to determine the wavelength scale (Sect. 3.3). The lines in the reference spectrum are produced by a light source at rest in the laboratory. If the reference spectrum contains some lines found also in the stellar spectrum, the displacements can be measured.

Also the spectra of distant galaxies show redshifts, but only a part of that is due to the Doppler effect. Instead, the redshifts are caused by the expansion of the Universe (Chap. 20). Also, gravitational fields cause redshifts as predicted by the general relativity (Appendix B).

Displacements of spectral lines give the radial velocity v_r of the star, and the proper motion μ can be measured from photographic plates or CCD images. To find the tangential velocity v_t, we have to know the distance r, obtainable from e.g. parallax measurements. Tangential velocity and proper motion are related by

$$v_t = \mu r. \tag{2.39}$$

If μ is given in arc seconds per year and r in parsecs we have to make the following unit transformations to get v_t in km/s:

$$1 \text{ rad} = 206{,}265'', \qquad 1 \text{ year} = 3.156 \times 10^7 \text{ s},$$
$$1 \text{ pc} = 3.086 \times 10^{13} \text{ km}.$$

Hence

$$\begin{aligned} &v_t = 4.74\,\mu r, \quad [v_t] = \text{km/s}, \\ &[\mu] = ''/\text{a}, \quad [r] = \text{pc}. \end{aligned} \tag{2.40}$$

The total velocity v of the star is then

$$v = \sqrt{v_r^2 + v_t^2}. \tag{2.41}$$

2.11 Constellations

At any one time, about 1000–1500 stars can be seen in the sky (above the horizon). Under ideal conditions, the number of stars visible to the naked eye can be as high as 3000 on a hemisphere, or 6000 altogether. Some stars seem to form figures vaguely resembling something; for ages they have been ascribed to various mythological and other animals. This grouping of stars into constellations is a product of human imagination without any physical basis. Different cultures have different constellations, depending on their mythology, history and environment.

About half of the shapes and names of the constellations we are familiar with date back to Mediterranean antiquity. But the names and boundaries were far from unambiguous as late as the 19th century. Therefore the International Astronomical Union (IAU) confirmed fixed boundaries at its 1928 meeting.

The official boundaries of the constellations were established along lines of constant right ascension and declination for the epoch 1875. During the time elapsed since then, precession has noticeably turned the equatorial frame. However, the boundaries remain fixed with respect to the stars. So a star belonging to a constellation will belong to it forever (unless it is moved across the boundary by its proper motion).

The names of the 88 constellations confirmed by the IAU are given in Table C.21 at the end of the book. The table also gives the abbreviation of the Latin name, its genitive (needed for names of stars) and the English name.

In his star atlas *Uranometria* (1603) *Johannes Bayer* started the current practice to denote the brightest stars of each constellation by Greek letters. The brightest star is usually α (alpha), e.g. Deneb in the constellation Cygnus is α Cygni, which is abbreviated as α Cyg. The second brightest star is β (beta), the next one γ (gamma) and so on. There are, however, several exceptions to this rule; for example, the stars of the Big Dipper are named in the order they appear in the constellation. After the Greek alphabet has been exhausted, Latin letters can be employed. Another method is to use numbers, which are assigned in the order of increasing right ascension; e.g. 30 Tau is a bright binary star in the constellation Taurus. These numbers are based on the star catalogue *Historia Cœlestis Britannica* (1725) by the first Astronomer Royal, John Flamsteed. Moreover, variable stars have their special identifiers (Sect. 13.1). About two hundred bright stars have a proper name; e.g. the bright α Aur is called also Capella.

As telescopes evolved, more and more stars were seen and catalogued. It soon became impractical to continue this method of naming. Thus most of the stars are known only by their catalogue index numbers. One star may have many different numbers; e.g. the abovementioned Capella (α Aur) is number BD+45° 1077 in the Bonner Durchmusterung and HD 34029 in the Henry Draper catalogue.

2.12 Star Catalogues and Maps

The first actual star catalogue was published by *Ptolemy* in the second century; this catalogue appeared in the book to be known later as *Almagest* (which is a Latin corruption of the name of the Arabic translation, *Al-mijisti*). It had 1025 entries; the positions of these bright stars had been measured by *Hipparchos* 250 years earlier. Ptolemy's catalogue was the only widely used one prior to the 17th century.

The first catalogues still being used by astronomers were prepared under the direction of *Friedrich Wilhelm August Argelander* (1799–1875). Argelander worked in Turku and later served as professor of astronomy in Helsinki, but he made his major contributions in Bonn. Using a 72 mm telescope, he and his assistants measured the positions and estimated the magnitudes of 320,000 stars. The catalogue, *Bonner Durchmusterung*, contains nearly all stars brighter than magnitude 9.5 between the north pole and declination −2°. (Magnitudes are further discussed in Chap. 4.) Argelander's work was later used as a model for two other large catalogues covering the whole sky. The total number of stars in these catalogues is close to one million (Fig. 2.30).

The purpose of these *Durchmusterungen* or general catalogues was to systematically list a great number of stars. In the *zone catalogues*, the main goal is to give the positions of stars as exactly as possible. A typical zone catalogue is the German *Katalog der Astronomischen Gesellschaft* (AGK). Twelve observatories, each measuring a certain region in the sky, contributed to this catalogue. The work was begun in the 1870's and completed at the turn of the century.

General and zone catalogues were based on visual observations with a telescope. The evolution of photography made this kind of work unnecessary at the end of the 19th century. Photographic plates could be stored for future purposes, and measuring the positions of stars became easier and faster, making it possible to measure many more stars.

A great international program was started at the end of the 19th century in order to photograph the entire sky. Eighteen observatories participated in this *Carte du Ciel* project, all using similar instruments and plates. The positions of stars were first measured with respect to a rectangular grid exposed on each plate (Fig. 2.25a). These coordinates could then be converted into declination and right ascension.

The old plates have been useful also much later. By observing the same areas again the proper motions of many stars can be measured.

Positions of stars in catalogues are measured with respect to certain comparison stars, the coordinates of which are known with high accuracy. The coordinates of these reference stars are published in fundamental catalogues. The first such catalogue was needed for the AGK catalogue; it was published in Germany in 1879. This *Fundamental Katalog* (FK 1) gives the positions of over 500 stars.

A widely used catalogue is the SAO catalogue, published by the Smithsonian Astrophysical Observatory in the 1960's. It contains the exact positions, magnitudes, proper motions, spectral classifications, etc. of 258,997 stars brighter than magnitude 9. The catalogue was accompanied by a star map containing all the stars in the catalogue.

In the 1990's a large astrometric catalogue, *PPM* (Positions and Proper Motions), was published to replace the AGK and SAO catalogues. It contained all stars brighter than 7.5 magnitudes, and was almost complete to magnitude 8.5. Altogether, the four volumes of the catalogue contained information on 378,910 stars.

The PPM was effectively replaced by the Tycho catalogue from Hipparcos satellite. Hipparcos was the first astrometric satellite, and was launched by the European Space Agency (ESA) in 1989. Although Hipparcos didn't reach the planned geosynchronous orbit, it gave exact positions of over a hundred thousand stars. The *Hipparcos catalogue*, based on the measurements of the satellite, contains astrometric and photometric data of 118,000 stars. The coordinates are precise to a couple of milliarcseconds. The less precise *Tycho catalogue* contains the data of about one million stars.

In 1999 and 2000, the sixth version of the Fundamental Katalog, the FK6, was published. It combined the Hipparcos data and FK5 for 4150 fundamental stars. The typical mean error in proper motion was 0.35 milliarcseconds per year for the basic stars. With the advance of the Internet, the printed versions of star catalogues were

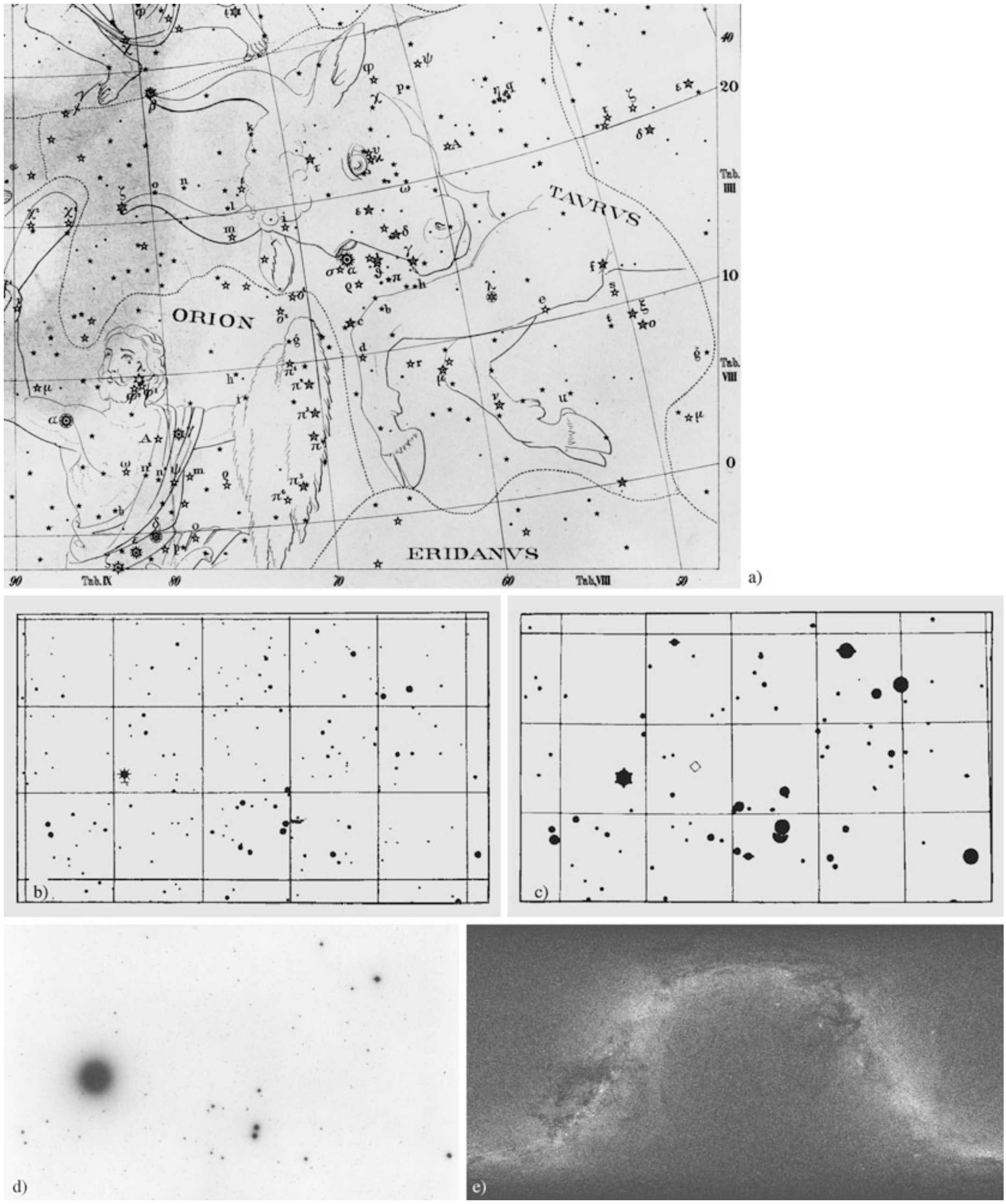


Fig. 2.30 The representations in four atlases of the Hyades cluster in the constellation Taurus. (**a**) Heis: Atlas Coelestis, published in 1872. (**b**) Bonner Durchmusterung. (**c**) SAO, (**d**) Palomar Sky Atlas, red plate. The big blob is the brightest star of Taurus, or α Tauri alias Aldebaran. (**e**) All the stars in the Tycho Catalog, numbering over one million, are marked on an all-sky chart. The bright lane is the Milky Way. (Picture David Seal, NASA/JPL/Caltech)

discontinued in the first years of the new millennium, and the catalogues were moved to the Internet.

In the current *ICRS* system (International Coordinate Reference System) the celestial coordinates are fixed to distant quasars, the positions of which remain pretty constant with time. The coordinates of the epoch J2000.0 correspond to the coordinates of this fundamental system quite closely.

With the new media, the size of the star catalogues exploded. The first Hubble Guide Star Catalog from the early 1990's contained 18 million stars and the second Guide Star Catalog from the year 2001, nearly 500 million stars. It was surpassed by the U.S. Naval Observatory USNO-B1.0 Catalog, which contains entries for 1,024,618,261 stars and galaxies from digitised images of several photographic sky surveys. The catalogue presents right ascension and declination, proper motion and magnitude estimates.

The next step in the accuracy of astrometry will be achieved in the 2010's with a new European astrometric satellite. The Gaia satellite launched in 2013 has improved the accuracy to about 10^{-5} seconds of arc.

The importance of astrometry is not limited to star maps. Astrometric observations give basic information with deeper physical content also. When the distance of a star is known, its true radiation power can be calculated from the observed brightness (Chap. 4), giving information about the structure and evolution of the star (Chaps. 11 and 12). Motions of stars are related e.g. to the mass distribution of the Milky Way (Chap. 18).

Star maps have been published since ancient times, but the earliest maps were globes showing the celestial sphere as seen from the outside. At the beginning of the 17th century, a German, Johannes Bayer, published the first map showing the stars as seen from inside the celestial sphere, as we see them in the sky. Constellations were usually decorated with drawings of mythological figures. The *Uranometria Nova* (1843) by Argelander represents a transition towards modern maps: mythological figures are beginning to fade away. The map accompanying the *Bonner Durchmusterung* carried this evolution to its extreme. The sheets contain nothing but stars and coordinate lines.

Most maps are based on star catalogues. Photography made it possible to produce star maps without the cataloguing stage. The most important of such maps is a photographic atlas the full name of which is *The National Geographic Society—Palomar Observatory Sky Atlas*. The plates for this atlas were taken with the 1.2 m Schmidt camera on Mount Palomar. The Palomar Sky Atlas was completed in the 1950's. It consists of 935 pairs of photographs: each region has been photographed in red and blue light. The size of each plate is about 35 cm $\times$ 35 cm, covering an area of $6.6^\circ \times 6.6^\circ$. The prints are negatives (black stars on a light background), because in this way, fainter objects are visible. The limiting magnitude is about 19 in blue and 20 in red.

The Palomar atlas covers the sky down to -30°. Work to map the rest of the sky was carried out later at two observatories in the southern hemisphere, at Siding Spring Observatory in Australia, and at the European Southern Observatory (ESO) in Chile. The instruments and the scale on the plates are similar to those used earlier for the Palomar plates, but the atlas is distributed on film transparencies instead of paper prints.

For amateurs there are several star maps of various kinds. Some of them are mentioned in the references.

Several star catalogues can nowadays copied freely from the Internet. Thus anyone with suitable programming skills can make star maps appropriate for her own purposes.

2.13 Sidereal and Solar Time

Time measurements can be based on the rotation of the Earth, orbital motion around the Sun, or on atomic clocks. The last-mentioned will be discussed in the next section. Here we consider the sidereal and solar times related to the rotation of the Earth.

We defined the sidereal time as the hour angle of the vernal equinox. A good basic unit is a *sidereal day*, which is the time between two successive upper culminations of the vernal equinox. After one sidereal day the celestial sphere with all its stars has returned to its original position with respect to the observer. The flow of sidereal time is as constant as the rotation of the Earth. The rotation rate is slowly decreasing, and thus the length of the sidereal day is increasing. In addition to the smooth slowing down irregular vari-

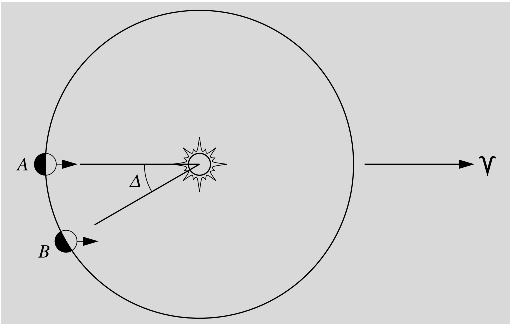


Fig. 2.31 One sidereal day is the time between two successive transits or upper culminations of the vernal equinox. By the time the Earth has moved from A to B, one sidereal day has elapsed. The angle Δ is greatly exaggerated; in reality, it is slightly less than one degree

ations of the order of one millisecond have been observed.

Unfortunately, also the sidereal time comes in two varieties, apparent and mean. The *apparent sidereal time* is determined by the true vernal equinox, and so it is obtained directly from observations.

Because of the precession the ecliptic longitude of the vernal equinox increases by about 50″ a year. This motion is very smooth. Nutation causes more complicated wobbling. The *mean equinox* is the point where the vernal equinox would be if there were no nutation. The *mean sidereal time* is the hour angle of this mean equinox.

The difference of the apparent and mean sidereal time is called the *equation of equinoxes*:

$$\Theta_a - \Theta_M = \Delta\psi \cos\varepsilon, \tag{2.42}$$

where ε is the obliquity of the ecliptic at the instant of the observation, and $\Delta\psi$, the nutation in longitude. This value is tabulated for each day e.g. in the *Astronomical Almanac*. It can also be computed from the formulae given in Box 2.1. It is at most about one second, so it has to be taken into account only in the most precise calculations.

Figure 2.31 shows the Sun and the Earth at vernal equinox. When the Earth is at the point A, the Sun culminates and, at the same time, a new sidereal day begins in the city with the huge black arrow standing in its central square. After one sidereal day, the Earth has moved along its orbit almost one degree of arc to the point B. Therefore the Earth has to turn almost a degree further before the Sun will culminate. The *solar* or *synodic day* is therefore 3 min 56.56 s (sidereal time) longer than the sidereal day. This means that the beginning of the sidereal day will move around the clock during the course of one year. After one year, sidereal and solar time will again be in phase. The number of sidereal days in one year is one higher than the number of solar days.

When we talk about rotation periods of planets, we usually mean sidereal periods. The length of day, on the other hand, means the rotation period with respect to the Sun. If the orbital period around the Sun is P, sidereal rotation period τ_* and synodic day τ, we now know that the number of sidereal days in time P, P/τ_*, is one higher than the number of synodic days, P/τ:

$$\frac{P}{\tau_*} - \frac{P}{\tau} = 1,$$

or

$$\frac{1}{\tau} = \frac{1}{\tau_*} - \frac{1}{P}. \tag{2.43}$$

This holds for a planet rotating in the direction of its orbital motion (counterclockwise). If the sense of rotation is opposite, or *retrograde*, the number of sidereal days in one orbital period is one less than the number of synodic days, and the equation becomes

$$\frac{1}{\tau} = \frac{1}{\tau_*} + \frac{1}{P}. \tag{2.44}$$

For the Earth, we have $P = 365.2564$ d, and $\tau = 1$ d, whence (2.43) gives $\tau_* = 0.99727$ d $=$ 23 h 56 min 4 s, solar time.

Since our everyday life follows the alternation of day and night, it is more convenient to base our timekeeping on the apparent motion of the Sun rather than that of the stars. Unfortunately, the solar time does not flow at a constant rate. There are two reasons for this. First, the orbit of the Earth is not exactly circular, but an ellipse, which means that the velocity of the Earth along its orbit is not constant. Second, the Sun moves along the ecliptic, not the equator. Thus its right ascension does not increase at a constant rate. The change is fastest at the end of December (4 min 27 s per day) and slowest in mid-September (3 min 35 s per day). As a consequence, the hour angle of the Sun (which determines the solar time) also grows at an uneven rate.

To find a solar time flowing at a constant rate, we define a fictitious *mean sun*, which moves along the celestial equator with constant angular velocity, making a complete revolution in one year. By year we mean here the *tropical year*, which is the time it takes for the Sun to move from one vernal equinox to the next. In one tropical year, the right ascension of the Sun increases exactly 24 hours. The length of the tropical year is 365 d 5 h 48 min 46 s = 365.2422 d. Since the direction of the vernal equinox moves due to precession, the tropical year differs from the sidereal year, during which the Sun makes one revolution with respect to the background stars. One sidereal year is 365.2564 d.

Using our artificial mean sun, we now define an evenly flowing solar time, the *mean solar time* (or simply *mean time*) T_M, which is equal to the hour angle h_M of the centre of the mean sun plus 12 hours (so that the date will change at midnight, to annoy astronomers):

$$T_M = h_M + 12\text{ h}. \tag{2.45}$$

The difference between the true solar time T and the mean time T_M is called the *equation of time*:

$$\text{E.T.} = T - T_M. \tag{2.46}$$

(In spite of the identical abbreviation, this has nothing to do with a certain species of little green men.) The greatest positive value of E.T. is about 16 minutes and the greatest negative value about −14 minutes (see Fig. 2.32). This is also the difference between the true noon (the meridian transit of the Sun) and the mean noon.

Both the true solar time and mean time are *local times*, depending on the hour angle of the Sun, real or artificial. If one observes the true solar time by direct measurement and computes the mean time from (2.46), a digital watch will probably be found to disagree with both of them. The reason for this is that we do not use local time in our everyday life; instead we use the *zonal time* of the nearest *time zone*.

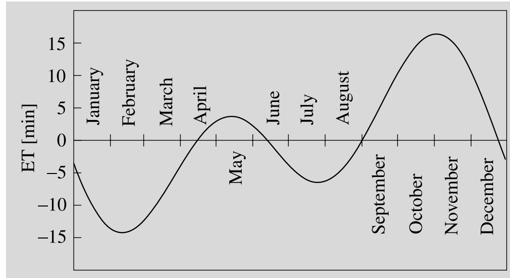


Fig. 2.32 Equation of time. A sundial always shows (if correctly installed) true local solar time. To find the local mean time the equation of time must be subtracted from the local solar time

In the past, each city had its own local time. When travelling became faster and more popular, the great variety of local times became an inconvenience. At the end of the 19th century, the Earth was divided into 24 zones, the time of each zone differing from the neighbouring ones by one hour. On the surface of the Earth, one hour in time corresponds to 15° in longitude; the time of each zone is determined by the local mean time at one of the longitudes 0°, 15°, ..., 345°.

The time of the zero meridian going through Greenwich is used as an international reference, Universal Time. In most European countries, time is one hour ahead of this (Fig. 2.33).

In summer, many countries switch to *daylight saving time*, during which time is one hour ahead of the ordinary time. The purpose of this is to make the time when people are awake coincide with daytime in order to save electricity, particularly in the evening, when people go to bed one hour earlier. During daylight saving time, the difference between the true solar time and the official time can grow even larger. It has also been criticised that the disadvantages of the daylight saving time are much more severe than its profits in energy saving. Many people have serious difficulties to adapt to a different daily rhythm twice every year. Also, confusingly, different countries use different rules for the daylight saving time.

In the EU countries the daylight saving time begins on the last Sunday of March, at 1 o'clock UTC in the morning, when the clocks are moved

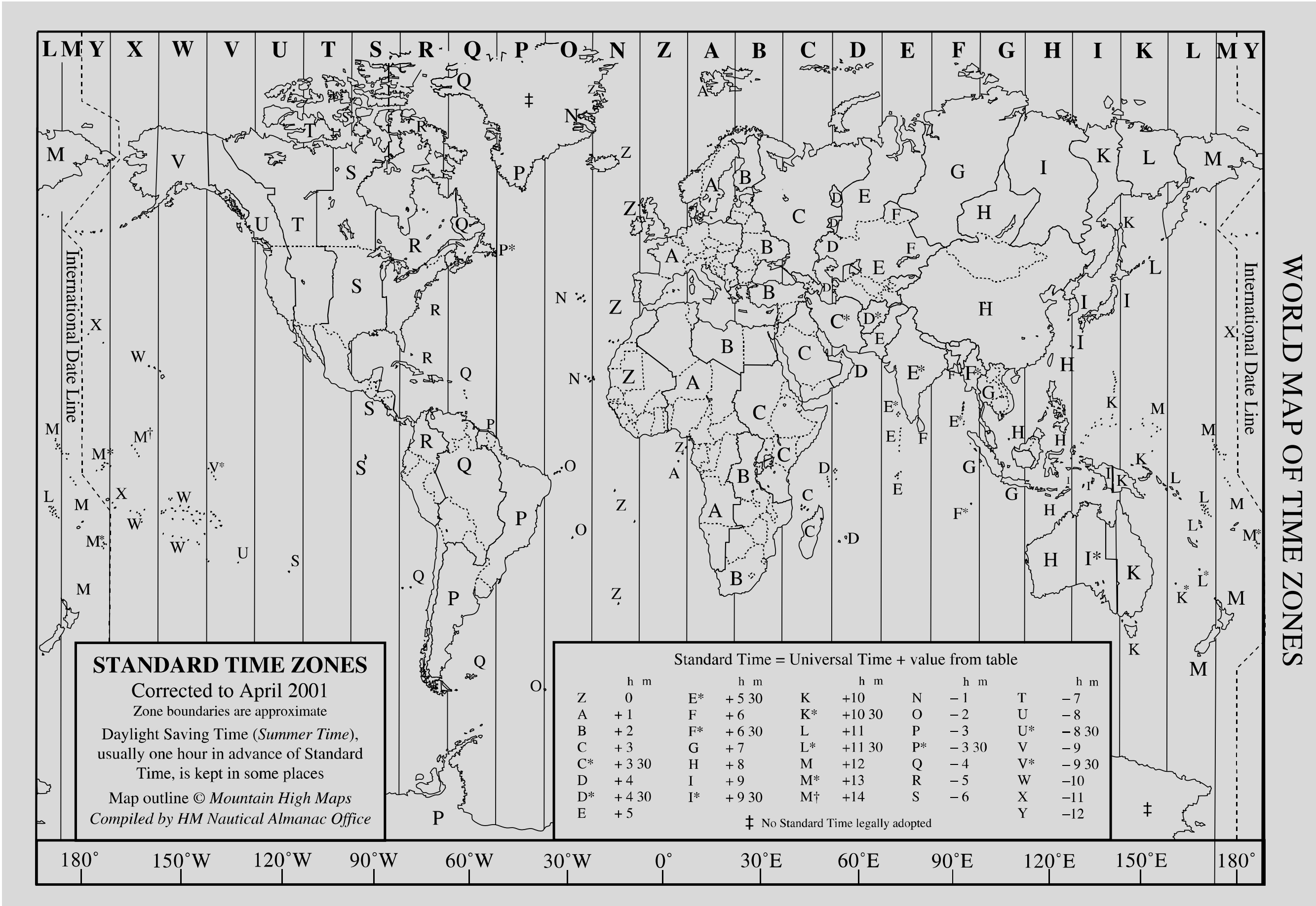


Fig. 2.33 The time zones. The map gives the difference of the local zonal time from the Greenwich mean time (UT). During daylight saving time, one hour must be added to the given figures. When travelling across the date line westward, the date must be incremented by one day, and decremented if going eastward. For example, a traveller taking a flight from Honolulu to Tokyo on Monday morning will arrive on Tuesday, even though (s)he does not see a single night en route. In 2015 North Korea adopted time zone UTC+8.5 h, which is half an hour behind the South Korean time. (Drawing U.S. Naval Observatory)

forward by one hour, and ends on the last Sunday of October at 1 o'clock UTC.

2.14 Astronomical Time Systems

Time can be defined using several different phenomena:

1. The solar and sidereal times are based on the rotation of the Earth.
2. The standard unit of time in the current SI system, the second, is based on quantum mechanical atomary phenomena.
3. Equations of physics like the ones describing the motions of celestial bodies involve a time variable corresponding to an ideal time running at a constant pace. The ephemeris time and dynamical time discussed a little later are such times.

Observations give directly the apparent sidereal time as the hour angle of the true vernal equinox. From the apparent sidereal time the mean sidereal time can be calculated.

The *universal time* UT is defined by the equation

$$\begin{aligned} \mathrm{GMST}(0\ \mathrm{UT}) = 24{,}110.54841\ \mathrm{s} \\ + T \times 8{,}640{,}184.812866\ \mathrm{s} \\ + T^2 \times 0.093104\ \mathrm{s} \\ - T^3 \times 0.0000062\ \mathrm{s}, \end{aligned} \tag{2.47}$$

where GMST is the Greenwich mean sidereal time and T the Julian century. The latter is obtained from the Julian date J, which is a running number of the day (Sects. 2.15 and Box 2.2):

$$T = \frac{J - 2{,}451{,}545.0}{36{,}525}. \tag{2.48}$$

This gives the time elapsed since January 1, 2000, in Julian centuries.

Sidereal time and hence also UT are related to the rotation of the Earth, and thus contain perturbations due to the irregular variations, mainly slowing down, of the rotation.

In (2.47) the constant 8,640,184.812866 s tells how much sidereal time runs fast compared to the UT in a Julian century. As the rotation of the Earth is slowing down the solar day becomes longer. Since the Julian century T contains a fixed number of days, it will also become longer. This gives rise to the small correction terms in (2.47).

Strictly speaking this universal time is the time denoted by UT1. Observations give UT0, which contains a small perturbation due to the wandering of the geographical pole, or *polar variation*. The direction of the axis with respect to the solid surface varies by about $0.1''$ (a few metres on the surface) with a period of about 430 days (*Chandler period*). In addition to this, the polar motion contains a slow nonperiodic part.

The z axis of the astronomical coordinates is aligned with the angular momentum vector of the Earth, but the terrestrial coordinates refer to the axis at the epoch 1903.5. In the most accurate calculations this has to be taken into account.

Due to the polar variation the UT0 does not grow at a constant rate, since also the direction of the origin, the Greenwich meridian, is varying. There are also additional irregular variations of the order of a millisecond, and the secular slowing-down caused by tidal forces.

Nowadays the SI unit of time, the second, is defined in a way that has nothing to do with celestial phenomena. Periods of quantum mechanical phenomena remain more stable than the motions of celestial bodies involving complicated perturbations.

In 1967, one second was defined as 9,192,631,770 times the period of the light emitted by cesium 133 isotope in its ground state, transiting from hyperfine level $F = 4$ to $F = 3$. Later, this definition was revised to include small relativistic effects produced by gravitational fields. The relative accuracy of this atomic time is about 10^{-12}. The *international atomic time*, TAI, was adopted as the basis of time signals in 1972. The time is maintained by the *Bureau International des Poids et Mesures* in Paris, and it is the average of several accurate atomic clocks.

Even before atomic clocks there was a need for an ideal time proceeding at a perfectly constant rate, corresponding to the time variable in the equations of Newtonian mechanics. The *ephemeris time* was such a time. It was used e.g. for tabulating ephemerides. The unit of ephemeris time was the *ephemeris second*, which is the length of the tropical year 1900 divided by 31,556,925.9747. Ephemeris time was not known in advance. Only afterwards was it possible to determine the difference of ET and UT from observational data.

In 1984 ephemeris time was replaced by *dynamical time*. It comes in two varieties.

The *terrestrial dynamical time* (TDT) corresponds to the proper time of an observer moving with the Earth. The time scale is affected by the relativistic time dilation due to the orbital speed of the Earth. The rotation velocity depends on the latitude, and thus in TDT it is assumed that the observer is not rotating with the Earth. The zero point of TDT was chosen so that the old ET changed without a jump to TDT.

In 1991 a new standard time, the *terrestrial time* (TT), was adopted. Practically it is equivalent to TDT.

TT (or TDT) is the time currently used for tabulating ephemerides of planets and other celestial bodies. For example, the *Astronomical Almanac* gives the coordinates of the planets for each day at 0 TT.

The *Astronomical Almanac* also gives the difference

$$\Delta T = \mathrm{TDT} - \mathrm{UT} \qquad (2.49)$$

for earlier years. For the present year and some future years a prediction extrapolated from the earlier years is given. Its accuracy is about 0.1 s. At the beginning of 1990 the difference was 56.7 s; it increases every year by an amount that is usually a little less than one second.

The terrestrial time differs from the atomic time by a constant offset

$$\mathrm{TT} = \mathrm{TAI} + 32.184\ \mathrm{s}. \qquad (2.50)$$

TT is well suited for ephemerides of phenomena as seen from the Earth. The equations of motion of the solar system, however, are solved in a frame the origin of which is the centre of mass or *barycentre* of the solar system. The coordinate time of this frame is called the *barycentric dynamical time*, TDB. The unit of TDB is defined so that, on the average, it runs at the same rate as TT, the difference containing only periodic terms depending on the orbital motion of the Earth. The difference can usually be neglected, since it is at most about 0.002 seconds.

Which of these many times should we use in our alarm-clocks? None of them. Yet another time is needed for that purpose. This official wall-clock time is called the *coordinated universal time*, UTC. The zonal time follows UTC but differs from it usually by an integral number of hours.

UTC is defined so that it proceeds at the same rate as TAI, but differs from it by an integral number of seconds. These *leap seconds* are used to adjust UTC so that the difference from UT1 never exceeds 0.9 seconds (Fig. 2.34). A leap second is added either at the beginning of a year or the night between June and July.

The difference

$$\Delta \mathrm{AT} = \mathrm{TAI} - \mathrm{UTC} \qquad (2.51)$$

is also tabulated in the *Astronomical Almanac*. According to the definition of UTC the difference in seconds is always an integer. The difference cannot be predicted very far to the future.

From (2.50) and (2.51) we get

$$\mathrm{TT} = \mathrm{UTC} + 32.184\ \mathrm{s} + \Delta \mathrm{AT}, \qquad (2.52)$$

which gives the terrestrial time TT corresponding to a given UTC. Table 2.2 gives this correction. The table is easy to extend to the future. When it is told in the news that a leap second will be added the difference will increase by one second. In case the number of leap seconds is not known, it can be approximated that a leap second will be added every 1.25 years.

The units of the coordinated universal time UTC, atomic time TAI and terrestrial time TT are the same second of the SI system. Hence all these times proceed at the same rate, the only difference being in their zero points. The difference of the TAI and TT is always the same, but due to the leap seconds the UTC will fall behind in a slightly irregular way.

Culminations and rising and setting times of celestial bodies are related to the rotation of the

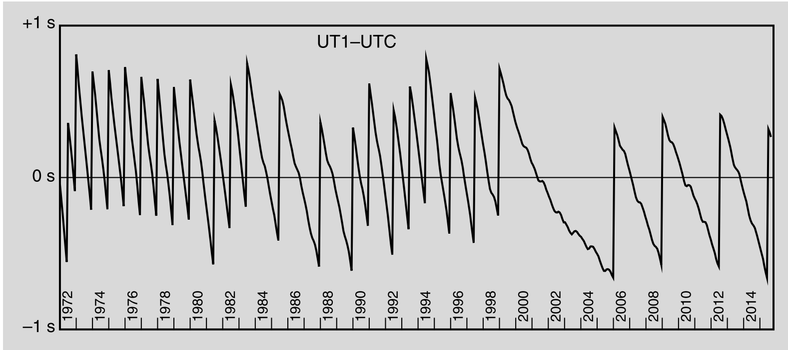


Fig. 2.34 The difference between the universal time UT1, based on the rotation of the Earth, and the coordinated universal time UTC during 1972–2002. Because the rotation of the Earth is slowing down, the UT1 will run slow of the UTC by about 0.8 seconds a year. Leap seconds are added to the UTC when necessary to keep the times approximately equal. In the graph these leap seconds are seen as one second jumps upward

Table 2.2 Differences of the atomic time and UTC (ΔAT) and the terrestrial time TT and UTC. The terrestrial time TT used in ephemerides is obtained by adding $\Delta \mathrm{AT} + 32.184$ s to the ordinary time UTC

	ΔAT	TT − UTC
1.1.1972– 30.6.1972	10 s	42.184 s
1.7.1972–31.12.1972	11 s	43.184 s
1.1.1973–31.12.1973	12 s	44.184 s
1.1.1974–31.12.1974	13 s	45.184 s
1.1.1975–31.12.1975	14 s	46.184 s
1.1.1976–31.12.1976	15 s	47.184 s
1.1.1977–31.12.1977	16 s	48.184 s
1.1.1978–31.12.1978	17 s	49.184 s
1.1.1979–31.12.1979	18 s	50.184 s
1.1.1980– 30.6.1981	19 s	51.184 s
1.7.1981– 30.6.1982	20 s	52.184 s
1.7.1982– 30.6.1983	21 s	53.184 s
1.7.1983– 30.6.1985	22 s	54.184 s
1.7.1985–31.12.1987	23 s	55.184 s
1.1.1988–31.12.1989	24 s	56.184 s
1.1.1990–31.12.1990	25 s	57.184 s
1.1.1991– 30.6.1992	26 s	58.184 s
1.7.1992– 30.6.1993	27 s	59.184 s
1.7.1993– 30.6.1994	28 s	60.184 s
1.7.1994–31.12.1995	29 s	61.184 s
1.1.1996– 31.6.1997	30 s	62.184 s
1.7.1997–31.12.1998	31 s	63.184 s
1.1.1999–31.12.2005	32 s	64.184 s
1.1.2006–	33 s	65.184 s

Earth. Thus the sidereal time and hence the UT of such an event can be calculated precisely. The corresponding UTC cannot differ from the UT by more than 0.9 seconds, but the exact value is not known in advance. The future coordinates of the Sun, Moon and planets can be calculated as functions of the TT, but the corresponding UTC can only be estimated.

2.15 Calendars

Our calendar is a result of long evolution. The main problem it must contend with is the incommensurability of the basic units, day, month and year: the numbers of days and months in a year are not integers. This makes it rather complicated to develop a calendar that takes correctly into account the alternation of seasons, day and night, and perhaps also the lunar phases.

Our calendar has its origin in the Roman calendar, which, in its earliest form, was based on the phases of the Moon. From around 700 B.C. on, the length of the year has followed the apparent motion of the Sun; thus originated the division of the year into twelve months. One month, however, still had a length roughly equal to the lunar cycle. Hence one year was only 354 days long. To keep the year synchronised with the seasons, a leap month had to be added to every other year.

Eventually the Roman calendar got mixed up. The mess was cleared by Julius Caesar in about 46 B.C., when the *Julian calendar* was developed upon his orders by the Alexandrian astronomer *Sosigenes*. The year had 365 days and a leap day was added to every fourth year.

In the Julian calendar, the average length of one year is 365 d 6 h, but the tropical year is 11 min 14 s shorter. After 128 years, the Julian year begins almost one day too late. The difference was already 10 days in 1582, when a calendar reform was carried out by Pope Gregory XIII. In the *Gregorian calendar*, every fourth year is a leap year, the years divisible by 100 being exceptions. Of these, only the years divisible by 400 are leap years. Thus 1900 was not a leap year, but 2000 was. The Gregorian calendar was adopted slowly, at different times in different countries. Although the calendar reform was seen necessary, orders given by the catholic pope were not easy to accept in e.g. protestant countries. The transition period did not end before the 20th century.

Even the Gregorian calendar is not perfect. The differences from the tropical year will accumulate to one day in about 3300 years.

Since years and months of variable length make it difficult to compute time differences, especially astronomers have employed various methods to give each day a running number. The most widely used numbers are the *Julian dates*. In spite of their name, they are not related to the Julian calendar. The only connection is the length of a *Julian century* of 36,525 days, a quantity appearing in many formulas involving Julian dates. The Julian day number 0 dawned in 4713 B.C. The day number changes always at 12 : 00 UT. For example, the Julian day 2,451,545 began at noon in January 1, 2000. The Julian date can be computed using the formulas given in Box 2.2.

Julian dates are uncomfortably big numbers, and therefore *modified Julian dates* are often used. The zero point can be e.g. January 1, 2000. Sometimes 0.5 is subtracted from the date to make it to coincide with the date corresponding to the UTC. When using such dates, the zero point should always be mentioned.

Box 2.1 (Reduction of Coordinates) Star catalogues give coordinates for some standard epoch. In the following we give the formulas needed to reduce the coordinates to a given date and time. The full reduction is rather laborious, but the following simplified version is sufficient for most practical purposes.

We assume that the coordinates are given for the epoch J2000.0.

1. First correct the place for proper motion unless it is negligible.
2. Precess the coordinates to the time of the observation. First we use the coordinates of the standard epoch (α_0, δ_0) to find a unit vector pointing in the direction of the star:

$$\boldsymbol{p}_0 = \begin{pmatrix} \cos\delta_0 \cos\alpha_0 \\ \cos\delta_0 \sin\alpha_0 \\ \sin\delta_0 \end{pmatrix}.$$

Precession changes the ecliptic longitude of the object. The effect on right ascension and declination can be calculated as three rotations, given by three rotation matrices. By multiplying these matrices we get the combined precession matrix that maps the previous unit vector to its precessed equivalent. A similar matrix can be derived for the nutation. The transformations and constants given here are based on the system standardised by the IAU in 1976.

The precession and nutation matrices contain several quantities depending on time. The time variables appearing in their expressions are

$$t = J - 2{,}451{,}545.0,$$

$$T = \frac{J - 2{,}451{,}545.0}{36{,}525}.$$

Here J is the Julian date of the observation, t the number of days since the epoch J2000.0 (i.e. noon of January 1, 2000), and T the same interval of time in Julian centuries.

The following three angles are needed for the precession matrix

$$\zeta = 2306.2181''T + 0.30188''T^2 + 0.017998''T^3,$$

$$z = 2306.2181''T + 1.09468''T^2 + 0.018203''T^3,$$
$$\theta = 2004.3109''T - 0.42665''T^2 - 0.041833''T^3.$$

The precession matrix is now

$$\boldsymbol{P} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix}.$$

The elements of this matrix in terms of the abovementioned angles are

$$\begin{aligned}
P_{11} &= \cos z \cos\theta \cos\zeta - \sin z \sin\zeta, \\
P_{12} &= -\cos z \cos\theta \sin\zeta - \sin z \cos\zeta, \\
P_{13} &= -\cos z \sin\theta, \\
P_{21} &= \sin z \cos\theta \cos\zeta + \cos z \sin\zeta, \\
P_{22} &= -\sin z \cos\theta \sin\zeta + \cos z \cos\zeta, \\
P_{23} &= -\sin z \sin\theta, \\
P_{31} &= \sin\theta \cos\zeta, \\
P_{32} &= -\sin\theta \sin\zeta, \\
P_{33} &= \cos\theta.
\end{aligned}$$

The new coordinates are now obtained by multiplying the coordinates of the standard epoch by the precession matrix:

$$\boldsymbol{p}_1 = \boldsymbol{P}\boldsymbol{p}_0.$$

This is the mean place at the given time and date.

If the standard epoch is not J2000.0, it is probably easiest to first transform the given coordinates to the epoch J2000.0. This can be done by computing the precession matrix for the given epoch and multiplying the coordinates by the inverse of this matrix. Inverting the precession matrix is easy: we just transpose it, i.e. interchange its rows and columns. Thus coordinates given for some epoch can be precessed to J2000.0 by multiplying them by

$$\boldsymbol{P}^{-1} = \begin{pmatrix} P_{11} & P_{21} & P_{31} \\ P_{12} & P_{22} & P_{32} \\ P_{13} & P_{23} & P_{33} \end{pmatrix}.$$

In case the required accuracy is higher than about one minute of arc, we have to do the following further corrections.

3. The full nutation correction is rather complicated. The nutation used in astronomical almanacs involves series expansions containing over a hundred terms. Very often, though, the following simple form is sufficient. We begin by finding the mean obliquity of the ecliptic at the observation time:

$$\varepsilon_0 = 23^\circ\,26'\,21.448'' - 46.8150''T - 0.00059''T^2 + 0.001813''T^3.$$

The mean obliquity means that periodic perturbations have been omitted. The formula is valid a few centuries before and after the year 2000.

The true obliquity of the ecliptic, ε, is obtained by adding the nutation correction to the mean obliquity:

$$\varepsilon = \varepsilon_0 + \Delta\varepsilon.$$

The effect of the nutation on the ecliptic longitude (denoted usually by $\Delta\psi$) and the obliquity of the ecliptic can be found from

$$\begin{aligned}
C_1 &= 125^\circ - 0.05295^\circ t, \\
C_2 &= 200.9^\circ + 1.97129^\circ t, \\
\Delta\psi &= -0.0048^\circ \sin C_1 - 0.0004^\circ \sin C_2, \\
\Delta\varepsilon &= 0.0026^\circ \cos C_1 + 0.0002^\circ \cos C_2.
\end{aligned}$$

Since $\Delta\psi$ and $\Delta\varepsilon$ are very small angles, we have, for example, $\sin\Delta\psi \approx \Delta\psi$ and $\cos\Delta\psi \approx 1$, when the angles are expressed in radians. Thus we get the nutation matrix

$$\boldsymbol{N} = \begin{pmatrix} 1 & -\Delta\psi\cos\varepsilon & -\Delta\psi\sin\varepsilon \\ \Delta\psi\cos\varepsilon & 1 & -\Delta\varepsilon \\ \Delta\psi\sin\varepsilon & \Delta\varepsilon & 1 \end{pmatrix}.$$

This is a linearised version of the full transformation. The angles here must be in radians. The place in the coordinate frame of the observing time is now

$$\boldsymbol{p}_2 = \boldsymbol{N}\boldsymbol{p}_1.$$

4. The annual aberration can affect the place about as much as the nutation. Approximate corrections are obtained from

$$\begin{aligned}\Delta\alpha\cos\delta &= -20.5''\sin\alpha\sin\lambda \\ &\quad - 18.8''\cos\alpha\cos\lambda, \\ \Delta\delta &= 20.5''\cos\alpha\sin\delta\sin\lambda \\ &\quad + 18.8''\sin\alpha\sin\delta\cos\lambda \\ &\quad - 8.1''\cos\delta\cos\lambda,\end{aligned}$$

where λ is the ecliptic longitude of the Sun. Sufficiently accurate value for this purpose is given by

$$\begin{aligned}G &= 357.528^\circ + 0.985600^\circ t, \\ \lambda &= 280.460^\circ + 0.985647^\circ t \\ &\quad + 1.915^\circ\sin G + 0.020^\circ\sin 2G.\end{aligned}$$

These reductions give the apparent place of the date with an accuracy of a few seconds of arc. The effects of parallax and diurnal aberration are even smaller.

Example The coordinates of Regulus (α Leo) for the epoch J2000.0 are

$$\begin{aligned}\alpha &= 10\text{ h }8\text{ min }22.2\text{ s} = 10.139500\text{ h}, \\ \delta &= 11^\circ\,58'\,02'' = 11.967222^\circ.\end{aligned}$$

Find the apparent place of Regulus on March 12, 1995.

We start by finding the unit vector corresponding to the catalogued place:

$$\boldsymbol{p}_0 = \begin{pmatrix}-0.86449829 \\ 0.45787318 \\ 0.20735204\end{pmatrix}.$$

The Julian date is $J = 2{,}449{,}789.0$, and thus $t = -1756$ and $T = -0.04807666$. The angles of the precession matrix are $\zeta = -0.03079849^\circ$, $z = -0.03079798^\circ$ and $\theta = -0.02676709^\circ$. The precession matrix is then

$$\boldsymbol{P} = \begin{pmatrix}0.99999931 & 0.00107506 & 0.00046717 \\ -0.00107506 & 0.99999942 & -0.00000025 \\ -0.00046717 & -0.00000025 & 0.99999989\end{pmatrix}.$$

The precessed unit vector is

$$\boldsymbol{p}_1 = \begin{pmatrix}-0.86390858 \\ 0.45880225 \\ 0.20775577\end{pmatrix}.$$

The angles needed for the nutation are $\Delta\psi = 0.00309516^\circ$, $\Delta\varepsilon = -0.00186227^\circ$, $\varepsilon = 23.43805403^\circ$, which give the nutation matrix

$$\boldsymbol{N} = \begin{pmatrix}1 & -0.00004956 & -0.00002149 \\ 0.00004956 & 1 & 0.00003250 \\ 0.00002149 & -0.00003250 & 1\end{pmatrix}.$$

The place in the frame of the date is

$$\boldsymbol{p}_2 = \begin{pmatrix}-0.86393578 \\ 0.45876618 \\ 0.20772230\end{pmatrix},$$

whence

$$\begin{aligned}\alpha &= 10.135390\text{ h}, \\ \delta &= 11.988906^\circ.\end{aligned}$$

To correct for the aberration we first find the longitude of the Sun: $G = -1373.2^\circ = 66.8^\circ$, $\lambda = -8.6^\circ$. The correction terms are then

$$\begin{aligned}\Delta\alpha &= 18.25'' = 0.0050^\circ, \\ \Delta\delta &= -5.46'' = -0.0015^\circ.\end{aligned}$$

Adding these to the previously obtained coordinates we get the apparent place of Regulus on March 12, 1995:

$$\begin{aligned}\alpha &= 10.1357\text{ h} = 10\text{ h }8\text{ min }8.5\text{ s}, \\ \delta &= 11.9874^\circ = 11^\circ\,59'\,15''.\end{aligned}$$

Comparison with the places given in the catalogue *Apparent Places of Fundamental Stars* shows that we are within about 3″ of the correct place, which is a satisfactory result.

Box 2.2 (Julian Date) There are several methods for finding the Julian date. The following one, developed by Fliegel and Van Flandern in 1968, is well adapted for computer programs. Let y be the year (with all four digits), m the month and d the day. The Julian date J at noon is then

$$\begin{aligned} J = 367y &- \{7[y + (m+9)/12]\}/4 \\ &- (3\{[y + (m-9)/7]/100 + 1\})/4 \\ &+ 275m/9 + d + 1{,}721{,}029. \end{aligned}$$

The division here means an integer division, the decimal part being truncated: e.g. $7/3 = 2$ and $-7/3 = -2$.

Example Find the Julian date on January 1, 1990.

Now $y = 1990$, $m = 1$ and $d = 1$.

$$\begin{aligned} J = 367 \times 1990 &- 7 \times [1990 + (1+9)/12]/4 \\ &- 3 \times \{[1990 + (1-9)/7]/100 + 1\}/4 \\ &+ 275 \times 1/9 + 1 + 1{,}721{,}029 \\ &= 730{,}330 - 3482 - 15 + 30 + 1 \\ &+ 1{,}721{,}029 \\ &= 2{,}447{,}893. \end{aligned}$$

Astronomical tables usually give the Julian date at 0 UT. In this case that would be 2,447,892.5.

The inverse procedure is a little more complicated. In the following J is the Julian date at noon (so that it will be an integer):

$$\begin{aligned} a &= J + 68{,}569, \\ b &= (4a)/146{,}097, \\ c &= a - (146{,}097b + 3)/4, \\ d &= [4000(c+1)]/1{,}461{,}001, \\ e &= c - (1461d)/4 + 31, \\ f &= (80e)/2447, \\ \text{day} &= e - (2447f)/80, \\ g &= f/11, \\ \text{month} &= f + 2 - 12g, \\ \text{year} &= 100(b-49) + d + g. \end{aligned}$$

Example In the previous example we got $J = 2{,}447{,}893$. Let's check this by calculating the corresponding calendar date:

$$\begin{aligned} a &= 2{,}447{,}893 + 68{,}569 = 2{,}516{,}462, \\ b &= (4 \times 2{,}516{,}462)/146{,}097 = 68, \\ c &= 2{,}516{,}462 - (146{,}097 \times 68 + 3)/4 \\ &= 32{,}813, \\ d &= [4000(32{,}813 + 1)]/1{,}461{,}001 = 89, \\ e &= 32{,}813 - (1461 \times 89)/4 + 31 = 337, \\ f &= (80 \times 337)/2447 = 11, \\ \text{day} &= 337 - (2447 \times 11)/80 = 1, \\ g &= 11/11 = 1, \\ \text{month} &= 11 + 2 - 12 \times 1 = 1, \\ \text{year} &= 100(68 - 49) + 89 + 1 = 1990. \end{aligned}$$

Thus we arrived back to the original date.

Since the days of the week repeat in seven day cycles, the remainder of the division $J/7$ unambiguously determines the day of the week. If J is the Julian date at noon, the remainder of $J/7$ tells the day of the week in the following way:

$$\begin{aligned} 0 &= \text{Monday}, \\ &\vdots \\ 5 &= \text{Saturday}, \\ 6 &= \text{Sunday}. \end{aligned}$$

Example The Julian date corresponding to January 1, 1990 was 2,447,893. Since $2{,}447{,}893 = 7 \times 349{,}699$, the remainder is zero, and the day was Monday.

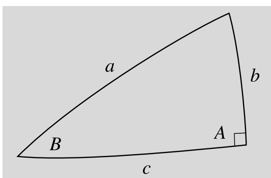
2.16 Examples

Example 2.1 (Trigonometric Functions in a Rectangular Spherical Triangle) Let the angle A be a right angle. When the figure is a plane triangle, the trigonometric functions of the angle B would be:

$$\sin B = b/a, \qquad \cos B = c/a, \qquad \tan B = b/c.$$

For the spherical triangle we have to use the equations in (2.7), which are now simply:

$$\sin B \sin a = \sin b,$$
$$\cos B \sin a = \cos b \sin c,$$
$$\cos a = \cos b \cos c.$$



The first equation gives the sine of B:

$$\sin B = \sin b / \sin a.$$

Dividing the second equation by the third one, we get the cosine of B:

$$\cos B = \tan c / \tan a.$$

And the tangent is obtained by dividing the first equation by the second one:

$$\tan B = \tan b / \sin c.$$

The third equation is the equivalent of the Pythagorean theorem for rectangular triangles.

Example 2.2 (Distance Between Two Locations) The latitude of Helsinki is about $\phi_1 = 60°$ and longitude $\lambda_1 = 25°$. The corresponding coordinates of La Palma are $\phi_2 = 28.7°$ and $\lambda_2 = -17.9°$. What is the distance between these locations measured along the surface of the Earth.

The distance can be found by applying the cosine formula to the spherical triangle *NHP*:

$$\begin{aligned}\cos a &= \cos(\lambda_2 - \lambda_1)\sin(90° - \phi_1) \\ &\quad \times \sin(90° - \phi_2) \\ &\quad + \cos(90° - \phi_1)\cos(90° - \phi_2) \\ &= \cos(\lambda_2 - \lambda_1)\cos\phi_1\cos\phi_2 \\ &\quad + \sin\phi_1 \sin\phi_2) \\ &= 0.7325 \times 0.5 \times 0.8771 \\ &\quad + 0.8660 \times 0.4802 \\ &= 0.7372,\end{aligned}$$

from which $a = 42.5°$.

Another method is to use vector calculus (Appendix A). The radius vector with respect to the centre of the Earth is

$$\boldsymbol{r} = R(\cos\phi\cos\lambda,\ \cos\phi\sin\lambda,\ \sin\phi),$$

where R is the radius of the Earth. In the following, only unit vectors are needed, and thus we can omit the constant R. In the example th unit vectors are

$$\boldsymbol{r}_1 = (0.4532,\ 0.2113,\ 0.8660),$$
$$\boldsymbol{r}_2 = (0.8347,\ -0.2696,\ 0.4802).$$

The scalar product of these vectors gives the cosine of the angle between the vectors:

$$\begin{aligned}\cos a &= \boldsymbol{r}_1 \cdot \boldsymbol{r}_2 \\ &= 0.4532 \times 0.8347 - 0.2113 \times 0.2696 \\ &\quad + 0.8660 \times 0.4802 \\ &= 0.7372,\end{aligned}$$

from which $a = 42.5° = 0.7419$ rad.

Both methods give the same angular separation between the two locations. The distance along the great circle on the surface of the Earth is then $Ra = 6400 \times 0.7419 = 4748$ km.

Example 2.3 (The Coordinates of New York City) The geographic coordinates are 41° north and 74° west of Greenwich, or $\phi = +41°$, $\lambda = -74°$. In time units, the longitude would be $74/15$ h $= 4$ h 56 min west of Greenwich. The geocentric latitude is obtained from

$$\tan\phi' = \frac{b^2}{a^2}\tan\phi = \left(\frac{6{,}356{,}752}{6{,}378{,}137}\right)^2 \tan 41^\circ$$
$$= 0.86347 \quad\Rightarrow\quad \phi' = 40^\circ\, 48'\, 34''.$$

The geocentric latitude is $11'\,26''$ less than the geographic latitude.

Example 2.4 The angular separation of two objects in the sky is quite different from their coordinate difference.

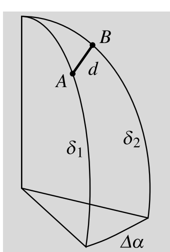
Suppose the coordinates of a star A are $\alpha_1 = 10$ h, $\delta_1 = 70^\circ$ and those of another star B, $\alpha_2 = 11$ h, $\delta_2 = 80^\circ$.

Using the Pythagorean theorem for plane triangles, we would get

$$d = \sqrt{(15^\circ)^2 + (10^\circ)^2} = 18^\circ.$$

But if we use the third equation in (2.7), we get

$$\begin{aligned}\cos d &= \cos(\alpha_1 - \alpha_2)\\ &\quad\times \sin(90^\circ - \delta_1)\sin(90^\circ - \delta_2)\\ &\quad+ \cos(90^\circ - \delta_1)\cos(90^\circ - \delta_2)\\ &= \cos(\alpha_1 - \alpha_2)\cos\delta_1\cos\delta_2\\ &\quad+ \sin\delta_1\sin\delta_2\\ &= \cos 15^\circ \cos 70^\circ \cos 80^\circ\\ &\quad+ \sin 70^\circ \sin 80^\circ\\ &= 0.983,\end{aligned}$$



which yields $d = 10.6^\circ$. The figure shows why the result obtained from the Pythagorean theorem is so far from being correct: hour circles (circles with $\alpha =$ constant) approach each other towards the poles and their angular separation becomes smaller, though the coordinate difference remains the same.

Example 2.5 Find the altitude and azimuth of the Moon in Helsinki at midnight at the beginning of 1996.

The right ascension is $\alpha = 2$ h 55 min 7 s $= 2.9186$ h and declination $\delta = 14^\circ\,42' = 14.70^\circ$, the sidereal time is $\Theta = 6$ h 19 min 26 s $= 6.3239$ h and latitude $\phi = 60.16^\circ$.

The hour angle is $h = \Theta - \alpha = 3.4053$ h $= 51.08^\circ$. Next we apply the equations in (2.16):

$$\begin{aligned}\sin A\cos a &= \sin 51.08^\circ \cos 14.70^\circ = 0.7526,\\ \cos A\cos a &= \cos 51.08^\circ \cos 14.70^\circ \sin 60.16^\circ\\ &\quad- \sin 14.70^\circ \cos 60.16^\circ\\ &= 0.4008,\\ \sin a &= \cos 51.08^\circ \cos 14.70^\circ \cos 60.16^\circ\\ &\quad+ \sin 14.70^\circ \sin 60.16^\circ\\ &= 0.5225.\end{aligned}$$

Thus the altitude is $a = 31.5^\circ$. To find the azimuth we have to compute its sine and cosine:

$$\sin A = 0.8827, \qquad \cos A = 0.4701.$$

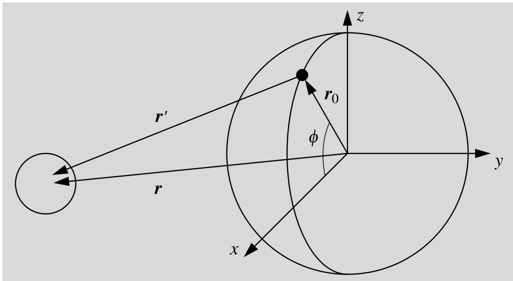
Hence the azimuth is $A = 62.0^\circ$. The Moon is in the southwest, 31.5 degrees above the horizon. Actually, this would be the direction if the Moon were infinitely distant.

Example 2.6 Find the topocentric place of the Moon in the case of the previous example.

The geocentric distance of the Moon at that time is $R = 62.58$ equatorial radii of the Earth. For simplicity, we can assume that the Earth is spherical.

We set up a rectangular coordinate frame in such a way that the z axis points towards the celestial pole and the observing site is in the xz plane. When the radius of the Earth is used as the unit of distance, the radius vector of the observing site is

$$\boldsymbol{r}_0 = \begin{pmatrix}\cos\phi\\ 0\\ \sin\phi\end{pmatrix} = \begin{pmatrix}0.4976\\ 0\\ 0.8674\end{pmatrix}.$$



The radius vector of the Moon is

$$\boldsymbol{r} = R\begin{pmatrix}\cos\delta\cos h\\ -\cos\delta\sin h\\ \sin\delta\end{pmatrix} = 62.58\begin{pmatrix}0.6077\\ -0.7526\\ 0.2538\end{pmatrix}.$$

The topocentric place of the Moon is

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{r}_0 = \begin{pmatrix}37.53\\ -47.10\\ 15.02\end{pmatrix}.$$

We divide this vector by its length 62.07 to get the unit vector $\boldsymbol{e}$ pointing to the direction of the Moon. This can be expressed in terms of the topocentric coordinates δ' and h':

$$\boldsymbol{e} = \begin{pmatrix}0.6047\\ -0.7588\\ 0.2420\end{pmatrix} = \begin{pmatrix}\cos\delta'\cos h'\\ -\cos\delta'\sin h'\\ \sin\delta'\end{pmatrix},$$

which gives $\delta' = 14.00°$ and $h' = 51.45°$. Next we can calculate the altitude and azimuth as in the previous example, and we get $a = 30.7°$, $A = 61.9°$.

Another way to find the altitude is to take the scalar product of the vectors $\boldsymbol{e}$ and $\boldsymbol{r}_0$, which gives the cosine of the zenith distance:

$$\begin{aligned}\cos z = \boldsymbol{e}\cdot\boldsymbol{r}_0 &= 0.6047 \times 0.4976\\ &\quad + 0.2420 \times 0.8674\\ &= 0.5108,\end{aligned}$$

whence $z = 59.3°$ and $a = 90° - z = 30.7°$. We see that this is 0.8° less than the geocentric altitude; i.e. the difference is more than the apparent diameter of the Moon.

Example 2.7 The coordinates of Arcturus are $\alpha = 14$ h 15.7 min, $\delta = 19°\,11'$. Find the sidereal time at the moment Arcturus rises or sets in Boston ($\phi = 42°\,19'$).

Neglecting refraction, we get

$$\begin{aligned}\cos h &= -\tan 19°\,11' \tan 42°\,19'\\ &= -0.348 \times 0.910 = -0.317.\end{aligned}$$

Hence, $h = \pm 108.47° = 7$ h 14 min. The more accurate result is

$$\begin{aligned}\cos h &= -\tan 19°\,11' \tan 42°\,19'\\ &\quad - \frac{\sin 35'}{\cos 19°\,11' \cos 42°19'}\\ &= -0.331,\end{aligned}$$

whence $h = \pm 109.35° = 7$ h 17 min. The plus and minus signs correspond to setting ant rising, respectively. When Arcturus rises, the sidereal time is

$$\begin{aligned}\Theta = \alpha + h &= 14 \text{ h } 16 \text{ min} - 7 \text{ h } 17 \text{ min}\\ &= 6 \text{ h } 59 \text{ min}\end{aligned}$$

and when it sets, the sidereal time is

$$\begin{aligned}\Theta &= 14 \text{ h } 16 \text{ min} + 7 \text{ h } 17 \text{ min}\\ &= 21 \text{ h } 33 \text{ min}.\end{aligned}$$

Note that the result is independent of the date: a star rises and sets at the same sidereal time every day.

Example 2.8 The proper motion of Aldebaran is $\mu = 0.20''/\text{a}$ and parallax $\pi = 0.048''$. The spectral line of iron at $\lambda = 440.5$ nm is displaced 0.079 nm towards the red. What are the radial and tangential velocities and the total velocity?

The radial velocity is found from

$$\begin{aligned}&\frac{\Delta\lambda}{\lambda} = \frac{v_\text{r}}{c}\\ \Rightarrow\quad &v_\text{r} = \frac{0.079}{440.5}\cdot 3\times 10^8 \text{ m/s}\\ &\quad = 5.4 \times 10^4 \text{ m/s} = 54 \text{ km/s}.\end{aligned}$$

The tangential velocity is now given by (2.40), since μ and π are in correct units:

$$v_t = 4.74\mu r = 4.74\mu/\pi = \frac{4.74 \times 0.20}{0.048}$$
$$= 20 \text{ km/s}.$$

The total velocity is

$$v = \sqrt{v_r^2 + v_t^2} = \sqrt{54^2 + 20^2}\,\text{km/s} = 58 \text{ km/s}.$$

Example 2.9 Find the local time in Paris (longitude $\lambda = 2°$) at 12:00.

Local time coincides with the zonal time along the meridian 15° east of Greenwich. Longitude difference $15° - 2° = 13°$ equals $(13°/15°) \times 60$ min $= 52$ minutes. The local time is 52 minutes less than the official time, or 11:08. This is mean solar time. To find the true solar time, we must add the equation of time. In early February, E.T. $= -14$ min and the true solar time is $11{:}08 - 14$ min $= 10{:}54$. At the beginning of November, ET $= +16$ min and the solar time would be 11:24. Since -14 min and $+16$ min are the extreme values of E.T., the true solar time is in the range 10:54–11:24, the exact time depending on the day of the year. During daylight saving time, we must still subtract one hour from these times.

Example 2.10 (Estimating Sidereal Time) Since the sidereal time is the hour angle of the vernal equinox Υ, it is 0 h when Υ culminates or transits the south meridian. At the moment of the vernal equinox, the Sun is in the direction of Υ and thus culminates at the same time as Υ. So the sidereal time at 12:00 local solar time is 0:00, and at the time of the vernal equinox, we have

$$\Theta = T + 12 \text{ h},$$

where T is the local solar time. This is accurate within a couple of minutes. Since the sidereal time runs about 4 minutes fast a day, the sidereal time, n days after the vernal equinox, is

$$\Theta \approx T + 12 \text{ h} + n \times 4 \text{ min}.$$

At autumnal equinox Υ culminates at 0:00 local time, and sidereal and solar times are equal.

Let us try to find the sidereal time in Paris on April 15 at 22:00, Central European standard time (= 23:00 daylight saving time). The vernal equinox occurs on the average on March 21; thus the time elapsed since the equinox is $10 + 15 = 25$ days. Neglecting the equation of time, the local time T is 52 minutes less than the zonal time. Hence

$$\Theta = T + 12 \text{ h} + n \times 4 \text{ min}$$
$$= 21 \text{ h } 8 \text{ min} + 12 \text{ h} + 25 \times 4 \text{ min}$$
$$= 34 \text{ h } 48 \text{ min} = 10 \text{ h } 48 \text{ min}.$$

The time of the vernal equinox can vary about one day in either direction from the average. Therefore the accuracy of the result is roughly 5 min.

Example 2.11 Find the rising time of Arcturus in Boston on January 10.

In Example 2.6 we found the sidereal time of this event, $\Theta = 6$ h 59 min. Since we do not know the year, we use the rough method of Example 2.9. The time between January 1 and vernal equinox (March 21) is about 70 days. Thus the sidereal time on January 1 is

$$\Theta \approx T + 12 \text{ h} - 70 \times 4 \text{ min} = T + 7 \text{ h } 20 \text{ min},$$

from which

$$T = \Theta - 7 \text{ h } 20 \text{ min} = 6 \text{ h } 59 \text{ min} - 7 \text{ h } 20 \text{ min}$$
$$= 30 \text{ h } 59 \text{ min} - 7 \text{ h } 20 \text{ min} = 23 \text{ h } 39 \text{ min}.$$

The longitude of Boston is 71°W, and the Eastern standard time is $(4°/15°) \times 60$ min $= 16$ minutes less, or 23:23.

Example 2.12 Find the sidereal time in Helsinki on April 15, 1982 at 20:00 UT.

The Julian date is $J = 2{,}445{,}074.5$ and

$$T = \frac{2{,}445{,}074.5 - 2{,}451{,}545.0}{36{,}525}$$
$$= -0.1771526.$$

Next, we use (2.47) to find the sidereal time at 0 UT:

$$\Theta_0 = -1{,}506{,}521.0 \text{ s} = -418 \text{ h } 28 \text{ min } 41 \text{ s}$$
$$= 13 \text{ h } 31 \text{ min } 19 \text{ s}.$$

Since the sidereal time runs 3 min 57 s fast a day as compared to the solar time, the difference in 20 hours will be

$$\frac{20}{24} \times 3\text{ min } 57\text{ s} = 3\text{ min } 17\text{ s},$$

and the sidereal time at 20 UT will be 13 h 31 min 19 s + 20 h 3 min 17 s = 33 h 34 min 36 s = 9 h 34 min 36 s.

At the same time (at 22:00 Finnish time, 23:00 daylight saving time) in Helsinki the sidereal time is ahead of this by the amount corresponding to the longitude of Helsinki, 25°, i.e. 1 h 40 min 00 s. Thus the sidereal time is 11 h 14 min 36 s.

2.17 Exercises

Exercise 2.1 Find the distance between Helsinki and Seattle along the shortest route. Where is the northernmost point of the route, and what is its distance from the North Pole? The longitude of Helsinki is 25°E and latitude 60°; the longitude of Seattle is 122°W and latitude 48°. Assume that the radius of the Earth is 6370 km.

Exercise 2.2 A star crosses the south meridian at an altitude of 85°, and the north meridian at 45°. Find the declination of the star and the latitude of the observer.

Exercise 2.3 Where are the following statements true?

(a) Castor (α Gem, declination $\delta = 31° 53'$) is circumpolar.
(b) Betelgeuze (α Ori, $\delta = 7° 24'$) culminates at zenith.
(c) α Cen ($\delta = -60° 50'$) rises to an altitude of 30°.

Exercise 2.4 In his *Old Man and the Sea* Hemingway wrote:

> *It was dark now as it becomes dark quickly after the Sun sets in September. He lay against the worn wood of the bow and rested all that he could. The first stars were out. He did not know the name of Rigel but he saw it and knew soon they would all be out and he would have all his distant friends.*

How was Hemingway's astronomy?

Exercise 2.5 The right ascension of the Sun on June 1, 1983, was 4 h 35 min and declination 22° 00′. Find the ecliptic longitude and latitude of the Sun and the Earth.

Exercise 2.6 Show that on the Arctic Circle the Sun

(a) rises at the same sidereal time Θ_0 between December 22 and June 22,
(b) sets at the same sidereal time Θ_0 between June 22 and December 22.

What is Θ_0?

Exercise 2.7 Derive the equations (2.24), which give the galactic coordinates as functions of the ecliptic coordinates.

Exercise 2.8 The coordinates of Sirius for the epoch 1900.0 were $\alpha = 6\text{ h } 40\text{ min } 45\text{ s}$, $\delta = -16° 35'$, and the components of its proper motion were $\mu_\alpha = -0.037$ s/a, $\mu_\delta = -1.12''\text{a}^{-1}$. Find the coordinates of Sirius for 2000.0. The precession must also be taken into account.

Exercise 2.9 The parallax of Sirius is 0.375″ and radial velocity −8 km/s.

(a) What are the tangential and total velocities of Sirius? (See also the previous exercise.)
(b) When will Sirius be closest to the Sun?
(c) What will its proper motion and parallax be then?

Exercise 2.10 The average period of the variable star Mira Ceti is 331.96 days. According to a catalogue the brightness was at maximum in September 22, 2000. When the star was brightest in 2010?