Chapter 16

Basic Random Processes

16.1 Introduction

So far we have studied the probabilistic description of a *finite* number of random variables. This is useful for random phenomena that have definite beginning and end times. Many physical phenomena, however, are more appropriately modeled as ongoing in time. Such is the case for the annual summer rainfall in Rhode Island as shown in Figure 1.1 and repeated for convenience in Figure 16.1. This physical

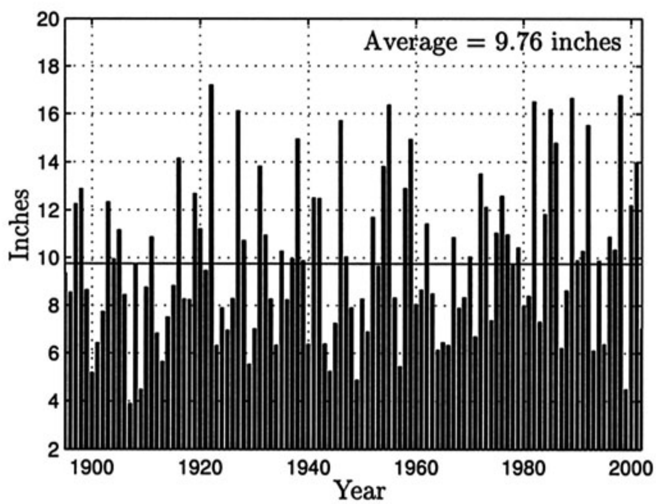

Figure 16.1: Annual summer rainfall in Rhode Island from 1895 to 2002.

process has been ongoing for all time and will undoubtedly continue into the future. It is only our limited ability to measure the rainfall over several lifetimes that has produced the data shown in Figure 16.1. It therefore seems more reasonable to attempt to study the probabilistic characteristics of the annual summer rainfall in

Rhode Island for *all time.* To do so let $X[n]$ be a random variable that denotes the annual summer rainfall for year n. Then, we will be interested in the behavior of the *infinite* tuple of random variables $(\ldots, X[-1], X[0], X[1], \ldots)$, where the corresponding year for $n = 0$ can be chosen for convenience (maybe according to the Christian or Hebrew calendars, as examples). Note that we cannot employ our previous probabilistic methods directly since the number of random variables is not finite or N-dimensional.

Given our interest in the annual summer rainfall, what types of questions are pertinent? A meterologist might wish to determine if the rainfall totals are increasing with time. Hence, he may question if the *average* rainfall is really constant. If it is not constant with time, then our estimate of the average, obtained by taking the sample mean of the values shown in Figure 16.1, is meaningless. As an example, we would also have obtained an average of 9.76 inches if the rainfall totals were increasing linearly with time, starting at 7.76 inches and ending at 11.76 inches. The meterologist might argue that due to global warming the rainfall totals should be increasing. We will return to this question in Section 16.8. Another question might be to assess the probability that the following year the rainfall will be 12 inches or more if we know the entire past history of rainfall totals. This is the problem of prediction, which is a fundamental problem in many scientific disciplines.

A second example of a random process, which is of intense interest, is a man-made one: the Dow-Jones industrial average (DJIA) for stocks. At the end of each trading day the average of the prices of a representative group of stocks is computed to give an indication of the health of the U.S. stock market. Its usefulness is that this value also gives an indication of the overall health of the U.S. economy. Some recent weekly values are shown in Figure 16.2. The overall trend beginning at week 10 is upward until about week 60, at which point it fluctuates up and down. Some questions of interest are whether the index will go back up again after week 92 and to what degree is it possible to predict the movement of the stock market, of which the DJIA is an indicator. The financial industry and in fact the health of the U.S. economy depends in a large degree upon the answers to these questions! In the remaining chapters we will describe the theory and application of random processes. As always, the theory will serve as a foundation upon which we will be able to analyze random processes. In any practical situation, however, the ideal theoretical analysis must be tempered with the constraints and additional complexities of the real world.

16.2 Summary

A random process is defined in Section 16.3. Four different types of random processes are described in Section 16.4. They are classified according to whether they are defined for all time or only for uniformly spaced time samples, and also according to their possible values as being discrete or continuous. Figure 16.5 illustrates the various types. A stationary random process is one for which its probabilistic

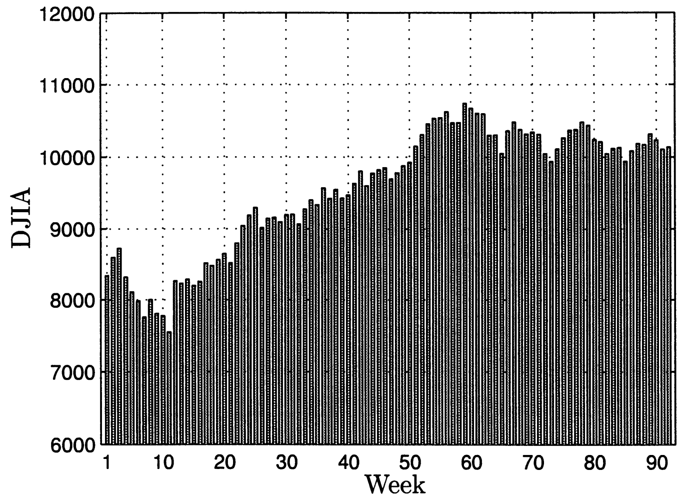

Figure 16.2: Dow-Jones industrial average at the end of each week from January 8, 2003 to September 29, 2004 [DowJones.com 2004].

description does not change with the chosen time origin, which is expressed mathematically by (16.3). An IID random process is stationary as shown in Example 16.3. The concept of a random process having stationary and independent increments is described in Section 16.5 with an illustration given in Example 16.5. Some more examples of random processes are given in Section 16.6. The most useful moments of a random process, the mean sequence and the covariance sequence, are defined by (16.5) and (16.7), respectively. Finally, in Section 16.8 an application of the estimation of the mean sequence to predicting average rainfall totals is described. The least squares estimator of the slope and intercept of a straight line is found using (16.9) and is commonly used in data analysis problems.

16.3 What Is a Random Process?

To define the concept of a random process we will begin by considering our usual example of a coin tossing experiment. Assume that at some start time we toss a coin and then repeat this subexperiment at one second intervals for all time. Letting n denote the time in seconds, we therefore generate successive outcomes at times $n = 0, 1, \ldots$. The experiment continues indefinitely. Since there are two possible outcomes for each coin toss and we will assume that the tosses are independent, we have an infinite sequence of Bernoulli trials. This is termed a *Bernoulli random process* and extends the finite Bernoulli set of random variables first introduced in Section 4.6.2, in which a finite number of trials were carried out. As usual, we let the probability of a head $(X = 1)$ be p and the probability of a tail ($X = 0$) be $1 - p$ for each trial. With this setup, a random process can be defined as a mapping from the original experimental sample space $\mathcal{S} = \{(H, H, T, \ldots), (H, T, H, \ldots), (T, T, H, \ldots), \ldots\}$ to the numerical sample space $\mathcal{S}_X = \{(1, 1, 0, \ldots), (1, 0, 1, \ldots), (0, 0, 1, \ldots), \ldots\}$. Note that each simple event or element of $\mathcal{S}$ is an infinite sequence of H's and T's which is then mapped into an infinite sequence of 1's and 0's, which is the corresponding simple event in $\mathcal{S}_X$. One may picture a random process as being generated by the "random process generator" shown in Figure 16.3. The random process is composed of the infinite (but

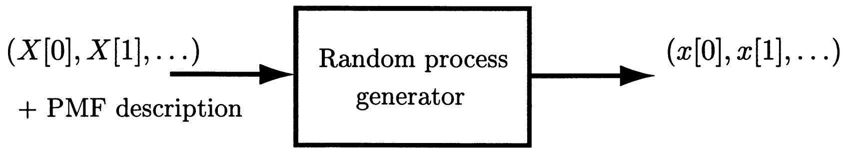

Figure 16.3: A conceptual random process generator. The input is an infinite sequence of random variables with their probabilistic description and the output is an infinite sequence of numbers.

countable) "vector" of random variables $(X[0], X[1], \ldots)$, each of which is a Bernoulli random variable, and each outcome of the random process is given by the infinite sequence of numerical values $(x[0], x[1], \ldots)$. As usual, uppercase letters are used for the random variables and lowercase letters for the values they take on. Some typical outcomes of the Bernoulli random process are shown in Figure 16.4. They were generated in MATLAB using `x=floor(rand(31,1)+0.5)` for each outcome. Each sequence in Figure 16.4 is called an *outcome* or by its synonyms of *realization* or *sample sequence*. We will prefer the use of the term "realization". Each realization is an infinite sequence of numbers. Hence, the random process is a mapping from $\mathcal{S}$, which is a set of infinite sequential experimental outcomes, to $\mathcal{S}_X$, which is a set of infinite sequences of 1's and 0's or realizations. The total number of realizations is not countable (see Problem 16.3). The set of all realizations is sometimes referred to as the *ensemble of realizations*. Just as for the case of a single random variable, which is a mapping from $\mathcal{S}$ to $\mathcal{S}_X$ and therefore is represented as the set function $X(s)$, a similar notation is used for random processes. Now, however, we will use $X[n, s]$ to represent the mapping from an element of $\mathcal{S}$ to a realization $x[n]$. In Figure 16.4 we see the result of the mapping for $s = s_1$, which is $X[n, s_1] = x_1[n]$, as well as others. It is important to note that if we fix n at $n = 18$, for example, then $X[18, s]$ is a *random variable* that has a Bernoulli PMF. Three of its outcomes are shown highlighted in Figure 16.4 with dashed boxes. Hence, all the methods developed for a single random variable are applicable. Likewise, if we fix two samples at $n = 20$ and $n = 22$, then $X[20, s]$ and $X[22, s]$ becomes a bivariate random vector. Again all our previous methods for two-dimensional random vectors apply.

To summarize, a random process is defined to be an infinite sequence of random

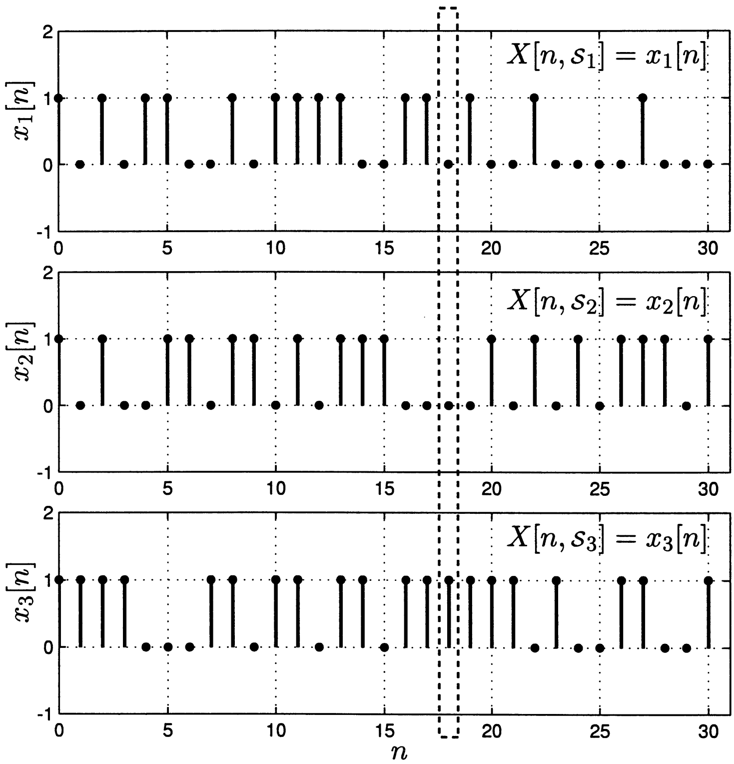

Figure 16.4: Typical outcomes of Bernoulli random process with $p = 0.5$. The realization starts at $n = 0$ and continues indefinitely. The dashed box indicates the realizations of the *random variable* $X[18, s]$.

variables $(X(0), X(1), \ldots)$, with one random variable for each time instant, and each realization of the random process takes on a value that is represented as an infinite sequence of numbers or $(x[0], x[1], \ldots)$. We will denote the random process more succinctly by $X[n]$ and the realization by $x[n]$ but *it is understood that the n denotes the values $n = 0, 1, \ldots$.* If we wish to indicate the random process at a *fixed time instant*, then we will use $n = n_0$ or $n = n_1$, etc. so that $X[n_0]$ is the random process at $n = n_0$ (which is just a random variable) and its realization at that time is $x[n_0]$ (which is a number). Finally, we have used the $[\cdot]$ notation to remind us that $X[n]$ is defined only for *discrete integer* times. This type of random process is known as a *discrete-time* random process. In the next section the *continuous-time* random process will be discussed. Before continuing, however, we look at a typical probability calculation for a random process.

Example 16.1 – Bernoulli random process

For the infinite coin tossing example, we might ask for the probability of the first

5 tosses coming up all heads. Thus, we wish to evaluate

$$P[X[0]=1, X[1]=1, X[2]=1, X[3]=1, X[4]=1, X[5]=0 \text{ or } 1, X[6]=0 \text{ or } 1, \ldots].$$

It would seem that since we don't care what the outcomes of $X[n]$ for $n = 5, 6, \ldots$ are, then the probability expression could be replaced by

$$P[X[0]=1, X[1]=1, X[2]=1, X[3]=1, X[4]=1]$$

and indeed this is the case, although it is not so easy to prove [Billingsley 1986]. Then, by using the assumption of independence of a Bernoulli random process we have

$$P[X[0]=1, X[1]=1, X[2]=1, X[3]=1, X[4]=1] = \prod_{n=0}^{4} P[X[n]=1] = p^5.$$

A related question is to determine the probability that we will *ever* observe 5 ones in a row. Intuitively, we expect this probability to be 1, but how do we prove this? It is not easy! Such is the difficulty encountered when we make the leap from a random vector, having a finite number of random variables, to a random process, having an infinite number of random variables.

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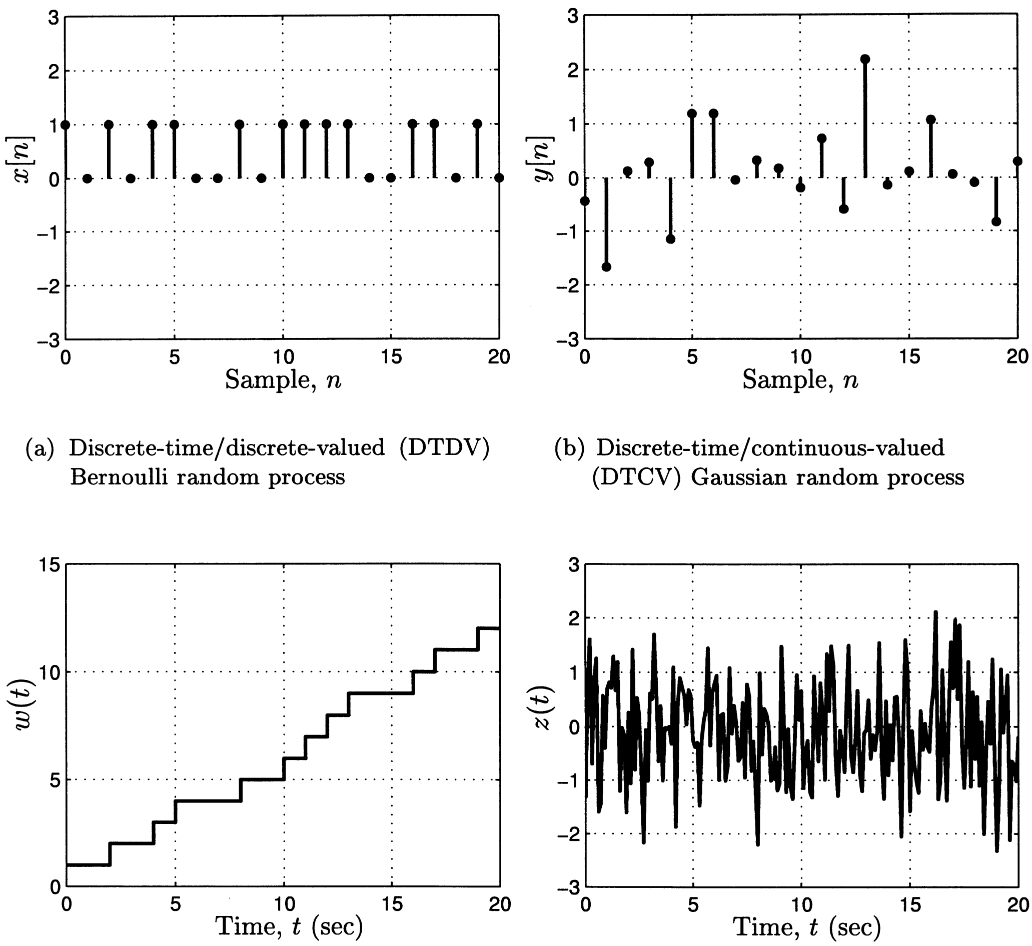
16.4 Types of Random Processes

The previous example of an infinite number of coin tosses produced a random process $X[n]$ for $n = 0, 1, \ldots$. In some cases, however, we wish to think of the random process as having started sometime in the infinite past. If $X[n]$ is defined for $n = \ldots, -1, 0, 1, \ldots$ or equivalently $-\infty < n < \infty$, where it is assumed that n is an integer, then $X[n]$ is called an *infinite* random process. In contrast, the previous example is referred to as a *semi-infinite* random process. Another categorization of random processes involves whether the times at which the random variables are defined and the values that they take on are either discrete or continuous. The infinite coin toss example is a *discrete-time* random process, since it is defined for $n = 0, 1, \ldots$, and is a *discrete-valued* random process, since it takes on values 0 and 1 only. It is referred to as a *discrete-time/discrete valued* (DTDV) random process. Other types of random processes are discrete-time/continuous-valued (DTCV), continuous-time/discrete-valued (CTDV), and continuous-time/continuous-valued (CTCV). A realization of each type is shown in Figure 16.5. In Figure 16.5a a realization of the Bernoulli random process, as previously described, is shown while in Figure 16.5b a realization of a Gaussian random process with $Y[n] \sim \mathcal{N}(0,1)$ is shown. The Bernoulli random process is defined for $n = 0, 1, \ldots$ (semi-infinite) while the Gaussian random process is defined for $-\infty < n < \infty$ and n an integer (infinite).



(a) Discrete-time/discrete-valued (DTDV) Bernoulli random process

(b) Discrete-time/continuous-valued (DTCV) Gaussian random process

(c) Continuous-time/discrete-valued (CTDV) binomial random process

(d) Continuous-time/continuous-valued (CTCV) Gaussian random process

Figure 16.5: Typical realizations of different types of random processes.

Both these random processes are discrete-time with the first one taking on only the values 0 and 1 and the second one taking on all real values. In Figure 16.5c is shown a random process, also known as a continuous-time binomial random process, which is defined as $W(t) = \sum_{n=0}^{[t]} X[n]$, where $X[n]$ is a Bernoulli random process and $[t]$ denotes the largest integer less than or equal to t. This process effectively counts the number of successes or ones of the Bernoulli random process (compare Figure 16.5c with Figure 16.5a). It is defined for all time; hence, it is a continuous-time random process, and it takes on only integer values in the range $\{0, 1, \ldots\}$; hence, it is discrete-valued. Finally, in Figure 16.5d is shown a realization of another

Gaussian process but with $Z(t) \sim \mathcal{N}(0,1)$ for all time t. This is a continuous-time random process that takes on all real values; hence, it is continuous-valued. We will generally use a discrete-time random process, with either discrete or continuous values, to introduce new concepts. This is because a continuous-time random process introduces a host of mathematical subtleties which in many cases are beyond the scope of this text. When possible, however, we will quote the analogous results for continuous-time random processes. Note finally that a realization of $X[n]$, which is $x[n]$, is also called a *sample sequence*, while a realization of $X(t)$, which is $x(t)$, is also called a *sample function*. We will, however, reserve the use of the word *sample* to refer to a *time sample* of the random process. Hence, a time sample will refer to either the random variable $X[n_0]$ ($X(t_0)$) or the realization $x[n_0]$ ($x(t_0)$) of the random process, with the meaning determined by the context of the discussion. We next revisit the random walk of Example 9.5.

Example 16.2 – Random walk (continued from Example 9.5)

Recall that

$$X_n = \sum_{i=1}^{n} U_i \qquad n = 1, 2, \ldots$$

where

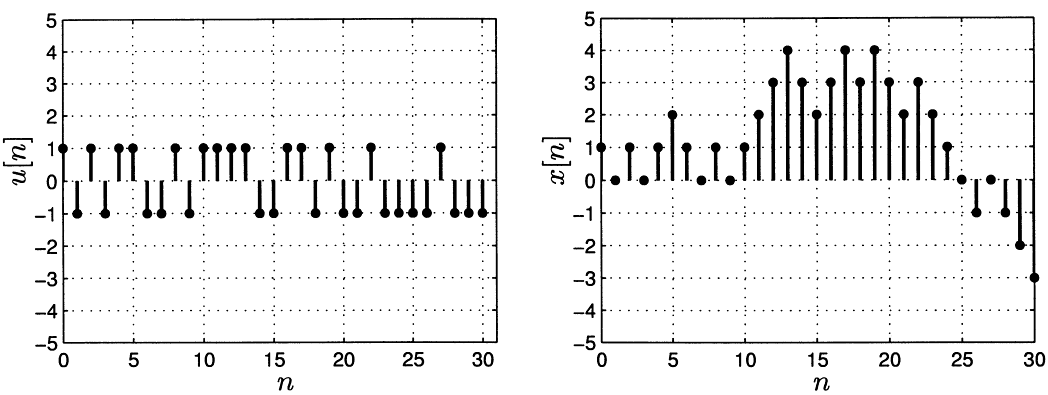
$$p_U[k] = \begin{cases} \frac{1}{2} & k = -1 \\ \frac{1}{2} & k = 1 \end{cases} \tag{16.1}$$

and the U_i's are IID. The random walk is a random process so that rewriting the definition in our new notation, we have

$$X[n] = \sum_{i=0}^{n} U[i] \qquad n = 0, 1, \ldots$$

where the $U[i]$'s are IID random variables having the PMF of (16.1). We also assume that the random walk starts at time $n = 0$. The $U[i]$'s comprise the random variables of a Bernoulli random process but with values of ± 1, instead of the usual 0 and 1. As such, we can view the $U[i]$'s as comprising a Bernoulli random process $U[n]$ for $n = 0, 1, \ldots$. Realizations of $U[n]$ and $X[n]$ are shown in Figure 16.6. One question that comes to mind is the behavior of the random walk for large n. For example, we might be interested in the PDF of $X[n]$ for large n. Relying on the central limit theorem (see Chapter 15), we can assert that the PDF is Gaussian, and therefore we need only determine the mean and variance. This easily follows from the definition of the random walk as

$$\begin{aligned} E[X[n]] &= \sum_{i=0}^{n} E[U[i]] = (n+1)E[U[0]] = 0 \\ \operatorname{var}(X[n]) &= \sum_{i=0}^{n} \operatorname{var}(U[i]) = (n+1)\operatorname{var}(U[0]) = n+1 \end{aligned}$$



(a) Realization of Bernoulli random process $U[n]$

(b) Realization of random walk $X[n]$

Figure 16.6: Typical realization of a random walk.

since $E[U[i]] = 0$ and $\text{var}(U[i]) = 1$. (Note that since the $U[i]$'s are identically distributed, they all have the same mean and variance. We have arbitrarily chosen $U[0]$ in the expression for the mean and variance of a single sample.) Hence, for large n we have approximately that $X[n] \sim \mathcal{N}(0, n+1)$. Does this appear to explain the behavior of $x[n]$ shown in Figure 16.6b?

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16.5 The Important Property of Stationarity

The simplest type of random process is an IID random process. The Bernoulli random process is an example of this. Each random variable $X[n_0]$ is independent of all the others and each random variable has the same marginal PMF. As such, the joint PMF of any finite number of samples can immediately be written as

$$p_{X[n_1],X[n_2],\ldots,X[n_N]}[x_1, x_2, \ldots, x_N] = \prod_{i=1}^{N} p_{X[n_i]}[x_i] \tag{16.2}$$

and used for probability calculations. For example, for a Bernoulli random process with values $0, 1$ the probability of the first 10 samples being $1, 0, 1, 0, 1, 0, 1, 0, 1, 0$ is $p^5(1-p)^5$. Note that we are able to specify the joint PMF for *any* finite number of sample times. This is sometimes referred to as being able to specify the *finite dimensional distribution* (FDD). It is the most complete probabilistic description that we can manage for a random process and reduces the analysis of a random process to the analysis of a *finite* but arbitrary set of random variables.

A generalization of the IID random process is a random process for which the FDD does not change with the time origin. This is to say that the PMF or PDF of the samples $\{X[n_1], X[n_2], \ldots, X[n_N]\}$ is the same as for $\{X[n_1+n_0], X[n_2+n_0], \ldots, X[n_N+n_0]\}$, where n_0 is an arbitrary integer. Alternatively, the set of samples can be shifted in time, with each one being shifted the same amount, without affecting the joint PMF or joint PDF. Mathematically, for the FDD not to change with the time origin, we must have that

$$p_{X[n_1+n_0],X[n_2+n_0],\ldots,X[n_N+n_0]} = p_{X[n_1],X[n_2],\ldots,X[n_N]} \tag{16.3}$$

for all n_0, and for any arbitrary choice of N and $n_1, n_2, \ldots, n_N$. Such a random process is said to be *stationary*. It is implicit from (16.3) that all joint and marginal PMFs or PDFs must have probabilities that do not depend on the time origin. For example, by letting $N = 1$ in (16.3) we have that $p_{X[n_1+n_0]} = p_{X[n_1]}$ and setting $n_1 = 0$, we have that $p_{X[n_0]} = p_{X[0]}$ for all n_0. This says that the marginal PMF or PDF is the same for every sample in a stationary random process. We next prove that an IID random process is stationary.

Example 16.3 – IID random process is stationary.

To prove that the IID random process is a special case of a stationary random process we must show that (16.3) is satisfied. This follows from

$$\begin{aligned}
p_{X[n_1+n_0],X[n_2+n_0],\ldots,X[n_N+n_0]} &= \prod_{i=1}^{N} p_{X[n_i+n_0]} && \text{(by independence)} \\
&= \prod_{i=1}^{N} p_{X[n_i]} && \text{(by identically distributed)} \\
&= p_{X[n_1],X[n_2],\ldots,X[n_N]} && \text{(by independence).}
\end{aligned}$$

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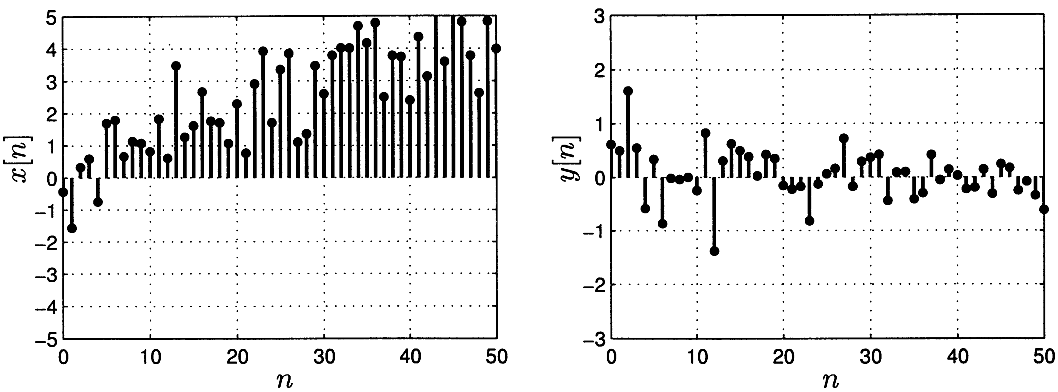
If a random process is stationary, then all its joint moments and more generally all expected values of functions of the random process, must also be stationary since

$$E_{X[n_1+n_0],\ldots,X[n_N+n_0]}[\cdot] = E_{X[n_1],\ldots,X[n_N]}[\cdot]$$

which follows from (16.3). Examples then of random processes that are not stationary are ones whose means and/or variances change in time, which implies that the marginal PMF or PDF change with time. In Figure 16.7 we show typical realizations of random processes whose mean in Figure 16.7a and whose variance in Figure 16.7b change with time. They were generated using the MATLAB code:

```
randn('state',0)
N=51;
x=randn(N,1)+0.1*[0:N-1]'; % for Figure 16.7a
y=sqrt(0.95.^[0:50]').*randn(N,1); % for Figure 16.7b
```



(a) Mean increasing with n

(b) Variance decreasing with n

Figure 16.7: Random processes that are not stationary.

In Figure 16.7a the true mean increases linearly from 0 to 5 while in Figure 16.7b the variance decreases exponentially as 0.95^n. It is clear then that the samples all have different moments and therefore $p_{X[n_1+n_0]} \neq p_{X[n_1]}$ which violates the condition for stationarity.

It is impossible to determine if a random process is stationary from a single realization.

A realization of a random process is a *single outcome* of the random process. This is analogous to observing a single outcome of a coin toss. We cannot determine if the coin is fair by observing that the outcome was a head. What is required are multiple realizations of the coin tossing experiment. So it is with random processes. In Figure 16.7b, although we generated the realization using a variance that decreased with time, and hence the random process is not stationary, the realization shown *could have been generated with a constant variance.* Then, the values of the realization near $n = 50$ just happen to be smaller than the ones near $n = 0$, which is possible, although maybe not very probable. To better discern whether a random process is stationary we require *multiple realizations.*

Another example of a random process that is not stationary follows.

Example 16.4 – Sum random process

A sum random process is a slight generalization of the random walk process of

Example 16.2. As before, $X[n] = \sum_{i=0}^{n} U[i]$, where the $U[i]$'s are IID but for the general sum process, the $U[i]$'s can have any, although the same, PMF or PDF. Thus, the sum random process is not stationary since

$$\begin{aligned} E[X[n]] &= (n+1)E_U[U[0]] \\ \text{var}(X[n]) &= (n+1)\text{var}(U[0]) \end{aligned}$$

both of which change with n. Hence, it violates the condition for stationarity.

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A random process that is not stationary is said to be *nonstationary*. In light of the fact that an IID random process lends itself to simple probability calculations, it is advantageous, if possible, to transform a nonstationary random process into a stationary one (see Problem 16.12 on transforming the random processes of Figure 16.7 into stationary ones). As an example, for the sum random process this can be done by "reversing" the summing operation. Specifically, we difference the random process. Then $X[n] - X[n-1] = U[n]$ for $n \geq 0$, where we define $X[-1] = 0$. This is an IID random process. The differences or *increment* random variables $U[n]$ are independent and identically distributed. More generally, for the sum random process any two *increments* of the form

$$\begin{aligned} X[n_2] - X[n_1] &= \sum_{i=n_1+1}^{n_2} U[i] \\ X[n_4] - X[n_3] &= \sum_{i=n_3+1}^{n_4} U[i] \end{aligned}$$

are independent if $n_4 > n_3 \geq n_2 > n_1$. Thus, nonoverlapping increments for a sum random process are independent. (Recall that functions of independent random variables are themselves independent.) If furthermore, $n_4 - n_3 = n_2 - n_1$, then they also have the same PMF or PDF since they are composed of the same number of IID random variables. It is then said that for the sum random process, the increments are independent and stationary (equivalent to being identically distributed) or that it has *stationary independent increments*. The reader may wish to ponder whether a random process can have independent but nonstationary increments (see Problem 16.13). Many random processes (an example of which follows) that we will encounter have this property and it allows us to more easily analyze the probabilistic behavior.

Example 16.5 – Binomial counting random process

Consider the repeated coin tossing experiment where we are interested in the number of heads that occurs. Letting $U[n]$ be a Bernoulli random process with $U[n] = 1$ with probability p and $U[n] = 0$ with probability $1-p$, the number of heads is given by the *binomial counting* or sum process

$$X[n] = \sum_{i=0}^{n} U[i] \qquad n = 0, 1, \ldots$$

or equivalently

$$X[n] = \begin{cases} U[0] & n = 0 \\ X[n-1] + U[n] & n \geq 1 . \end{cases}$$

A typical realization is shown in Figure 16.8. The random process has stationary

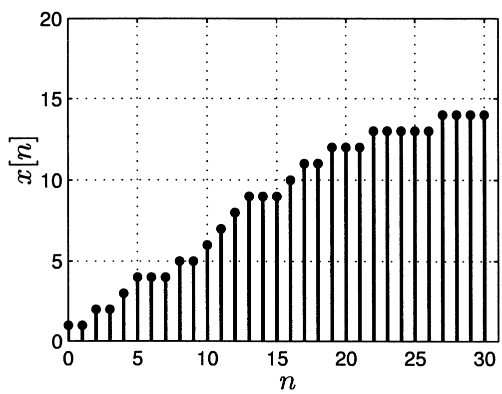

Figure 16.8: Typical realization of binomial counting random process with $p = 0.5$.

and independent increments since the changes over two nonoverlapping intervals are composed of different sets of identically distributed $U[i]$'s. We can use this property to more easily determine probabilities of events. For example, to determine $p_{X[1],X[2]}[1,2] = P[X[1] = 1, X[2] = 2]$, we can note that the event $X[1] = 1, X[2] = 2$ is equivalent to the event $Y_1 = X[1] - X[-1] = 1$, $Y_2 = X[2] - X[1] = 1$, where $X[-1]$ is defined to be identically zero. But Y_1 and Y_2 are nonoverlapping increments (but of unequal length), making them independent random variables. Thus,

$$\begin{aligned} P[X[1] = 1, X[2] = 2] &= P[Y_1 = 1, Y_2 = 1] = P[Y_1 = 1]P[Y_2 = 1] \\ &= P[\underbrace{U[0] + U[1]}_{\text{bin}(2,p)} = 1]P[U[2] = 1] \\ &= \binom{2}{1} p^1 (1-p)^1 \cdot p \\ &= 2p^2(1-p). \end{aligned}$$

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16.6 Some More Examples

We continue our discussion by examining some random processes of practical interest.

Example 16.6 – White Gaussian noise

A common model for physical noise, such as resistor noise due to electron motion fluctuations in an electric field, is termed *white Gaussian noise* (WGN). It is assumed that the noise has been sampled in time to yield a DTCV random process $X[n]$. The WGN random process is defined to be an IID one whose marginal PDF is Gaussian so that $X[n] \sim \mathcal{N}(0, \sigma^2)$ for $-\infty < n < \infty$. Each random variable $X[n_0]$ has a mean of zero, consistent with our notion of a noise process, and the same variance or because the mean is zero, the same power $E[X^2[n_0]]$. A typical realization is shown in Figure 16.5b for $\sigma^2 = 1$. The WGN random process is stationary since it is an IID random process. Its joint PDF is

$$\begin{aligned} p_{X[n_1],X[n_2],\ldots,X[n_N]}(x_1, x_2, \ldots, x_N) &= \prod_{i=1}^{N} p_{X[n_i]}(x_i) \\ &= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} x_i^2\right) \\ &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{N} x_i^2\right). \qquad (16.4) \end{aligned}$$

Note that the joint PDF is $\mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$, which is a special form of the multivariate Gaussian PDF (see Problem 16.15). The terminology of "white" derives from the property that such a random process may be synthesized from a sum of different frequency random sinusoids each having the same power, much the same as white light is composed of equal contributions of each visible wavelength of light. We will justify this property in Chapter 17 when we discuss the power spectral density.

◇

Example 16.7 – Moving average random process

The moving average (MA) random process is a DTCV random process defined as

$$X[n] = \tfrac{1}{2}(U[n] + U[n-1]) \qquad -\infty < n < \infty$$

where $U[n]$ is a WGN random process with variance σ_U^2. (To avoid confusion with the variance of other random variables we will sometimes use a subscript on σ^2, in this case σ_U^2, to refer to the variance of the $U[n_0]$ random variable.) The terminology of moving average refers to the averaging of the current random variable $U[n]$ with the previous random variable $U[n-1]$ to form the current moving average random variable. Also, this averaging "moves" in time, as for example,

$$\begin{aligned} X[0] &= \tfrac{1}{2}(U[0] + U[-1]) \\ X[1] &= \tfrac{1}{2}(U[1] + U[0]) \\ X[2] &= \tfrac{1}{2}(U[2] + U[1]) \\ &\text{etc.} \end{aligned}$$

A typical realization of $X[n]$ is shown in Figure 16.9 and should be compared to the realization of $U[n]$ shown in Figure 16.5b. It is seen that the moving average random process is "smoother" than the WGN random process, from which it was obtained. Further smoothing is possible by averaging more WGN samples together (see Problem 16.17). The MATLAB code shown below was used to generate the realization.

```
randn('state',0)
u=randn(21,1);
for i=1:21
  if i==1
    x(i,1)=0.5*(u(1)+randn(1,1)); % needed to initialize sequence
  else
    x(i,1)=0.5*(u(i)+u(i-1));
  end
end
```

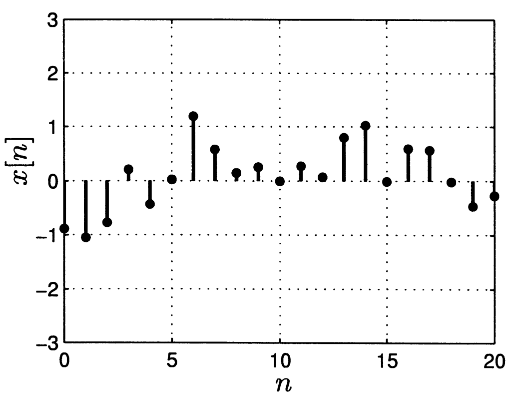

Figure 16.9: Typical realization of moving average random process. The realization of the $U[n]$ random process is shown in Figure 16.5b.

The joint PDF of $X[n]$ can be determined by observing that it is a linearly transformed version of $U[n]$. As an example, to determine the joint PDF of the random vector $[X[0]\, X[1]]^T$, we have from the definition of the MA random process

$$\begin{bmatrix} X[0] \\ X[1] \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} U[-1] \\ U[0] \\ U[1] \end{bmatrix}$$

or in matrix/vector notation $\mathbf{X} = \mathbf{G}\mathbf{U}$. Now recalling that $\mathbf{U}$ is a Gaussian random vector (see (16.4)) and that a linear transformation of a Gaussian random vector produces another Gaussian random vector, we have from Example 14.3 that

$$\mathbf{X} \sim \mathcal{N}(\mathbf{G}E[\mathbf{U}], \mathbf{G}\mathbf{C}_U\mathbf{G}^T).$$

Explicitly, since each sample of $U[n]$ is zero mean with variance σ_U^2 and all samples are independent, we have that $E[\mathbf{U}] = \mathbf{0}$ and $\mathbf{C}_U = \sigma_U^2\mathbf{I}$. This results in

$$\mathbf{X} = \begin{bmatrix} X[0] \\ X[1] \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \sigma_U^2\mathbf{G}\mathbf{G}^T)$$

where

$$\mathbf{G}\mathbf{G}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

It can furthermore be shown that the MA random process is stationary (see Example 20.2 and Property 20.2).

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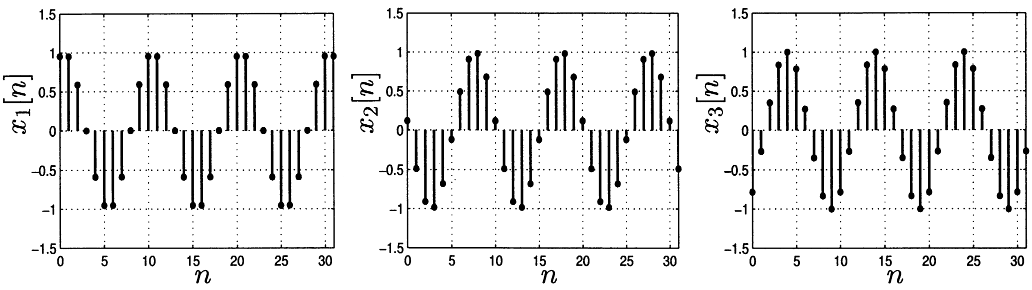
Example 16.8 – Randomly phased sinusoid (or sine wave)

Consider the DTCV random process given as

$$X[n] = \cos(2\pi(0.1)n + \Theta) \qquad -\infty < n < \infty$$

where $\Theta \sim \mathcal{U}(0, 2\pi)$. Some typical realizations are shown in Figure 16.10. The MATLAB statements `n=[0:31]'` and `x=cos(2*pi*0.1*n+2*pi*rand(1,1))` can be used to generate each realization. This random process is frequently used to model an analog sinusoid whose phase is unknown and that has been sampled by an analog-to-digital convertor. It is nearly a deterministic signal, except for the phase uncertainty, and is therefore perfectly predictable. This is to say that once we observe two successive samples, then all the remaining ones are known (see Problem 16.20). This is in contrast to the WGN random process, for which regardless of how many samples we observe, we cannot predict any of the remaining ones due to the independence of the samples. Because of the predictability of the randomly phased sinusoidal process, the joint PDF can only be represented using impulsive functions. As an example, you might try to find the PDF of (X, Y) if (X, Y) has the bivariate Gaussian



(a) $\theta = 5.9698$ (b) $\theta = 1.4523$ (c) $\theta = 3.8129$

Figure 16.10: Typical realizations for randomly phased sinusoid.

PDF with $\rho = 1$. We will not pursue this further. However, we can determine the marginal PDF $p_{X[n]}$. To do so we use the transformation formula of (10.30), where the Y random variable is $X[n_0]$ (considering the random process at a fixed time) and the X random variable is Θ. The transformation is shown in Figure 16.11 for $n_0 = 0$. Note that there are two solutions for any given $x[n_0] = y$ (except for the

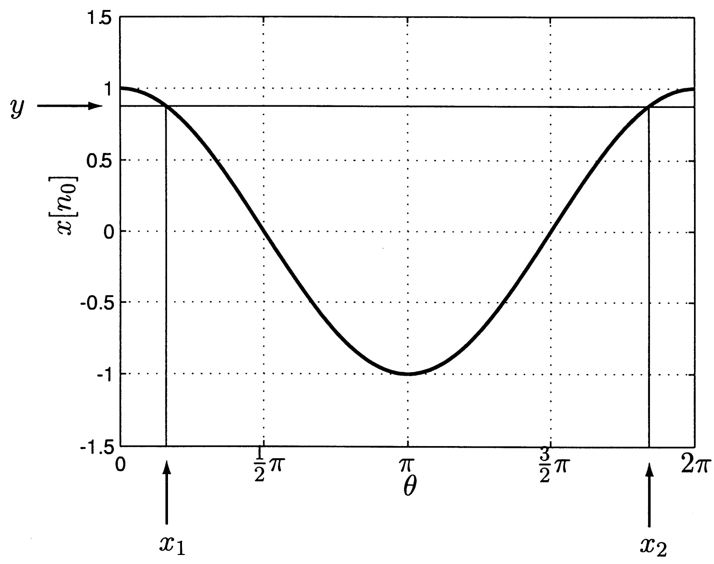


Figure 16.11: Function transforming Θ into $X[n_0]$ for the value $n_0 = 0$, where $X[n_0] = \cos(2\pi(0.1)n_0 + \Theta)$.

point at $\theta = \pi$, which has probability zero). We denote the solutions as $\theta = x_1, x_2$. Using our previous notation of $y = g(x)$ for a transformation of a single random variable we have that

$$y = \cos(2\pi(0.1)n_0 + x)$$

so that the solutions are

$$\begin{aligned} x_1 &= \arccos(y) - 2\pi(0.1)n_0 = g_1^{-1}(y) \\ x_2 &= 2\pi - [\arccos(y) - 2\pi(0.1)n_0] = g_2^{-1}(y) \end{aligned}$$

for $-1 < y < 1$ and thus $0 < \arccos(y) < \pi$. Using $d\arccos(y)/dy = 1/\sqrt{1-y^2}$, we have

$$\begin{aligned} p_Y(y) &= p_X(g_1^{-1}(y)) \left| \frac{dg_1^{-1}(y)}{dy} \right| + p_X(g_2^{-1}(y)) \left| \frac{dg_2^{-1}(y)}{dy} \right| \\ &= \frac{1}{2\pi} \left| \frac{1}{\sqrt{1-y^2}} \right| + \frac{1}{2\pi} \left| -\frac{1}{\sqrt{1-y^2}} \right| \\ &= \frac{1}{\pi\sqrt{1-y^2}}. \end{aligned}$$

Finally, in our original notation we have the marginal PDF for $X[n]$ for any n

$$p_{X[n]}(x) = \begin{cases} \frac{1}{\pi\sqrt{1-x^2}} & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

This PDF is shown in Figure 16.12. Note that the values of $X[n]$ that are most probable are near $x = \pm 1$. Can you explain why? (Hint: Determine the values of θ for which $0.9 < \cos\theta < 1$ and also $0 < \cos\theta < 0.1$ in Figure 16.11.)

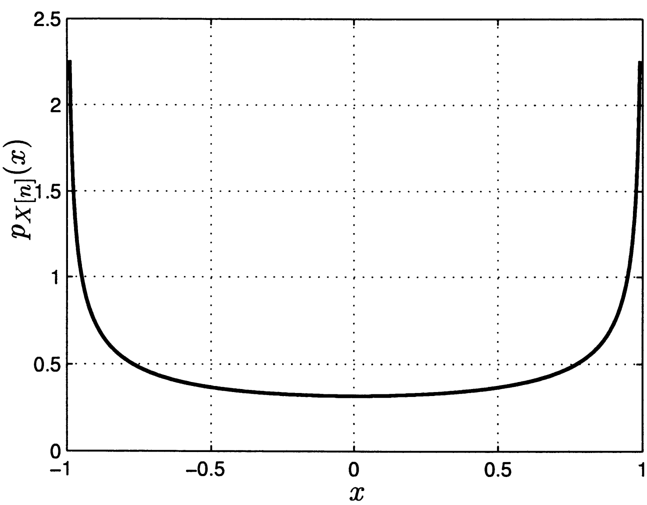


Figure 16.12: Marginal PDF for randomly phased sinusoid.

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16.7 Joint Moments

The first and second moments or equivalently the mean and variance of a random process at a given sample time are of great practical importance since they are easily determined. Also, the covariance between two samples of the random process at two different times is easily found. At worst, the first and second moments can always be estimated in practice. This is in contrast to the joint PMF or joint PDF, which in practice may be difficult to determine. Hence, we next define and give some examples of the mean, variance, and covariance sequences for a DTCV random process. The mean sequence is defined as

$$\mu_X[n] = E[X[n]] \qquad -\infty < n < \infty \tag{16.5}$$

while the variance sequence is defined as

$$\sigma_X^2[n] = \mathrm{var}(X[n]) \qquad -\infty < n < \infty \tag{16.6}$$

and finally the covariance sequence is defined as

$$\begin{aligned} c_X[n_1, n_2] &= \mathrm{cov}(X[n_1], X[n_2]) \\ &= E[(X[n_1] - \mu_X[n_1])(X[n_2] - \mu_X[n_2])] \qquad \begin{array}{c} -\infty < n_1 < \infty \\ -\infty < n_2 < \infty \, . \end{array} \end{aligned} \tag{16.7}$$

The expectations for the mean and variance are taken with respect to the PMF or PDF $p_{X[n]}$ for a particular value of n. Similarly, the expectation needed for the evaluation of the covariance is with respect to the joint PMF or PDF $p_{X[n_1],X[n_2]}$ for particular values of n_1 and n_2. Since the required PMF or PDF should be clear from the context, we henceforth do not subscript the expectation operator as we have done so previously. Note that the usual symmetry property of the covariance holds, which results in $c_X[n_2, n_1] = c_X[n_1, n_2]$. Also, it follows from the definition of the covariance sequence that $c_X[n, n] = \sigma_X^2[n]$. The actual evaluation of the moments proceeds exactly the same as for random variables.

If the random process is a continuous-time one, then the corresponding definitions are

$$\begin{aligned} \mu_X(t) &= E[X(t)] \\ \sigma_X^2(t) &= \mathrm{var}(X(t)) \\ c_X(t_1, t_2) &= E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))]. \end{aligned}$$

These are called the *mean function*, *variance function*, and *covariance function*, respectively. We next examine the moments for the examples of the previous section. Noting that the variance is just the covariance sequence evaluated at $n_1 = n_2 = n$, we need only determine the mean and covariance sequences.

Example 16.9 – White Gaussian noise
Since $X[n] \sim \mathcal{N}(0, \sigma^2)$ for all n, we have that

$$\begin{aligned} \mu_X[n] &= 0 \qquad -\infty < n < \infty \\ \sigma_X^2[n] &= \sigma^2 \qquad -\infty < n < \infty. \end{aligned}$$

The covariance sequence for $n_1 \neq n_2$ must be zero since the random variables are all independent. Recalling that the covariance between $X[n]$ and itself is just the variance, we have that

$$c_X[n_1, n_2] = \begin{cases} 0 & n_1 \neq n_2 \\ \sigma^2 & n_1 = n_2 \,. \end{cases}$$

This can be written in more succinct form by using the discrete delta function as

$$c_X[n_1, n_2] = \sigma^2 \delta[n_2 - n_1].$$

In summary, for a WGN random process we have that $\mu_X[n] = 0$ for all n and $c_X[n_1, n_2] = \sigma^2 \delta[n_2 - n_1]$.

◇

Example 16.10 – Moving average random process
The mean sequence is

$$\mu_X[n] = E[X[n]] = E[\tfrac{1}{2}(U[n] + U[n-1])] = 0 \qquad -\infty < n < \infty$$

since $U[n]$ is white Gaussian noise, which has a zero mean for all n. To find the covariance sequence using $X[n] = (U[n] + U[n-1])/2$, we have

$$\begin{aligned} c_X[n_1, n_2] &= E[(X[n_1] - \mu_X[n_1])(X[n_2] - \mu_X[n_2])] \\ &= E[X[n_1]X[n_2]] \\ &= \frac{1}{4} E[(U[n_1] + U[n_1 - 1])(U[n_2] + U[n_2 - 1])] \\ &= \frac{1}{4}\left(E[U[n_1]U[n_2]] + E[U[n_1]U[n_2 - 1]]\right. \\ &\quad \left. + E[U[n_1 - 1]U[n_2]] + E[U[n_1 - 1]U[n_2 - 1]]\right). \end{aligned}$$

But $E[U[k]U[l]] = \sigma_U^2 \delta[l - k]$ since $U[n]$ is WGN, and as a result

$$\begin{aligned} c_X[n_1, n_2] &= \frac{1}{4}\left(\sigma_U^2 \delta[n_2 - n_1] + \sigma_U^2 \delta[n_2 - 1 - n_1] + \sigma_U^2 \delta[n_2 - n_1 + 1] + \sigma_U^2 \delta[n_2 - n_1]\right) \\ &= \frac{\sigma_U^2}{2} \delta[n_2 - n_1] + \frac{\sigma_U^2}{4} \delta[n_2 - n_1 - 1] + \frac{\sigma_U^2}{4} \delta[n_2 - n_1 + 1]. \end{aligned}$$

This is plotted in Figure 16.13 versus $\Delta n = n_2 - n_1$. It is seen that the covariance sequence is zero unless the two samples are at most one unit apart or $\Delta n = n_2 - n_1 =$

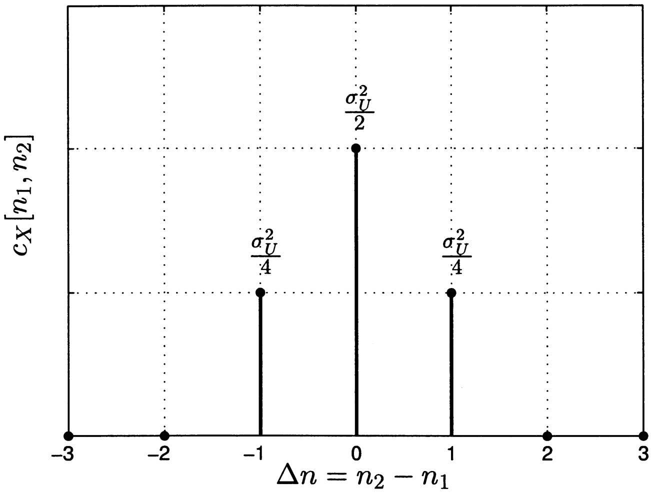

Figure 16.13: Covariance sequence for moving average random process.

± 1. Note that the covariance between any two samples spaced one unit apart is the same. Thus, for example, $X[1]$ and $X[2]$ have the covariance $c_X[1,2] = \sigma_U^2/4$, as do $X[9]$ and $X[10]$ since $c_X[9,10] = \sigma_U^2/4$, and as do $X[-3]$ and $X[-2]$ since $c_X[-3,-2] = \sigma_U^2/4$ (see Figure 16.13). Any samples that are spaced *more than* one unit apart are *uncorrelated.* This is because for $|n_2 - n_1| > 1$, $X[n_1]$ and $X[n_2]$ are independent, being composed of two sets of *different* WGN samples (recall that functions of independent random variables are independent). In summary, we have that

$$\begin{aligned} \mu_X[n] &= 0 \\ c_X[n_1, n_2] &= \begin{cases} \frac{\sigma_U^2}{2} & n_1 = n_2 \\ \frac{\sigma_U^2}{4} & |n_2 - n_1| = 1 \\ 0 & |n_2 - n_1| > 1 \,. \end{cases} \end{aligned}$$

and the variance is $c_X[n,n] = \sigma_U^2/2$ for all n. Also, note from Figure 16.13 that the covariance sequence is symmetric about $\Delta n = 0$.

◇

Example 16.11 – Randomly phased sinusoid

Recalling that the phase is uniformly distributed on $(0, 2\pi)$ we have that the mean sequence is

$$\begin{aligned}\mu_X[n] &= E[X[n]] = E[\cos(2\pi(0.1)n + \Theta)]\\ &= \int_0^{2\pi} \cos(2\pi(0.1)n + \theta)\frac{1}{2\pi}d\theta \qquad \text{(use (11.10))}\\ &= \frac{1}{2\pi}\sin(2\pi(0.1)n+\theta)\Big|_0^{2\pi} = 0\end{aligned}$$

for all n. Noting that the mean sequence is zero, the covariance sequence becomes

$$\begin{aligned}c_X[n_1,n_2] &= E[X[n_1]X[n_2]]\\ &= \int_0^{2\pi} [\cos(2\pi(0.1)n_1 + \theta)\cos(2\pi(0.1)n_2+\theta)]\frac{1}{2\pi}d\theta\\ &= \int_0^{2\pi}\left[\frac{1}{2}\cos[2\pi(0.1)(n_2-n_1)] + \frac{1}{2}\cos[2\pi(0.1)(n_1+n_2)+2\theta]\right]\frac{1}{2\pi}d\theta\\ &= \frac{1}{2}\cos[2\pi(0.1)(n_2-n_1)] + \frac{1}{8\pi}\sin[2\pi(0.1)(n_1+n_2)+2\theta]\Big|_0^{2\pi}\\ &= \frac{1}{2}\cos[2\pi(0.1)(n_2-n_1)].\end{aligned}$$

Once again the covariance sequence depends only on the spacing between the two

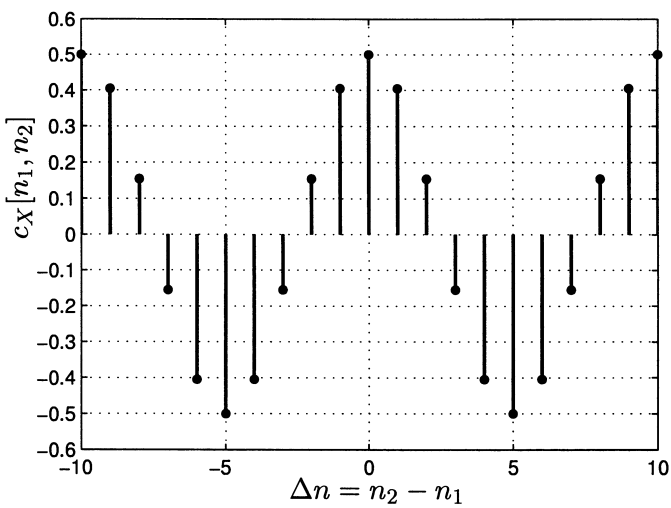

Figure 16.14: Covariance sequence for randomly phased sinusoid.

samples or on $n_2 - n_1$. The covariance sequence is shown in Figure 16.14. The reader should note the symmetry of the covariance sequence about $\Delta n = 0$. Also, the variance follows as $\sigma_X^2[n] = c_X[n,n] = 1/2$ for all n. It is interesting to observe that in this example the fact that the mean sequence is zero makes intuitive sense.

To see this we have plotted 50 realizations of the random process in an overlaid fashion in Figure 16.15. This representation is called a *scatter diagram.* Also is

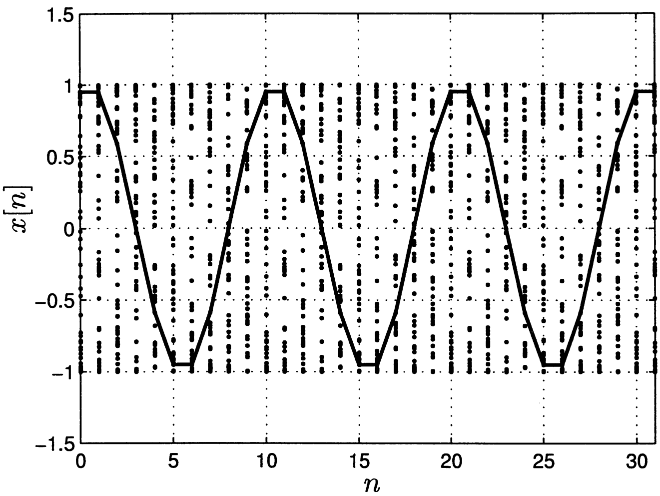

Figure 16.15: Fifty realizations of randomly phased sinusoid plotted in an overlaid format with one realization shown with its points connected by straight lines.

plotted the first realization with the values connected by straight lines for easier viewing. The difference in the realizations is due to the different values of phase realized. It is seen that for a given time instant the values are nearly symmetric about zero, as is predicted by the PDF shown in Figure 16.12 and that the majority of the values are near ± 1, again in agreement with the PDF. The MATLAB code used to generate Figure 16.15 (but omitting the solid curve) is given below.

```
clear all
rand('state',0)
n=[0:31]';
nreal=50;
for i=1:nreal
   x(:,i)=cos(2*pi*0.1*n+2*pi*rand(1,1));
end
plot(n,x(:,1),'.')
grid
hold on
for i=2:nreal
   plot(n,x(:,i),'.')
end
axis([0 31 -1.5 1.5])
```

$\diamond$

In these three examples the covariance sequence only depends on $|n_2 - n_1|$. This is not always the case, as is illustrated in Problem 16.26. Also, another counterexample is the random process whose realization is shown in Figure 16.7b. This random process has $\text{var}(X[n]) = c_X[n,n]$ which is not a function of $n_2 - n_1 = n - n = 0$ since otherwise its variance would be a constant for all n.

16.8 Real-World Example – Statistical Data Analysis

It was mentioned in the introduction that some meterologists argue that the annual summer rainfall totals are increasing due to global warming. Referring to Figure 16.1 this supposition asserts that if $X[n]$ is the annual summer rainfall total for year n, then $\mu_X[n_2] > \mu_X[n_1]$ for $n_2 > n_1$. One way to attempt to confirm or dispute this supposition is to assume that $\mu_X[n] = an + b$ and then determine if $a > 0$, as would be the case if the mean were increasing. From the data shown in Figure 16.1 we can estimate a. To do so we let the year 1895, which is the beginning of our data set, be indexed as $n = 0$ and note that $an + b$ when plotted versus n is a straight line. We estimate a by fitting a straight line to the data set using a *least squares* procedure [Kay 1993]. The least squares estimate chooses as estimates of a and b the values that minimize the *least squares error*

$$J(a,b) = \sum_{n=0}^{N-1} (x[n] - (an+b))^2 \tag{16.8}$$

where $N = 108$ for our data set. This approach can be shown to be an optimal one under the condition that the random process is actually given by $X[n] = an + b + U[n]$, where $U[n]$ is a WGN random process [Kay 1993]. Note that if we did not suspect that the mean rainfall totals were changing, then we might assume that $\mu_X[n] = b$ and the least squares estimate of b would result from minimizing

$$J(b) = \sum_{n=0}^{N-1} (x[n] - b)^2.$$

If we differentiate $J(b)$ with respect to b, set the derivative equal to zero, and solve for b, we obtain (see Problem 16.32)

$$\hat{b} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

or $\hat{b} = \bar{x}$, where $\bar{x}$ is the sample mean, which for our data set is 9.76. Now, however, we obtain the least squares estimates of a and b by differentiating (16.8) with respect to b and a to yield

$$\begin{aligned}\frac{\partial J}{\partial b} &= -2\sum_{n=0}^{N-1}(x[n]-an-b)=0 \\ \frac{\partial J}{\partial a} &= -2\sum_{n=0}^{N-1}(x[n]-an-b)n=0.\end{aligned}$$

This results in two simultaneous linear equations

$$\begin{aligned}bN + a\sum_{n=0}^{N-1} n &= \sum_{n=0}^{N-1} x[n] \\ b\sum_{n=0}^{N-1} n + a\sum_{n=0}^{N-1} n^2 &= \sum_{n=0}^{N-1} nx[n].\end{aligned}$$

In vector/matrix form this is

$$\begin{bmatrix} N & \sum_{n=0}^{N-1} n \\ \sum_{n=0}^{N-1} n & \sum_{n=0}^{N-1} n^2 \end{bmatrix}\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{N-1} x[n] \\ \sum_{n=0}^{N-1} nx[n] \end{bmatrix} \tag{16.9}$$

which is easily solved to yield the estimates $\hat{b}$ and $\hat{a}$. For the data of Figure 16.1 the estimates are $\hat{a} = 0.0173$ and $\hat{b} = 8.8336$. The data along with the estimated mean sequence $\hat{\mu}_X[n] = 0.0173n + 8.8336$ are shown in Figure 16.16. Note that the

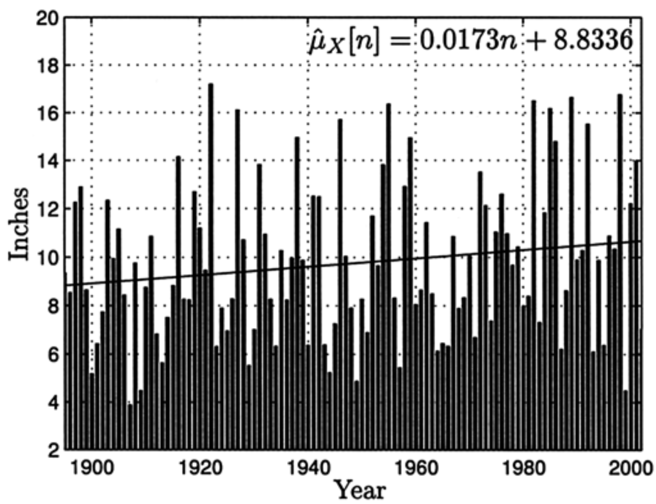

Figure 16.16: Annual summer rainfall in Rhode Island and the estimated mean sequence, $\hat{\mu}_X[n] = 0.0173n + 8.8336$, where $n = 0$ corresponds to the year 1895.

mean indeed appears to be increasing with time. The least squares error sequence, which is defined as $e[n] = x[n] - (\hat{a}n + \hat{b})$, is shown in Figure 16.17. It is sometimes referred to as the *fitting error*. Note that the error can be quite large. In fact, we

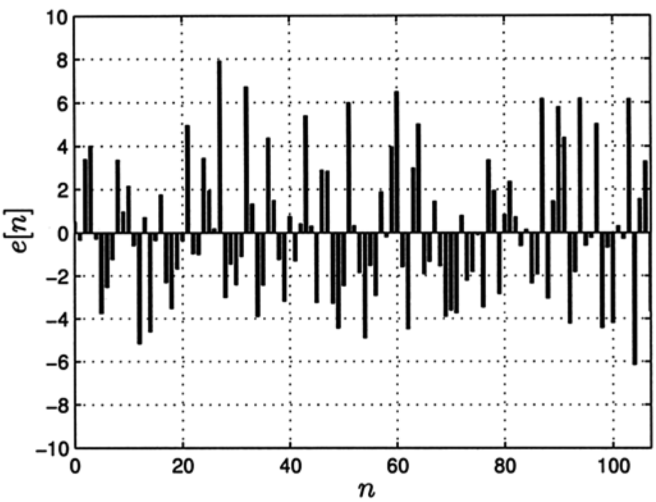

Figure 16.17: Least squares error sequence for annual summer rainfall in Rhode Island fitted with a straight line.

have that $(1/N)\sum_{n=0}^{N-1} e^2[n] = 10.05$.

Now the real question is whether the estimated mean increase in rainfall is significant. The increase is $\hat{a} = 0.0173$ *per year* for a total increase of about 1.85 inches over the course of 108 years. Is it possible that the true mean rainfall has not changed, or that it is really $\mu_X[n] = b$ with the true value of a being zero? In effect, is the value of $\hat{a} = 0.0173$ only due to estimation error? One way to answer this question is to hypothesize that $a = 0$ and then determine the probability density function of $\hat{a}$ as obtained from (16.9). This can be done analytically by assuming $X[n] = b + U[n]$, where $U[n]$ is white Gaussian noise (see Problem 16.33). However, we can gain some quick insight into the problem by resorting to a computer simulation. To do so we assume that the true model for the rainfall data is $X[n] = b + U[n] = 9.76 + U[n]$, where $U[n]$ is white Gaussian noise with variance σ^2. Since we do not know the value of σ^2, we estimate it by using the results shown in Figure 16.17. The least squares error sequence $e[n]$, which is the original data with its estimated mean sequence subtracted, should then be an estimate of $U[n]$. Therefore, we use $\hat{\sigma^2} = (1/N)\sum_{n=0}^{N-1} e^2[n] = 10.05$ in our simulation. In summary, we generate 20 realizations of the random process $X[n] = 9.76 + U[n]$, where $U[n]$ is WGN with $\sigma^2 = 10.05$. Then, we use (16.9) to estimate a and b and finally we plot our mean sequence estimate, which is $\hat{\mu}_X[n] = \hat{a}n + \hat{b}$ for each realization. Using the MATLAB code shown at the end of this section, the results are shown in Figure 16.18. It is seen that even though the true value of a is zero, the estimated value will take on nonzero values with a high probability. Since some of the lines are decreasing, some

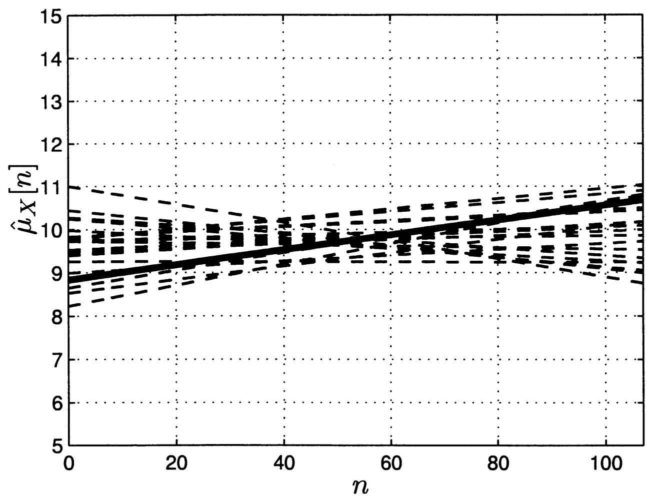

Figure 16.18: Twenty realizations of the estimated mean sequence $\hat{\mu}_X[n] = \hat{a}n + \hat{b}$ based on the random process $X[n] = 9.76 + U[n]$ with $U[n]$ being WGN with $\sigma^2 = 10.05$. The realizations are shown as dashed lines. The estimated mean sequence from Figure 16.16 is shown as the solid line.

of the estimated values of a are even negative. Hence, we would be hard pressed to say that the mean rainfall totals are indeed increasing. Such is the quandry that scientists must deal with on an everyday basis. The only way out of this dilemma is to accumulate more data so that hopefully our estimate of a will be more accurate (see also Problem 16.34).

```
clear all
randn('state',0)
years=[1895:2002]';
N=length(years);
n=[0:N-1]';
A=[N sum(n);sum(n) sum(n.^2)]; % precompute matrix (see (16.9))
B=inv(A); % invert matrix
for i=1:20
  xn=9.76+sqrt(10.05)*randn(N,1); % generate realizations
  baest=B*[sum(xn);sum(n.*xn)]; % estimate a and b using (16.9)
  aest=baest(2);best=baest(1);
  meanest(:,i)=aest*n+best; % determine mean sequence estimate
end
figure % plot mean sequence estimates and overlay
plot(n,meanest(:,1))
grid
xlabel('n')
```

```
ylabel('Estimated mean')
axis([0 107 5 15])
hold on
for i=2:20
  plot(n,meanest(:,i))
end
```

References

Billingsley, P., *Probability and Measure*, John Wiley & Sons, New York, 1986.

DowJones.com, "DowJones Averages," http://averages.dowjones.com/jsp/uiHistoricalIndexRep.jsp, 2004.

Kay, S., *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice-Hall, Englewood Cliffs, NJ, 1993.

Problems

16.1 (⌣) **(w)** Describe a random process that you are likely to encounter in the following situations:

a. listening to the daily weather forecast

b. paying the monthly telephone bill

c. leaving for work in the morning

Why is each process a random one?

16.2 (w) A single die is tossed repeatedly. What are $\mathcal{S}$ and $\mathcal{S}_X$? Also, can you determine the joint PMF for any N sample times?

16.3 (t) An infinite sequence of 0's and 1's, denoted as $b_1, b_2, \ldots$, can be used to represent any number x in the interval $[0,1]$ using the binary representation formula

$$x = \sum_{i=1}^{\infty} b_i 2^{-i}.$$

For example, we can represent 3/4 as $0.b_1 b_2 \ldots = 0.11000\ldots$ and 1/16 as $0.b_1 b_2 \ldots = 0.0001000\ldots$. Find the representations for 7/8 and 5/8. Is the total number of infinite sequences of 0's and 1's countable?

16.4 (⌣) **(w)** For a Bernoulli random process determine the probability that we will observe an alternating sequence of 1's and 0's for the first 100 samples with the first sample being a 1. What is the probability that we will observe an alternating sequence of 1's and 0's for all n?

16.5 (w) Classify the following random processes as either DTDV, DTCV, CTDV, or CTCV:

a. temperature in Rhode Island

b. outcomes for continued spins of a roulette wheel

c. daily weight of person

d. number of cars stopped at an intersection

16.6 (c) Simulate a realization of the random walk process described in Example 16.2 on a computer. What happens as n becomes large?

16.7 (⌣) (c,f) A *biased* random walk process is defined as $X[n] = \sum_{i=0}^{n} U[i]$, where $U[i]$ is a Bernoulli random process with

$$p_U[k] = \begin{cases} \frac{1}{4} & k = -1 \\ \frac{3}{4} & k = 1 . \end{cases}$$

What is $E[X[n]]$ and $\text{var}(X[n])$ as a function of n? Next, simulate on a computer a realization of this random process. What happens as $n \to \infty$ and why?

16.8 (w) A random process $X[n]$ is stationary. If it is known that $E[X[10]] = 10$ and $\text{var}(X[10]) = 1$, then determine $E[X[100]]$ and $\text{var}(X[100])$.

16.9 (⌣) (f) The IID random process $X[n]$ has the marginal PDF $p_X(x) = \exp(-x)u(x)$. What is the probability that $X[0], X[1], X[2]$ will all be greater than 1?

16.10 (w) If an IID random process $X[n]$ is transformed to the random process $Y[n] = X^2[n]$, is the transformed random process also IID?

16.11 (w) A Bernoulli random process $X[n]$ that takes on values 0 or 1, each with probability of $p = 1/2$, is transformed using $Y[n] = (-1)^n X[n]$. Is the random process $Y[n]$ IID?

16.12 (w,f) A nonstationary random process is defined as $X[n] = a^{|n|}U[n]$, where $0 < a < 1$ and $U[n]$ is WGN with variance σ_U^2. Find the mean and covariance sequences of $X[n]$. Can you transform the $X[n]$ random process to make it stationary?

16.13 (⌣) (w) Consider the random process $X[n] = \sum_{i=0}^{n} U[i]$, which is defined for $n \geq 0$. The $U[n]$ random process consists of independent Gaussian random variables with marginal PDF $U[n] \sim \mathcal{N}(0, (1/2)^n)$. Are the increments independent? Are the increments stationary?

16.14 (c) Plot 50 realizations of a WGN random process $X[n]$ with $\sigma^2 = 1$ for $n = 0, 1, \ldots, 49$ using a scatter diagram (see Figure 16.15 for an example). Use the MATLAB commands `plot(x,y,'.')` and `hold on` to plot each realization as dots and to overlay the realizations on the same graph, respectively. For a fixed n can you explain the observed distribution of the dots?

16.15 (f) Prove that

$$\frac{1}{(2\pi)^{N/2} \det^{1/2}(\mathbf{C})} \exp\left(-\tfrac{1}{2}\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x}\right)$$

where $\mathbf{x} = [x_1\, x_2 \ldots x_N]^T$ and $\mathbf{C} = \sigma^2\mathbf{I}$ for $\mathbf{I}$ an $N \times N$ identity matrix, reduces to (16.4).

16.16 (⌣) (f) A "white" uniform random process is defined to be an IID random process with $X[n] \sim \mathcal{U}(-\sqrt{3}, \sqrt{3})$ for all n. Determine the mean and covariance sequences for this random process and compare them to those of the WGN random process. Explain your results.

16.17 (w) A moving average random process can be defined more generally as one for which N samples of WGN are averaged, instead of only $N = 2$ samples as in Example 16.7. It is given by $X[n] = (1/N)\sum_{i=0}^{N-1} U[n-i]$ for all n, where $U[n]$ is a WGN random process with variance σ_U^2. Determine the correlation coefficient for $X[0]$ and $X[1]$. What happens as N increases?

16.18 (⌣) (f) For the moving average random process defined in Example 16.7 determine $P[X[n] > 3]$ and compare it to $P[U[n] > 3]$. Explain the difference in terms of "smoothing". Assume that $\sigma_U^2 = 1$.

16.19 (c) For the randomly phased sinusoid defined in Example 16.8 determine the mean sequence using a computer simulation.

16.20 (t) For the randomly phased sinusoid of Example 16.8 assume that the realization $x[n] = \cos(2\pi(0.1)n + 0)$ is generated. Prove that if we observe only the samples $x[0] = 1$ and $x[1] = \cos(2\pi(0.1)) = 0.8090$, then all the future samples can be found by using the recursive formula $x[n] = 2\cos(2\pi(0.1))x[n-1] - x[n-2]$ for $n \geq 2$. Could you also find the past samples or $x[n]$ for $n \leq -1$? See also Problem 18.25 for prediction of a sinusoidal random process.

16.21 (c) Verify the PDF of the randomly phased sinusoid given in Figure 16.12 by using a computer simulation.

16.22 (⌣) (f,c) A continuous-time random process known as the random *amplitude* sinusoid is defined as $X(t) = A\cos(2\pi t)$ for $-\infty < t < \infty$ and $A \sim \mathcal{N}(0, 1)$. Find the mean and covariance functions. Then, plot some realizations of $X(t)$ in an overlaid fashion.

16.23 (f) A random process is the sum of WGN and a deterministic sinusoid and is given as $X[n] = U[n] + \sin(2\pi f_0 n)$ for all n, where $U[n]$ is WGN with variance σ_U^2. Determine the mean and covariance sequences.

16.24 (⌣) (w) A random process is IID with samples $X[n] \sim \mathcal{N}(\mu, 1)$. It is desired to remove the mean of the random process by forming the new random process $Y[n] = X[n] - X[n-1]$. First determine the mean sequence of $Y[n]$. Next find $\text{cov}(Y[0], Y[1])$. Is $Y[n]$ an IID random process with a zero mean sequence?

16.25 (f) If a random process is defined as $X[n] = h[0]U[n] + h[1]U[n-1]$, where $h[0]$ and $h[1]$ are constants and $U[n]$ is WGN with variance σ_U^2, find the covariance for $X[0]$ and $X[1]$. Repeat for $X[9]$ and $X[10]$. How do they compare?

16.26 (⌣) (f) If a sum random process is defined as $X[n] = \sum_{i=0}^{n} U[i]$ for $n \geq 0$, where $E[U[i]] = 0$ and $\text{var}(U[i]) = \sigma_U^2$ for $i \geq 0$ and the $U[i]$ are IID, find the mean and covariance sequences of $X[n]$.

16.27 (⌣) (c) For the MA random process defined in Example 16.7 find $c_X[1,1]$, $c_X[1,2]$ and $c_X[1,3]$ if $\sigma_U^2 = 1$. Next simulate on a computer $M = 10{,}000$ realizations of the random process $X[n]$ for $n = 0, 1, \ldots, 10$. Estimate the previous covariance sequence samples using $\hat{c}_X[n_1, n_2] = (1/M)\sum_{i=1}^{M} x_i[n_1]x_i[n_2]$, where $x_i[n]$ is the ith realization of $X[n]$. Note that since $X[n]$ is zero mean, $c_X[n_1, n_2] = E[X[n_1]X[n_2]]$.

16.28 (w) For the randomly phased sinusoid described in Example 16.11 determine the minimum mean square estimate of $X[10]$ based on observing $x[0]$. How accurate do you think this prediction will be?

16.29 (f) For a random process $X[n]$ the mean sequence $\mu_X[n]$ and covariance sequence $c_X[n_1, n_2]$ are known. It is desired to predict k samples into the future. If $x[n_0]$ is observed, find the minimum mean square estimate of $X[n_0 + k]$. Next assume that $\mu_X[n] = \cos(2\pi f_0 n)$ and $c_X[n_1, n_2] = 0.9^{|n_2 - n_1|}$ and evaluate the estimate. Finally, what happens to your prediction as $k \to \infty$ and why?

16.30 (f) A random process is defined as $X[n] = As[n]$ for all n, where $A \sim \mathcal{N}(0, 1)$ and $s[n]$ is a deterministic signal. Find the mean and covariance sequences.

16.31 (⌣) (f) A random process is defined as $X[n] = AU[n]$ for all n, where $A \sim \mathcal{N}(0, \sigma_A^2)$ and $U[n]$ is WGN with variance σ_U^2, and A is independent of $U[n]$ for all n. Find the mean and covariance sequences. What type of random process is $X[n]$?

16.32 (f) Verify that by differentiating $\sum_{n=0}^{N-1}(x[n] - b)^2$ with respect to b, setting the derivative equal to zero, and solving for b, we obtain the sample mean.

16.33 (t) In this problem we show how to obtain the variance of $\hat{a}$ as obtained by solving (16.9). The variance of $\hat{a}$ is derived under the assumption that $X[n] = b + U[n]$, where $U[n]$ is WGN with variance σ^2. This says that we assume the true value of a is zero. The steps are as follows:

a. Let

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & N-1 \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}$$

where $\mathbf{H}$ is an $N \times 2$ matrix and $\mathbf{X}$ is an $N \times 1$ random vector. Now show that that the equations of (16.9) can be written as

$$\mathbf{H}^T\mathbf{H} \begin{bmatrix} b \\ a \end{bmatrix} = \mathbf{H}^T\mathbf{X}.$$

b. The solution for b and a can now be written symbolically as

$$\begin{bmatrix} \hat{b} \\ \hat{a} \end{bmatrix} = \underbrace{(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T}_{\mathbf{G}}\mathbf{X}$$

Since $\mathbf{X}$ is a Gaussian random vector, show that $[\hat{b}\,\hat{a}]^T$ is also a Gaussian random vector with mean $[b\,0]^T$ and covariance matrix $\sigma^2(\mathbf{H}^T\mathbf{H})^{-1}$.

c. As a result we can assert that the marginal PDF of $\hat{a}$ is Gaussian with mean zero and variance equal to the $(2,2)$ element of $\sigma^2(\mathbf{H}^T\mathbf{H})^{-1}$. Show then that $\hat{a} \sim \mathcal{N}(0, \text{var}(\hat{a}))$, where

$$\text{var}(\hat{a}) = \frac{\sigma^2}{\sum_{n=0}^{N-1} n^2 - \frac{1}{N}(\sum_{n=0}^{N-1} n)^2}.$$

Next assume that $\sigma^2 = 10.05$, $N = 108$ and find the probability that $\hat{a} > 0.0173$. Can we assert that the estimated mean sequence shown in Figure 16.16 is not just due to estimation error?

16.34 (⌣̇̇) (f) Using the results of Problem 16.33 determine the required value of N so that the probability that $\hat{a} > 0.0173$ is less than 10^{-6}.