11 Stellar Structure

The stars are huge gas spheres, hundreds of thousands or millions of times more massive than the Earth. A star such as the Sun can go on shining steadily for thousands of millions of years. This is shown by studies of the prehistory of the Earth, which indicate that the energy radiated by the Sun has not changed by much during the last four thousand million years. The equilibrium of a star must remain stable for such periods.

11.1 Internal Equilibrium Conditions

Mathematically the conditions for the internal equilibrium of a star can be expressed as four differential equations governing the distribution of mass, gas pressure and energy production and transport in the star. These equations will now be derived.

Hydrostatic Equilibrium The force of gravity pulls the stellar material towards the centre. It is resisted by the pressure force due to the thermal motions of the gas molecules. The first equilibrium condition is that these forces be in equilibrium.

Consider a cylindrical volume element at the distance r from the centre of the star (Fig. 11.1). Its volume is $\mathrm{d}V = \mathrm{d}A\,\mathrm{d}r$, where $\mathrm{d}A$ is its base area and $\mathrm{d}r$ its height; its mass is $\mathrm{d}m = \rho\,\mathrm{d}A\,\mathrm{d}r$, where $\rho = \rho(r)$ is the gas density at the radius r. If the mass inside radius r is M_r, the gravitational

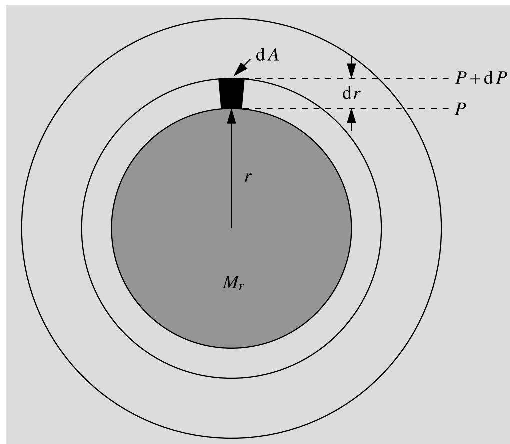

Fig. 11.1 In hydrostatic equilibrium the sum of the gravitational and pressure force acting on a volume element is zero

force on the volume element will be

$$\mathrm{d}F_\mathrm{g} = -\frac{GM_r\,\mathrm{d}m}{r^2} = -\frac{GM_r\rho}{r^2}\,\mathrm{d}A\,\mathrm{d}r,$$

where G is the gravitational constant. The minus sign in this expression means that the force is directed towards the centre of the star. If the pressure at the lower surface of the volume element is P and at its upper surface $P + \mathrm{d}P$, the net force of pressure acting on the element is

$$\begin{aligned}\mathrm{d}F_\mathrm{p} &= P\,\mathrm{d}A - (P + \mathrm{d}P)\,\mathrm{d}A\\ &= -\mathrm{d}P\,\mathrm{d}A.\end{aligned}$$

Since the pressure decreases outwards, $\mathrm{d}P$ will be negative and the force $\mathrm{d}F_\mathrm{p}$ positive. The equi-

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librium condition is that the total force acting on the volume element should be zero, i.e.

$$0 = dF_g + dF_p$$
$$= -\frac{GM_r\rho}{r^2}\,dA\,dr - dP\,dA$$

or

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}. \qquad (11.1)$$

This is the *equation of hydrostatic equilibrium*.

Mass Distribution The second equation gives the mass contained within a given radius. Consider a spherical shell of thickness dr at the distance r from the centre (Fig. 11.2). Its mass is

$$dM_r = 4\pi r^2\rho\,dr,$$

giving the *mass continuity equation*

$$\frac{dM_r}{dr} = 4\pi r^2\rho. \qquad (11.2)$$

Energy Production The third equilibrium condition expresses the conservation of energy, requiring that any energy produced in the star has to be carried to the surface and radiated away. We again consider a spherical shell of thickness dr and mass dM_r at the radius r (Fig. 11.3). Let L_r be the energy flux, i.e. the amount of energy passing through the surface r per unit time. If ε is the energy production coefficient, i.e. the amount of energy released in the star per unit time and mass, then

$$dL_r = L_{r+dr} - L_r = \varepsilon\,dM_r = 4\pi r^2\rho\varepsilon\,dr.$$

Thus the *energy conservation equation* is

$$\frac{dL_r}{dr} = 4\pi r^2\rho\varepsilon. \qquad (11.3)$$

The rate at which energy is produced depends on the distance to the centre. Essentially all of the energy radiated by the star is produced in the hot and dense core. In the outer layers the energy production is negligible and L_r is almost constant.

The Temperature Gradient The fourth equilibrium equation gives the temperature change as a function of the radius, i.e. the temperature gradient dT/dr. The form of the equation depends on how the energy is transported: by *conduction*, *convection* or *radiation*.

In the interiors of normal stars conduction is very inefficient, since the electrons carrying the energy can only travel a short distance without colliding with other particles. Conduction

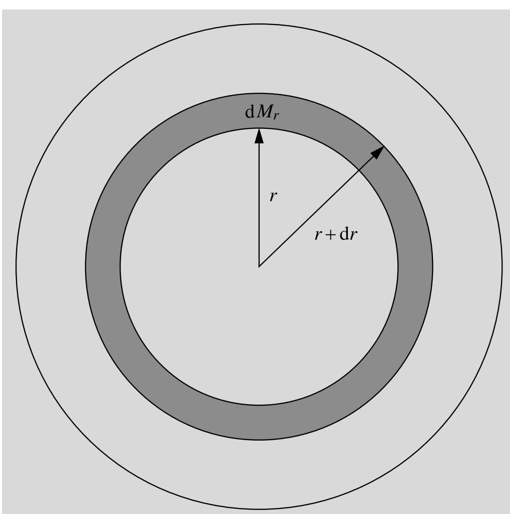


Fig. 11.2 The mass of a thin spherical shell is the product of its volume and its density

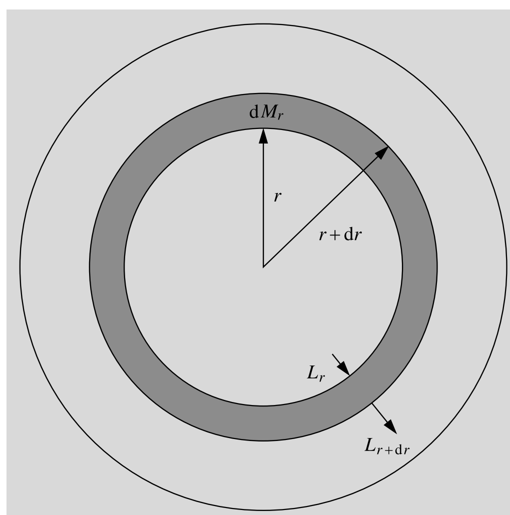


Fig. 11.3 The energy flowing out of a spherical shell is the sum of the energy flowing into it and the energy generated within the shell

only becomes important in compact stars, white dwarfs and neutron stars, where the mean free path of photons is extremely short, but that of some electrons can be relatively large. In normal stars conductive energy transport can be neglected.

In radiative energy transport, photons emitted in hotter parts of the star are absorbed in cooler regions, which they heat. The star is said to be in radiative equilibrium, when the energy released in the stellar interior is carried outwards entirely by radiation.

The radiative temperature gradient is related to the energy flux L_r according to

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \left(-\frac{3}{4ac}\right)\left(\frac{\kappa\rho}{T^3}\right)\left(\frac{L_r}{4\pi r^2}\right), \tag{11.4}$$

where $a = 4\sigma/c = 7.564 \times 10^{-16}\ \mathrm{J\,m^{-3}\,K^{-4}}$ is the radiation constant, c the speed of light, and ρ the density. The *mass absorption coefficient* κ gives the amount of absorption per unit mass. Its value depends on the temperature, density and chemical composition.

In order to derive (11.4), we consider the equation of radiative transfer (5.44). In terms of the variables used in the present chapter, it may be written

$$\cos\theta\,\frac{\mathrm{d}I_\nu}{\mathrm{d}r} = -\kappa_\nu\rho I_\nu + j_\nu .$$

In this equation κ_ν is replaced with a suitable mean value κ. The equation is then multiplied with $\cos\theta$ and integrated over all directions and frequencies. On the left hand side, I_ν may be approximated with the Planck function B_ν. The frequency integral may then be evaluated by means of (5.16). On the right-hand side, the first term can be expressed in terms of the flux density according to (4.2) and the integral over directions of the second gives zero, since j_ν does not depend on θ. One thus obtains

$$\frac{4\pi}{3}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{ac}{4\pi}T^4\right) = -\kappa\rho F_r .$$

Finally, using the relation

$$F_r = \frac{L_r}{4\pi r^2},$$

between the flux density F_r and the energy flux L_r, one obtains (11.4).

The derivative $\mathrm{d}T/\mathrm{d}r$ is negative, since the temperature increases inwards. Clearly there has to be a temperature gradient, if energy is to be transported by radiation: otherwise the radiation field would be the same in all directions and the net flux F_r would vanish.

If the radiative transfer of energy becomes inefficient, the absolute value of the radiative temperature gradient becomes very large. In that case motions are set up in the gas, which carry the energy outwards more efficiently than the radiation. In these convective motions, hot gas rises upwards into cooler layers, where it loses its energy and sinks again. The rising and sinking gas elements also mix the stellar material, and the composition of the convective parts of a star becomes homogeneous. Radiation and conduction, on the other hand, do not mix the material, since they move only energy, not gas.

In order to derive the temperature gradient for the convective case, consider a rising bubble. Assume that the gas moving with the bubble obeys the adiabatic equation of state

$$T \propto P^{1-\frac{1}{\gamma}}, \tag{11.5}$$

where P is the pressure of the gas and γ, the *adiabatic exponent*

$$\gamma = C_\mathrm{P}/C_\mathrm{V}, \tag{11.6}$$

is the ratio of the specific heats in constant pressure and constant volume. This ratio of the specific heats depends on the ionisation of the gas, and can be computed when the temperature, density and chemical composition are known.

Taking the derivative of (11.5) we get the expression for the convective temperature gradient

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{\mathrm{d}P}{\mathrm{d}r}. \tag{11.7}$$

In the practical computation of stellar structure, one uses either (11.4) or (11.7), depending on which equation gives a less steep temperature gradient. In the outermost layers of stars heat exchange with the surroundings must be taken into account, and (11.7) is no longer a good approximation. An often used method for calculating the convective temperature gradient in that case is the mixing-length theory. The theory of convection is a difficult and still imperfectly understood problem, which is beyond the scope of this presentation.

The convective motions set in when the radiative temperature gradient becomes larger in absolute value than the adiabatic gradient, i.e. if either the radiative gradient becomes steep or if the convective gradient becomes small. From (11.4) it can be seen that a steep radiative gradient is expected, if either the energy flux density or the mass absorption coefficient becomes large. The convective gradient may become small, if the adiabatic exponent approaches 1.

Boundary Conditions In order to obtain a well-posed problem, some boundary conditions have to be prescribed for the preceding differential equations:

– There are no sources of energy or mass at the centre inside the radius $r = 0$; thus $M_0 = 0$ and $L_0 = 0$.
– The total mass within the radius R of the star is fixed, $M_R = M$.
– The temperature and pressure at the stellar surface have some determinate values, T_R and P_R. These will be very small compared to those in the interior, and thus it is usually sufficient to take $T_R = 0$ and $P_R = 0$.

In addition to these boundary conditions one needs an expression for the pressure, which is given by the equation of state as well as expressions for the mass absorption coefficient and the energy generation rate, which will be considered later. The solution of the basic differential equations give the mass, temperature, density and energy flux as functions of the radius. The stellar radius and luminosity can then be calculated and compared with the observations.

The properties of a stellar equilibrium model are essentially determined once the mass and the chemical composition have been given. This result is known as the *Vogt–Russell theorem*.

11.2 Physical State of the Gas

Due to the high temperature the gas in the stars is almost completely ionised. The interactions between individual particles are small, so that, to a good approximation, the gas obeys the perfect gas equation of state,

$$P = \frac{k}{\mu m_{\mathrm{H}}} \rho T, \tag{11.8}$$

where k is Boltzmann's constant, μ the mean molecular weight in units of m_{H}, and m_{H} the mass of the hydrogen atom.

The mean molecular weight can be approximately calculated assuming complete ionisation. An atom with nuclear charge Z then produces $Z+1$ free particles (the nucleus and Z electrons). Hydrogen gives rise to two particles per atomic mass unit; helium gives rise to three particles per four atomic mass units. For all elements heavier than hydrogen and helium it is usually sufficient to take $Z+1$ to be half the atomic weight. (Exact values could easily be calculated, but the abundance of heavy elements is so small that this is usually not necessary.) In astrophysics the relative mass fraction of hydrogen is conventionally denoted by X, that of helium by Y and that of all heavier elements by Z, so that

$$X + Y + Z = 1. \tag{11.9}$$

(The Z occurring in this equation should not be confused with the nuclear charge, which is unfortunately denoted by the same letter.) Thus the mean molecular weight will be

$$\mu = \frac{1}{2X + \frac{3}{4}Y + \frac{1}{2}Z}. \tag{11.10}$$

At high temperatures the radiation pressure has to be added to the gas pressure described by the perfect gas equation. The pressure exerted by radiation is (see Box 11.2)

$$P_{\mathrm{rad}} = \frac{1}{3} a T^4, \tag{11.11}$$

where a is the radiation constant. Thus the total pressure is

$$P = \frac{k}{\mu m_{\mathrm{H}}} \rho T + \frac{1}{3} a T^4. \tag{11.12}$$

The perfect gas law does not apply at very high densities.

The *Pauli exclusion principle* states that an atom with several electrons cannot have more than one electron with all four quantum numbers equal. This can also be generalised to a gas consisting of electrons (or other fermions). A *phase space* can be used to describe the electrons. The phase space is a 6-dimensional space, three coordinates of which give the position of the particle and the other three coordinates the momenta in x, y and z directions. A volume element of the phase space is

$$\Delta V = \Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z . \qquad (11.13)$$

From the uncertainty principle it follows that the smallest meaningful volume element is of the order of h^3. According to the exclusion principle there can be only two electrons with opposite spins in such a volume element. When density becomes high enough, all volume elements of the phase space will be filled up to a certain limiting momentum. Such matter is called *degenerate*.

Electron gas begins to degenerate when the density is of the order 10^7 kg/m^3. In ordinary stars the gas is usually nondegenerate, but in white dwarfs and in neutron stars, degeneracy is of central importance. The pressure of a degenerate electron gas is (see Box 11.2)

$$P \approx \left(\frac{h^2}{m_e}\right)\left(\frac{N}{V}\right)^{5/3}, \qquad (11.14)$$

where m_e is the electron mass and N/V the number of electrons per unit volume. This equation may be written in terms of the density

$$\rho = N\mu_e m_H / V,$$

where μ_e is the mean molecular weight per free electron in units of m_H. An expression for μ_e may be derived in analogy with (11.10):

$$\mu_e = \frac{1}{X + \frac{2}{4}Y + \frac{1}{2}Z} = \frac{2}{X+1} . \qquad (11.15)$$

For the solar hydrogen abundance this yields

$$\mu_e = 2/(0.71 + 1) = 1.17.$$

The final expression for the pressure is

$$P \approx \left(\frac{h^2}{m_e}\right)\left(\frac{\rho}{\mu_e m_H}\right)^{5/3} . \qquad (11.16)$$

This is the equation of state of a degenerate electron gas. In contrast to the perfect gas law the pressure no longer depends on the temperature, only on the density and on the particle masses.

In normal stars the degenerate gas pressure is negligible, but in the central parts of giant stars and in white dwarfs, where the density is of the order of 10^8 kg/m^3, the degenerate gas pressure is dominant, in spite of the high temperature.

At even higher densities the electron momenta become so large that their velocities approach the speed of light. In this case the formulas of the special theory of relativity have to be used. The pressure of a relativistic degenerate gas is

$$P \approx hc\left(\frac{N}{V}\right)^{4/3} = hc\left(\frac{\rho}{\mu_e m_H}\right)^{4/3} . \qquad (11.17)$$

In the relativistic case the pressure is proportional to the density to the power 4/3, rather than 5/3 as for the nonrelativistic case. The transition to the relativistic situation takes place roughly at the density 10^9 kg/m^3.

In general the pressure inside a star depends on the temperature (except for a completely degenerate gas), density and chemical composition. In actual stars the gas will never be totally ionised or completely degenerate. The pressure will then be given by more complicated expressions. Still it can be calculated for each case of interest. One may then write

$$P = P(T, \rho, X, Y, Z), \qquad (11.18)$$

giving the pressure as a known function of the temperature, density and chemical composition.

The *opacity* of the gas describes how difficult it is for radiation to propagate through it. The change $\mathrm{d}I$ of the intensity in a distance $\mathrm{d}r$ can be expressed as

$$\mathrm{d}I = -I\alpha\,\mathrm{d}r,$$

where α is the opacity (Sect. 4.5). The opacity depends on the chemical composition, temperature and density of the gas. It is usually written as $\alpha = \kappa\rho$, where ρ is the density of the gas and κ the *mass absorption coefficient* ($[\kappa] = \mathrm{m}^2/\mathrm{kg}$).

The inverse of the opacity represents the mean free path of radiation in the medium, i.e. the distance it can propagate without being scattered or absorbed. The different types of absorption processes (bound–bound, bound–free, free–free) have been described in Sect. 5.1. The opacity of the stellar material due to each process can be calculated for relevant values of temperature and density.

11.3 Stellar Energy Sources

When the equations of stellar structure were derived, the character of the source of stellar energy was left unspecified. Knowing a typical stellar luminosity, one can calculate how long different energy sources would last. For instance, normal chemical burning could produce energy for only a few thousand years. The energy released by the contraction of a star would last slightly longer, but after a few million years this energy source would also run out.

Terrestrial biological and geological evidence shows that the solar luminosity has remained fairly constant for at least a few thousand million years. Since the age of the Earth is about 5000 million years, the Sun has presumably existed at least for that time. Since the solar luminosity is 4×10^{26} W, it has radiated about 6×10^{43} J in 5×10^9 years. The Sun's mass is 2×10^{30} kg; thus it must be able to produce at least 3×10^{13} J/kg.

The general conditions in the solar interior are known, regardless of the exact energy source. Thus, in Example 11.5, it will be estimated that the temperature at half the radius is about 5 million degrees. The central temperature must be about ten million kelvins, which is high enough for *thermonuclear fusion reactions* to take place.

In fusion reactions light elements are transformed into heavier ones. The final reaction products have a smaller total mass than the initial nuclei. This mass difference is released as energy according to Einstein's relation $E = mc^2$. Thermonuclear reactions are commonly referred to as burning, although they have no relation to the chemical burning of ordinary fuels.

The atomic nucleus consists of protons and neutrons, together referred to as nucleons. We define

$$
\begin{aligned}
m_\mathrm{p} &= \text{proton mass},\\
m_\mathrm{n} &= \text{neutron mass},\\
Z &= \text{nuclear charge} = \text{atomic number},\\
N &= \text{neutron number},\\
A &= Z + N = \text{atomic weight},\\
m(Z, N) &= \text{mass of the nucleus}.
\end{aligned}
$$

The mass of the nucleus is smaller than the sum of the masses of all its nucleons. The difference is called the *binding energy*. The binding energy per nucleon is

$$Q = \frac{1}{A}\big(Zm_\mathrm{p} + Nm_\mathrm{n} - m(Z, N)\big)c^2. \quad (11.19)$$

It turns out that Q increases towards heavier elements up to iron ($Z = 26$). Beyond iron the binding energy again begins to decrease (Fig. 11.4).

It is known that the stars consist mostly of hydrogen. Let us consider how much energy would be released by the fusion of four hydrogen nuclei into a helium nucleus. The mass of a proton is 1.672×10^{-27} kg and that of a helium nucleus is 6.644×10^{-27} kg. The mass difference, 4.6×10^{-29} kg, corresponds to an energy difference $E = 4.1 \times 10^{-12}$ J. Thus 0.7 % of the mass is turned into energy in the reaction, corresponding to an energy release of 6.4×10^{14} J per one kilogram of hydrogen. This should be compared with our previous estimate that 3×10^{13} J/kg is needed.

Already in the 1930's it was generally accepted that stellar energy had to be produced by nuclear fusion. In 1938 *Hans Bethe* and independently *Carl Friedrich von Weizsäcker* put forward the first detailed mechanism for energy production in the stars, the *carbon–nitrogen–oxygen (CNO) cycle*. The other important energy gener-

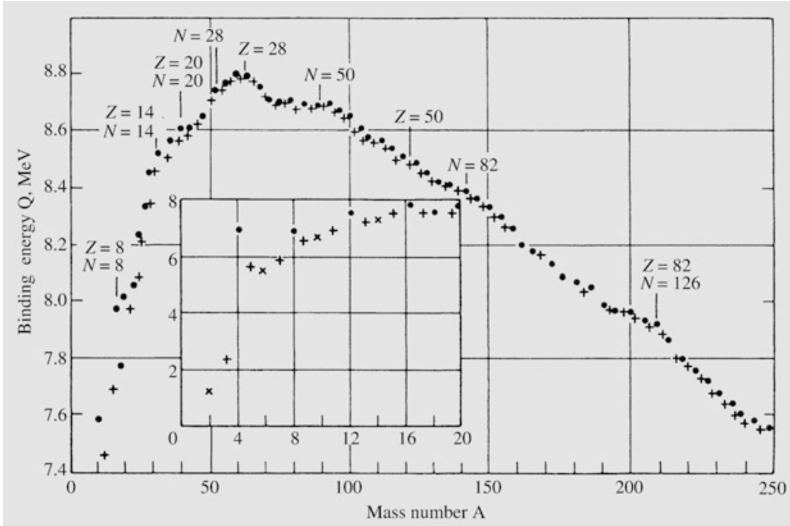


Fig. 11.4 The nuclear binding energy per nucleon as a function of the atomic weight. Among isotopes with the same atomic weight the one with the largest binding energy is shown. *The points* correspond to nuclei with even proton and neutron numbers, *the crosses* to nuclei with odd mass numbers. Preston, M.A. (1962): *Physics of the Nucleus* (Addison-Wesley Publishing Company, Inc., Reading, Mass.)

ation processes (the *proton–proton chain* and the *triple-alpha reaction*) were not proposed until the 1950's.

The Proton–Proton Chain (Fig. 11.5). In stars with masses of about that of the Sun or smaller, the energy is produced by the proton–proton (pp) chain. It consists of the following steps:

$$\begin{array}{lll} \text{ppI:} & (1) & {}^1\mathrm{H} + {}^1\mathrm{H} \rightarrow {}^2\mathrm{H} + \mathrm{e}^+ + \nu_\mathrm{e}, \\ & & {}^1\mathrm{H} + {}^1\mathrm{H} + \mathrm{e}^- \rightarrow {}^2\mathrm{H} + \nu_\mathrm{e}, \\ & (2) & {}^2\mathrm{H} + {}^1\mathrm{H} \rightarrow {}^3\mathrm{He} + \gamma, \\ & (3) & {}^3\mathrm{He} + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + 2\,{}^1\mathrm{H}. \end{array}$$

For each reaction (3) the reactions (1) and (2) have to take place twice. The first reaction step has a very small probability, which has not been measured in the laboratory. At the central density and temperature of the Sun, the expected time for a proton to collide with another one to form a deuteron is 10^{10} years on the average. It is only thanks to the slowness of this reaction that the Sun is still shining. If it were faster, the Sun would have burnt out long ago. The neutrino produced in the reaction (1) can escape freely from the star and carries away some of the energy released. The positron e^+ is immediately annihilated together with an electron, giving rise to two gamma quanta.

The second reaction, where a deuteron and a proton unite to form the helium isotope ^{3}He, is very fast compared to the preceding one. Thus the abundance of deuterons inside stars is very small.

The last step in the pp chain can take three different forms. The ppI chain shown above is the most probable one. In the Sun 91 % of the energy is produced by the ppI chain. It is also possible for ^{3}He nuclei to unite into ^{4}He nuclei in two additional branches of the pp chain.

$$\begin{array}{lll} \text{ppII:} & (3) & {}^3\mathrm{He} + {}^4\mathrm{He} \rightarrow {}^7\mathrm{Be} + \gamma, \\ & (4) & {}^7\mathrm{Be} + \mathrm{e}^- \rightarrow {}^7\mathrm{Li} + \nu_\mathrm{e}, \\ & (5) & {}^7\mathrm{Li} + {}^1\mathrm{H} \rightarrow {}^4\mathrm{He} + {}^4\mathrm{He}, \\ \text{ppIII:} & (3) & {}^3\mathrm{He} + {}^4\mathrm{He} \rightarrow {}^7\mathrm{Be} + \gamma, \\ & (4) & {}^7\mathrm{Be} + {}^1\mathrm{H} \rightarrow {}^8\mathrm{B} + \gamma, \\ & (5) & {}^8\mathrm{B} \rightarrow {}^8\mathrm{Be} + \mathrm{e}^+ + \nu_\mathrm{e}, \\ & (6) & {}^8\mathrm{Be} \rightarrow {}^4\mathrm{He} + {}^4\mathrm{He}. \end{array}$$

The Carbon Cycle (See Fig. 11.6.) At temperatures below 20 million degrees the pp chain is the main energy production mechanism. At higher temperatures corresponding to stars with masses above $1.5\,M_\odot$, the carbon (CNO) cycle becomes dominant, because its reaction rate increases more rapidly with temperature. In the CNO cycle carbon, oxygen and nitrogen act as

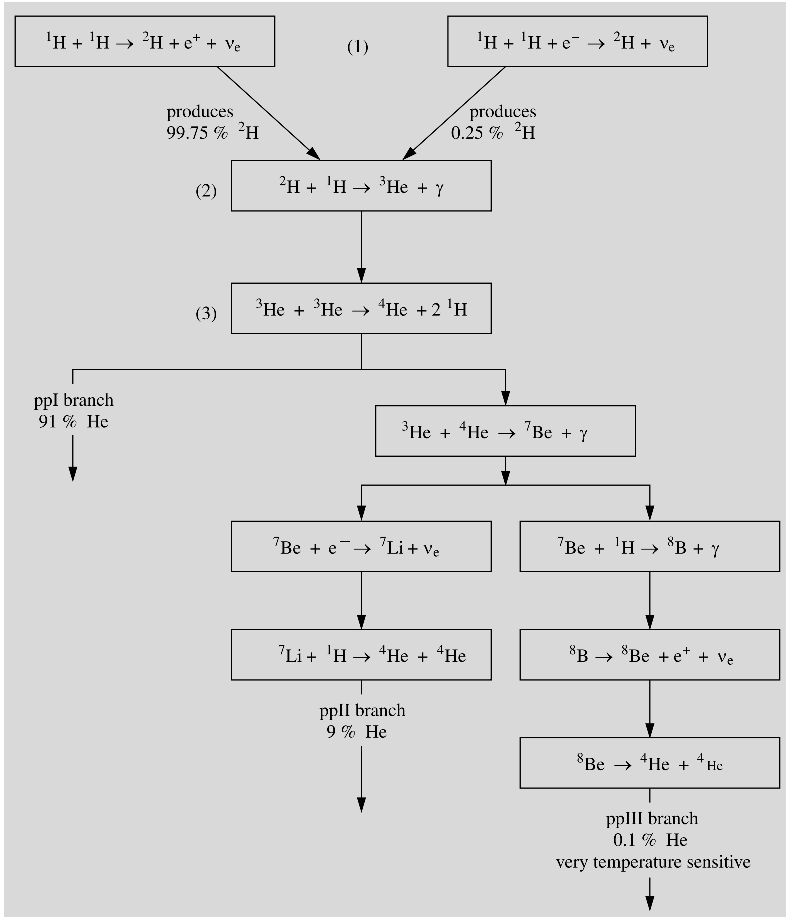


Fig. 11.5 The proton–proton chain. In the ppI branch, four protons are transformed into one helium nucleus, two positrons, two neutrinos and radiation. The relative weights of the reactions are given for conditions in the Sun. The pp chain is the most important energy source in stars with mass below $1.5\,M_\odot$

catalysts. The reaction cycle is the following:

$$
\begin{array}{lll}
(1) & ^{12}\mathrm{C} + {}^{1}\mathrm{H} & \rightarrow {}^{13}\mathrm{N} + \gamma, \\
(2) & ^{13}\mathrm{N} & \rightarrow {}^{13}\mathrm{C} + \mathrm{e}^{+} + \nu_{\mathrm{e}}, \\
(3) & ^{13}\mathrm{C} + {}^{1}\mathrm{H} & \rightarrow {}^{14}\mathrm{N} + \gamma, \\
(4) & ^{14}\mathrm{N} + {}^{1}\mathrm{H} & \rightarrow {}^{15}\mathrm{O} + \gamma, \\
(5) & ^{15}\mathrm{O} & \rightarrow {}^{15}\mathrm{N} + \gamma + \nu_{\mathrm{e}}, \\
(6) & ^{15}\mathrm{N} + {}^{1}\mathrm{H} & \rightarrow {}^{12}\mathrm{C} + {}^{4}\mathrm{He}.
\end{array}
$$

Reaction (4) is the slowest, and thus determines the rate of the CNO cycle. At a temperature of 20 million degrees the reaction time for the reaction (4) is a million years.

The fraction of energy released as radiation in the CNO cycle is slightly smaller than in the pp chain, because more energy is carried away by neutrinos.

The Triple Alpha Reaction As a result of the preceding reactions, the abundance of helium in the stellar interior increases. At a temperature above 10^8 degrees the helium can be transformed into carbon in the triple alpha reaction:

$$
\begin{array}{ll}
(1) & ^{4}\mathrm{He} + {}^{4}\mathrm{He} \leftrightarrow {}^{8}\mathrm{Be}, \\
(2) & ^{8}\mathrm{Be} + {}^{4}\mathrm{He} \rightarrow {}^{12}\mathrm{C} + \gamma.
\end{array}
$$

Here ^{8}Be is unstable and decays into two helium nuclei or alpha particles in 2.6×10^{-16} seconds. The production of carbon thus requires the almost simultaneous collision of three particles. The reaction is often written

$$3\,{}^{4}\mathrm{He} \rightarrow {}^{12}\mathrm{C} + \gamma.$$

Once helium burning has been completed, at higher temperatures other reactions become pos-

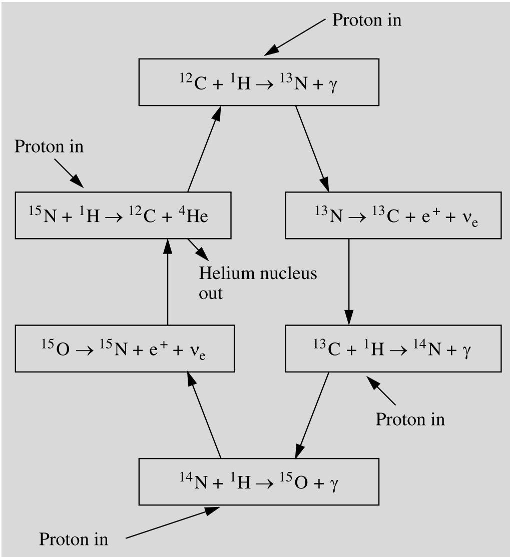


Fig. 11.6 The CNO cycle is catalysed by ^{12}C. It transforms four protons into a helium nucleus, two positrons, two neutrinos and radiation. It is the dominant energy source for stars more massive than 1.5 $M_\odot$

sible, in which heavier elements up to iron and nickel are built up. Examples of such reactions are various alpha reactions and oxygen, carbon and silicon burning.

Alpha Reactions During helium burning some of the carbon nuclei produced react with helium nuclei to form oxygen, which in turn reacts to form neon, etc. These reactions are fairly rare and thus are not important as stellar energy sources. Examples are

$$\begin{aligned} ^{12}\text{C} + {}^{4}\text{He} &\rightarrow {}^{16}\text{O} + \gamma, \\ ^{16}\text{O} + {}^{4}\text{He} &\rightarrow {}^{20}\text{Ne} + \gamma, \\ ^{20}\text{Ne} + {}^{4}\text{He} &\rightarrow {}^{24}\text{Mg} + \gamma. \end{aligned}$$

Carbon Burning After the helium is exhausted, carbon burning sets in at the temperature (5–8) × 10^{10} K:

$$\begin{aligned} ^{12}\text{C} + {}^{12}\text{C} &\rightarrow {}^{24}\text{Mg} + \gamma \\ &\rightarrow {}^{23}\text{Na} + {}^{1}\text{H} \\ &\rightarrow {}^{20}\text{Ne} + {}^{4}\text{He} \\ &\rightarrow {}^{23}\text{Mg} + \text{n} \\ &\rightarrow {}^{16}\text{O} + 2\,{}^{4}\text{He}. \end{aligned}$$

Oxygen Burning Oxygen is consumed at slightly higher temperatures in the reactions

$$\begin{aligned} ^{16}\text{O} + {}^{16}\text{O} &\rightarrow {}^{32}\text{S} + \gamma \\ &\rightarrow {}^{31}\text{P} + {}^{1}\text{H} \\ &\rightarrow {}^{28}\text{Si} + {}^{4}\text{He} \\ &\rightarrow {}^{31}\text{S} + \text{n} \\ &\rightarrow {}^{24}\text{Mg} + 2\,{}^{4}\text{He}. \end{aligned}$$

Silicon Burning After several intermediate steps the burning of silicon produces nickel and iron. The total process may be expressed as

$$\begin{aligned} ^{28}\text{Si} + {}^{28}\text{Si} &\rightarrow {}^{56}\text{Ni} + \gamma, \\ ^{56}\text{Ni} &\rightarrow {}^{56}\text{Fe} + 2\,\text{e}^{+} + 2\,\nu_\text{e}. \end{aligned}$$

When the temperature becomes higher than about 10^9 K, the energy of the photons becomes large enough to destroy certain nuclei. Such reactions are called *photonuclear reactions* or *photodissociations*.

The production of elements heavier than iron requires an input of energy, and therefore such elements cannot be produced by thermonuclear reactions. Elements heavier than iron are almost exclusively produced by *neutron capture* during the final violent stages of stellar evolution (Sect. 11.5).

The rates of the reactions presented above can be determined by laboratory experiments or by theoretical calculations. Knowing them, one can calculate the rate at which energy is released per unit mass and time as a function of the density, temperature and chemical composition:

$$\varepsilon = \varepsilon(T, \rho, X, Y, Z). \tag{11.20}$$

In reality the relative abundance of each of the heavier nuclei needs to be known, not just their total abundance Z.

11.4 Stellar Models

A theoretical stellar model is obtained if one solves the differential equations for stellar structure. As we have already noted, the model is uniquely defined once the chemical composition and the mass of the star have been given.

Table 11.1 Properties of zero age main sequence stars. (T_c = central temperature; ρ_c = central density; M_{ci} = relative mass of convective interior; M_{ce} = relative mass of convective envelope)

M [$M_\odot$]	R [$R_\odot$]	L [$L_\odot$]	T_e [K]	T_c [10^6 K]	ρ_c [kg/m^3]	M_{ci} [M]	M_{ce} [M]
30	6.6	140,000	44,000	36	3000	0.60	0
15	4.7	21,000	32,000	34	6200	0.39	0
9	3.5	4500	26,000	31	7900	0.26	0
5	2.2	630	20,000	27	26,000	0.22	0
3	1.7	93	14,000	24	42,000	0.18	0
1.5	1.2	5.4	8100	19	95,000	0.06	0
1.0	0.87	0.74	5800	14	89,000	0	0.01
0.5	0.44	0.038	3900	9.1	78,000	0	0.41

Stars just formed out of the interstellar medium are chemically homogeneous. When stellar models for homogeneous stars are plotted in the HR diagram, they fall along the lower edge of the main sequence. The theoretical sequence obtained in this way is called the *zero age main sequence*, ZAMS. The exact position of the ZAMS depends on the initial chemical composition. For stars with an initial abundance of heavy elements like that in the Sun, the computed ZAMS is in good agreement with the observations. If the initial value of Z is smaller, the theoretical ZAMS falls below the main sequence in the subdwarf region of the HR diagram. This is related to the classification of stars into populations I and II, which is discussed in Sect. 17.2.

The theoretical models also provide an explanation for the mass–luminosity relation. The computed properties of zero age main sequence stars of different masses are given in Table 11.1. The chemical composition assumed is $X = 0.71$ (hydrogen mass fraction), $Y = 0.27$ (helium) and $Z = 0.02$ (heavier elements), except for the $30\,M_\odot$ star, which has $X = 0.70$ and $Y = 0.28$. The luminosity of a one solar mass star is $0.74\,L_\odot$ and the radius $0.87\,R_\odot$. Thus the Sun has brightened and expanded to some extent during its evolution. However, these changes are small and do not conflict with the evidence for a steady solar energy output. In addition the biological evidence only goes back about 3000 million years.

The model calculations show that the central temperature in the smallest stars ($M \approx 0.08\,M_\odot$) is about 4×10^6 K, which is the minimum temperature required for the onset of thermonuclear reactions. In the biggest stars ($M \approx 50\,M_\odot$), the central temperature reaches 4×10^7 K.

The changes in chemical composition caused by the nuclear reactions can be computed, since the rates of the various reactions at different depths in the star are known. For example, the change ΔX of the hydrogen abundance in the time interval Δt is proportional to the rate of energy generation ε and to Δt:

$$\Delta X \propto -\varepsilon \Delta t. \tag{11.21}$$

The constant of proportionality is clearly the amount of hydrogen consumed per unit energy [kg/J]. The value of this constant of proportionality is different for the pp chain and the CNO cycle. Therefore the contribution from each reaction chain must be calculated separately in (11.21). For elements that are produced by the nuclear reactions, the right-hand side contribution in (11.21) is positive. If the star is convective, the change in composition is obtained by taking the average of (11.21) over the convection zone.

Box 11.1 (Gas Pressure and Radiation Pressure) Let us consider noninteracting particles in a rectangular box. The particles may also be photons. Let the sides of the box be Δx, Δy and Δz, and the number of particles, N. The pressure is caused by the collisions of the particles with the sides of the box. When a particle hits a wall perpendicular to the x axis, its momentum in the x direction, p_x, changes by $\Delta p = 2p_x$. The particle will return to the same wall after the time $\Delta t = 2\Delta x/v_x$. Thus the pressure exerted by the particles on the wall (surface area $A = \Delta y \Delta z$) is

$$P = \frac{F}{A} = \frac{\sum \Delta p/\Delta t}{A} = \frac{\sum p_x v_x}{\Delta x \Delta y \Delta z} = \frac{N\langle p_x v_x\rangle}{V},$$

where $V = \Delta x \Delta y \Delta z$ is the volume of the box and the angular brackets represent the average value. The momentum is $p_x = m v_x$ (where $m = h\nu/c^2$ for photons), and hence

$$P = \frac{Nm\langle v_x^2\rangle}{V}.$$

Suppose the velocities of the particles are isotropically distributed. Then $\langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle$, and thus

$$\langle v^2\rangle = \langle v_x^2\rangle + \langle v_y^2\rangle + \langle v_z^2\rangle = 3\langle v_x^2\rangle$$

and

$$P = \frac{Nm\langle v^2\rangle}{3V}.$$

If the particles are gas molecules, the energy of a molecule is $\varepsilon = \frac{1}{2}mv^2$. The total energy of the gas is $E = N\langle\varepsilon\rangle = \frac{1}{2}Nm\langle v^2\rangle$, and the pressure may be written

$$P = \frac{2}{3}\frac{E}{V} \quad \text{(gas)}.$$

If the particles are photons, they move with the speed of light and their energy is $\varepsilon = mc^2$. The total energy of a photon gas is thus $E = N\langle\varepsilon\rangle = Nmc^2$ and the pressure is

$$P = \frac{1}{3}\frac{E}{V} \quad \text{(radiation)}.$$

According to (4.7), (4.4) and (5.16) the energy density of blackbody radiation is

$$\frac{E}{V} = u = \frac{4\pi}{c} I = \frac{4}{c} F = \frac{4}{c}\sigma T^4 \equiv aT^4,$$

where $a = 4\sigma/c$ is the radiation constant. Thus the radiation pressure is

$$P_{\text{rad}} = a\, T^4/3.$$

Box 11.2 (The Pressure of a Degenerate Gas) A gas where all available energy levels up to a limiting momentum p_0, known as the Fermi momentum, are filled is called degenerate. We shall determine the pressure of a completely degenerate electron gas.

Let the volume of the gas be V. We consider electrons with momenta in the range $[p, p + \mathrm{d}p]$. Their available phase space volume is $4\pi p^2 \mathrm{d}p V$. According to the Heisenberg uncertainty relation the elementary volume in phase space is h^3 and, according to the Pauli exclusion principle, this volume can contain two electrons with opposite spins. Thus the number of electrons in the momentum interval $[p, p + \mathrm{d}p]$ is

$$\mathrm{d}N = 2\frac{4\pi p^2\,\mathrm{d}p\, V}{h^3}.$$

The total number of electrons with momenta smaller than p_0 is

$$N = \int \mathrm{d}N = \frac{8\pi V}{h^3}\int_0^{p_0} p^2\,\mathrm{d}p = \frac{8\pi V}{3h^3} p_0^3.$$

Hence the Fermi momentum p_0 is

$$p_0 = \left(\frac{3}{\pi}\right)^{1/3}\frac{h}{2}\left(\frac{N}{V}\right)^{1/3}.$$

Nonrelativistic Gas The kinetic energy of an electron is $\varepsilon = p^2/2m_e$. The total energy of the gas is

$$E = \int \varepsilon\,\mathrm{d}N = \frac{4\pi V}{m_e h^3}\int_0^{p_0} p^4\,\mathrm{d}p = \frac{4\pi V}{5 m_e h^3} p_0^5.$$

Introducing the expression for the Fermi momentum p_0, one obtains

$$E = \frac{\pi}{40}\left(\frac{3}{\pi}\right)^{5/3}\frac{h^2}{m_e} V\left(\frac{N}{V}\right)^{5/3}.$$

The pressure of the gas was derived in Box 11.1:

$$P = \frac{2}{3}\frac{E}{V}$$

$$= \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_e}\left(\frac{N}{V}\right)^{5/3}$$
(nonrelativistic).

Here N/V is the number density of electrons.

Relativistic Gas If the density becomes so large that the electron kinetic energy ϵ corresponding to the Fermi momentum exceeds the rest energy $m_e c^2$, the relativistic expression for the electron energy has to be used. In the extreme relativistic case $\varepsilon = cp$ and the total energy

$$E = \int \varepsilon \, dN = \frac{8\pi cV}{h^3}\int_0^{p_0} p^3 \, dp = \frac{2\pi cV}{h^3} p_0^4.$$

The expression for the Fermi momentum remains unchanged, and hence

$$E = \frac{\pi}{8}\left(\frac{3}{\pi}\right)^{4/3} hcV\left(\frac{N}{V}\right)^{4/3}.$$

The pressure of the relativistic electron gas is obtained from the formula derived for a photon gas in Box 11.1:

$$P = \frac{1}{3}\frac{E}{V} = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hc\left(\frac{N}{V}\right)^{4/3} \quad \text{(relativistic)}.$$

We have obtained the nonrelativistic and extreme relativistic approximations to the pressure. In intermediate cases the exact expression for the electron energy,

$$\varepsilon = \left(m_e^2 c^4 + p^2 c^2\right)^{1/2},$$

has to be used.

The preceding derivations are rigorously valid only at zero temperature. However, the densities in compact stars are so high that the effects of a nonzero temperature are negligible, and the gas may be considered completely degenerate.

11.5 Examples

Example 11.1 (The Gravitational Acceleration at the Solar Surface) The expression for the gravitational acceleration is

$$g = \frac{GM_\odot}{R^2}.$$

Using the solar mass $M = 1.989 \times 10^{30}$ kg and radius $R = 6.96 \times 10^8$ m, one obtains

$$g = 274 \text{ m s}^{-2} \approx 28\, g_0,$$

where $g_0 = 9.81 \text{ m s}^{-2}$ is the gravitational acceleration at the surface of the Earth.

Example 11.2 (The Average Density of the Sun) The volume of a sphere with radius R is

$$V = \frac{4}{3}\pi R^3;$$

thus the average density of the Sun is

$$\overline{\rho} = \frac{M}{V} = \frac{3M}{4\pi R^3} \approx 1410 \text{ kg m}^{-3}.$$

Example 11.3 (Pressure at Half the Solar Radius) The pressure can be estimated from the condition for the hydrostatic equilibrium (11.1). Suppose the density is constant and equal to the average density $\overline{\rho}$. Then the mass within the radius r is

$$M_r = \frac{4}{3}\pi\overline{\rho} r^3,$$

and the hydrostatic equation can be written

$$\frac{dP}{dr} = -\frac{GM_r\overline{\rho}}{r^2} = -\frac{4\pi G\overline{\rho}^2 r}{3}.$$

This can be integrated from half the solar radius, $r = R_\odot/2$, to the surface, where the pressure vanishes:

$$\int_P^0 dP = -\frac{4}{3}\pi G\overline{\rho}^2 \int_{R_\odot/2}^{R_\odot} r \, dr,$$

which gives

$$\begin{aligned}P &= \frac{1}{2}\pi G\overline{\rho}^2 R_\odot^2\\ &\approx \frac{1}{2}\pi\, 6.67\times 10^{-11}\times 1410^2\\ &\quad \times (6.96\times 10^8)^2\ \mathrm{N/m^2}\\ &\approx 10^{14}\ \mathrm{Pa}.\end{aligned}$$

This estimate is extremely rough, since the density increases strongly inwards.

Example 11.4 (The Mean Molecular Weight of the Sun) In the outer layers of the Sun the initial chemical composition has not been changed by nuclear reactions. In this case one can use the values $X = 0.71$, $Y = 0.27$ and $Z = 0.02$. The mean molecular weight (11.10) is then

$$\begin{aligned}\mu &= \frac{1}{2\times 0.71 + 0.75\times 0.27 + 0.5\times 0.02}\\ &\approx 0.61.\end{aligned}$$

When the hydrogen is completely exhausted, $X = 0$ and $Y = 0.98$, and hence

$$\mu = \frac{1}{0.75\times 0.98 + 0.5\times 0.02} \approx 1.34.$$

Example 11.5 (The Temperature of the Sun at $r = R_\odot/2$) Using the density from Example 11.2 and the pressure from Example 11.3, the temperature can be estimated from the perfect gas law (11.8). Assuming the surface value for the mean molecular weight (Example 11.4), one obtains the temperature

$$\begin{aligned}T &= \frac{\mu m_\mathrm{H} P}{k\rho}\\ &= \frac{0.61\times 1.67\times 10^{-27}\times 1.0\times 10^{14}}{1.38\times 10^{-23}\times 1410}\\ &\approx 5\times 10^6\ \mathrm{K}.\end{aligned}$$

Example 11.6 (The Radiation Pressure in the Sun at $r = R_\odot/2$) In the previous example we found that the temperature is $T \approx 5\times 10^6$ K. Thus the radiation pressure given by (11.11) is

$$\begin{aligned}P_\mathrm{rad} &= \frac{1}{3}aT^4\\ &= \frac{1}{3}\times 7.564\times 10^{-16}\times (5\times 10^6)^4\\ &\approx 2\times 10^{11}\ \mathrm{Pa}.\end{aligned}$$

This is about a thousand times smaller than the gas pressure estimated in Example 11.3. Thus it confirms that the use of the perfect gas law in Example 11.5 was correct.

Example 11.7 (The Path of a Photon from the Centre of a Star to Its Surface) Radiative energy transport can be described as a random walk, where a photon is repeatedly absorbed and re-emitted in a random direction. Let the step length of the walk (the mean free path) be d. Consider, for simplicity, the random walk in a plane. After one step the photon is absorbed at

$$x_1 = d\cos\theta_1, \qquad y_1 = d\sin\theta_1,$$

where θ_1 is an angle giving the direction of the step. After N steps the coordinates are

$$x = \sum_{i=1}^{N} d\cos\theta_i, \qquad y = \sum_{i=1}^{N} d\sin\theta_i,$$

and the distance from the starting point is

$$\begin{aligned}r^2 &= x^2 + y^2\\ &= d^2\left[\left(\sum_1^N \cos\theta_i\right)^2 + \left(\sum_1^N \sin\theta_i\right)^2\right].\end{aligned}$$

The first term in square brackets can be written

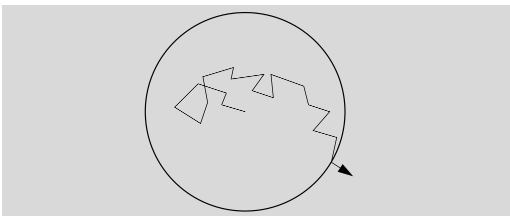
$$\begin{aligned}&\left(\sum_1^N \cos\theta_i\right)^2\\ &\quad = (\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_N)^2\\ &\quad = \sum_1^N \cos^2\theta_i + \sum_{i\neq j}\cos\theta_i\cos\theta_j.\end{aligned}$$

Since the directions θ_i are randomly distributed and independent,

$$\sum_{i \neq j} \cos\theta_i \cos\theta_j = 0.$$

The same result applies for the second term in square brackets. Thus

$$r^2 = d^2 \sum_1^N \left(\cos^2\theta_i + \sin^2\theta_i\right) = Nd^2.$$



After N steps the photon is at the distance $r = d\sqrt{N}$ from the starting point. Similarly, a drunkard taking a hundred one-metre steps at random will have wandered ten metres from his/her starting point. The same result applies in three dimensions.

The time taken by a photon to reach the surface from the centre depends on the mean free path $d = 1/\alpha = 1/\kappa\rho$. The value of κ at half the solar radius can be estimated from the values of density and temperature obtained in Examples 11.2 and 11.5. The mass absorption coefficient in these conditions is found to be $\kappa = 10\ \mathrm{m^2/kg}$. (We shall not enter on how it is calculated.) The photon mean free path is then

$$d = \frac{1}{\kappa\rho} \approx 10^{-4}\ \mathrm{m}.$$

This should be a reasonable estimate in most of the solar interior. Since the solar radius $r = 10^9$ m, the number of steps needed to reach the surface will be $N = (r/d)^2 = 10^{26}$. The total path travelled by the photon is $s = Nd = 10^{22}$ m, and the time taken is $t = s/c = 10^6$ years; a more careful calculation gives $t = 10^7$ years. Thus it takes 10 million years for the energy generated at the centre to radiate into space. Of course the radiation that leaves the surface does not consist of the same gamma photons that were produced near the centre. The intervening scattering, emission and absorption processes have transformed the radiation into visible light (as can easily be seen).

11.6 Exercises

Exercise 11.1 How many hydrogen atoms are there in the Sun per each helium atom?

Exercise 11.2 (a) How many pp reactions take place in the Sun every second? The luminosity of the Sun is 3.9×10^{26} W, the mass of a proton is 1.00728 amu, and that of the α particle 4.001514 amu (1 amu is 1.6604×10^{-27} kg).

(b) How many neutrinos produced in these pp reactions will hit the Earth in one second?

Exercise 11.3 The mass absorption coefficient of a neutrino is $\kappa = 10^{-21}\ \mathrm{m^2\,kg^{-1}}$. Find the mean free path at the centre of the Sun.