Chapter 20

Gaussian Random Processes

20.1 Introduction

There are several types of random processes that have found wide application because of their realistic physical modeling yet relative mathematical simplicity. In this and the next two chapters we describe these important random processes. They are the Gaussian random process, the subject of this chapter; the Poisson random process, described in Chapter 21; and the Markov chain, described in Chapter 22. Concentrating now on the Gaussian random process, we will see that it has many important properties. These properties have been inherited from those of the N-dimensional Gaussian PDF, which was discussed in Section 14.3. Specifically, the important characteristics of a Gaussian random process are:

1. It is physically motivated by the central limit theorem (see Chapter 15).

2. It is a mathematically tractable model.

3. The joint PDF of any set of samples is a multivariate Gaussian PDF, which enjoys many useful properties (see Chapter 14).

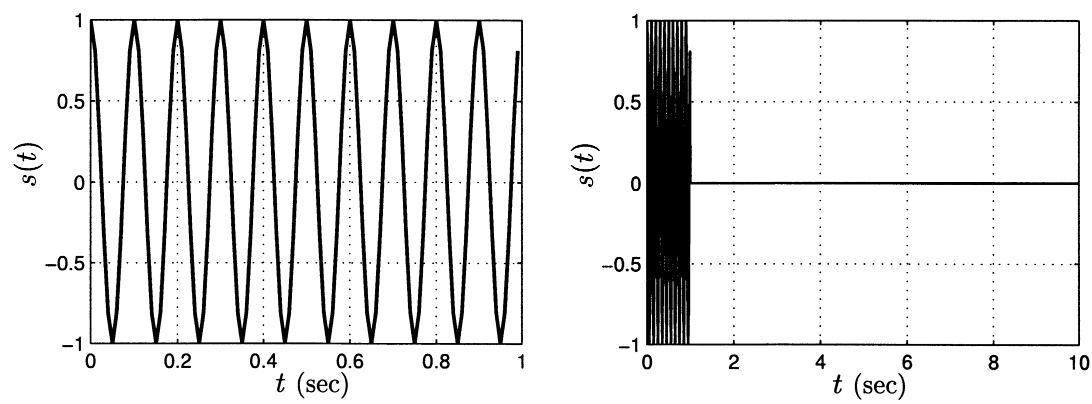
4. Only the first two moments, the mean sequence and the covariance sequence, are required to completely describe it. As a result,

   a. In practice the joint PDF can be estimated by estimating only the first two moments.

   b. If the Gaussian random process is wide sense stationary, then it is also stationary.

5. The processing of a Gaussian random process by a linear filter does not alter its Gaussian nature, but only modifies the first two moments. The modified moments are easily found.

In effect, the Gaussian random process has so many useful properties that it is always the first model to be proposed in the solution of a problem. It finds application as a model for electronic noise [Bell Labs 1970], ambient ocean noise [Urick 1975], scattering phenomena such as reverberation of sound in the ocean or electromagnetic clutter in the atmosphere [Van Trees 1971], and financial time series [Taylor 1986], just to name a few. Any time a random process can be modeled as due to the sum of a large number of independent and similar type effects, a Gaussian random process results due to the central limit theorem. One example that we will explore in detail is the use of the scattering of a sound pulse from a school of fish to determine their numbers (see Section 20.9). In this case, the received waveform is the sum of a large number of scattered pulses that have been added together. The addition occurs because the leading edge of a pulse that is reflected from a fish farther away will coincide in time with the trailing edge of the pulse that is reflected from a fish that is nearer (see Figure 20.14). If the fish are about the same size and type, then the *average intensity* of the returned echos will be relatively constant. However, the echo amplitudes will be different due to the different reflection characteristics of each fish, i.e., its exact position, orientation, and motion will all determine how the incoming pulse is scattered. These characteristics cannot be predicted in advance and so the amplitudes are modeled as random variables. When overlapped in time, these random echos are well modeled by a Gaussian random process. As an example, consider a transmitted pulse $s(t) = \cos(2\pi F_0 t)$, where $F_0 = 10$ Hz, over the time interval $0 \leq t \leq 1$ second as shown in Figure 20.1. Assuming a single reflection



(a) Transmit pulse

(b) Transmit pulse shown in receive waveform observation window

Figure 20.1: Transmitted sinusoidal pulse.

for every 0.1 second interval with the starting time being a uniformly distributed random variable within the interval and an amplitude A that is a random variable with $A \sim \mathcal{U}(0,1)$ to account for the unknown reflection coefficient of each fish, a typical received waveform is shown in Figure 20.2. If we now estimate the marginal

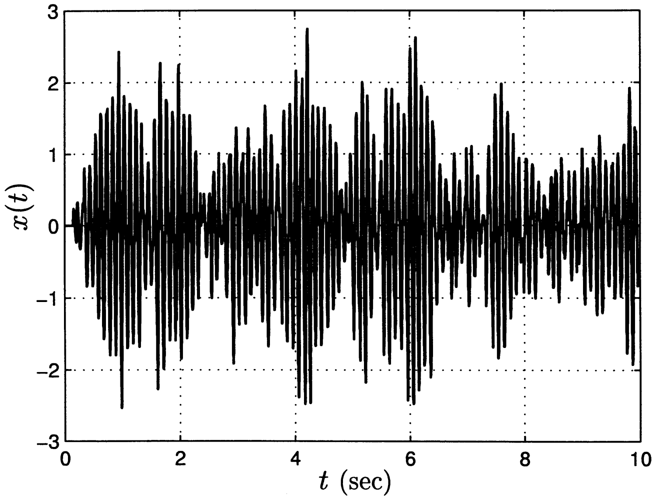


Figure 20.2: Received waveform consisting of many randomly overlapped and random amplitude echos.

PDF for $x(t)$ as shown in Figure 20.2 by assuming that each sample has the same marginal PDF, we have the estimated PDF shown in Figure 20.3 (see Section 10.9 on how to estimate the PDF). Also shown is the Gaussian PDF with its mean and variance estimated from uniformly spaced samples of $x(t)$. It is seen that the Gaussian PDF is very accurate as we would expect from the central limit theorem. The MATLAB code used to generate Figure 20.2 is given in Appendix 20A. In Section 20.3 we formally define the Gaussian random process.

20.2 Summary

Section 20.1 gives an example of why the Gaussian random process arises quite frequently in practice. The discrete-time Gaussian random process is defined in Section 20.3 as one whose samples comprise a Gaussian random vector as characterized by the PDF of (20.1). Also, some examples are given and are shown to exhibit two important properties, which are summarized in that section. Any linear transformation of a Gaussian random process produces another Gaussian random process. In particular for a discrete-time WSS Gaussian random process that is filtered by an LSI filter, the output random process is Gaussian with PDF given in Theorem 20.4.1. A nonlinear transformation does not maintain the Gaussian random process but its effect can be found in terms of the output moments using (20.12). An example of a squaring operation on a discrete-time WSS Gaussian random process produces an output random process that is still WSS with moments

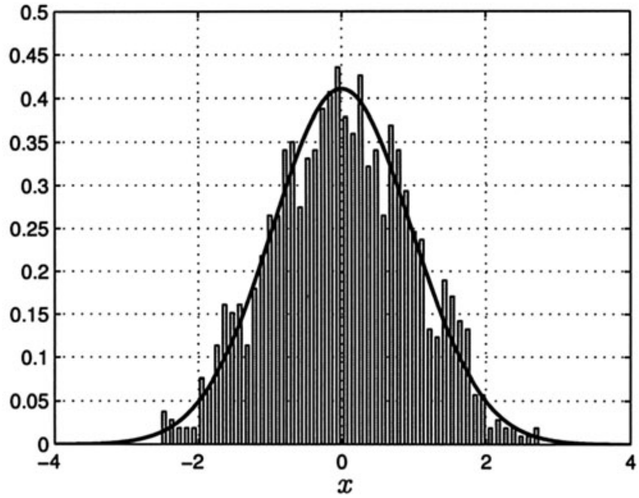


Figure 20.3: Marginal PDF of samples of received waveform shown in Figure 20.2 and Gaussian PDF fit.

given by (20.14). A continuous-time Gaussian random process is defined in Section 20.6 and examples are given. An important one is the Wiener random process examined in Example 20.7. Its covariance matrix is found using (20.16). Some special continuous-time Gaussian random processes are described in Section 20.7. The Rayleigh fading sinusoid is described in Section 20.7.1. It has the ACF given by (20.17) and corresponding PSD given by (20.18). A continuous-time bandpass Gaussian random process is described in Section 20.7.2. It has an ACF given by (20.21) and a corresponding PSD given by (20.22). The important example of bandpass "white" Gaussian noise is discussed in Example 20.8. The computer generation of a discrete-time WSS Gaussian random process realization is described in Section 20.8. Finally, an application of the theory to estimating fish populations using a sonar is the subject of Section 20.9.

20.3 Definition of the Gaussian Random Process

We will consider here the *discrete-time* Gaussian random process, an example of which was given in Figure 16.5b as the discrete-time/continuous-valued (DTCV) random process. The continuous-time/continuous-valued (CTCV) Gaussian random process, an example of which was given in Figure 16.5d, will be discussed in Section 20.6. Before defining the Gaussian random process we briefly review the N-dimensional multivariate Gaussian PDF as described in Section 14.3. An $N \times 1$ random vector $\mathbf{X} = [X_1\, X_2 \ldots X_N]^T$ is defined to be a Gaussian random vector if its joint PDF is given by the multivariate Gaussian PDF

$$p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{N/2}\det^{1/2}(\mathbf{C})}\exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right] \tag{20.1}$$

where $\boldsymbol{\mu} = [\mu_1\,\mu_2\ldots\mu_N]^T$ is the mean vector defined as

$$\boldsymbol{\mu} = E_{\mathbf{X}}[\mathbf{X}] = \begin{bmatrix} E_{X_1}[X_1] \\ E_{X_2}[X_2] \\ \vdots \\ E_{X_N}[X_N] \end{bmatrix} \tag{20.2}$$

and $\mathbf{C}$ is the $N \times N$ covariance matrix defined as

$$\mathbf{C} = \begin{bmatrix} \mathrm{var}(X_1) & \mathrm{cov}(X_1,X_2) & \ldots & \mathrm{cov}(X_1,X_N) \\ \mathrm{cov}(X_2,X_1) & \mathrm{var}(X_2) & \ldots & \mathrm{cov}(X_2,X_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}(X_N,X_1) & \mathrm{cov}(X_N,X_2) & \ldots & \mathrm{var}(X_N) \end{bmatrix}. \tag{20.3}$$

In shorthand notation $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{C})$. The important properties of a Gaussian random vector are:

1. Only the first two moments $\boldsymbol{\mu}$ and $\mathbf{C}$ are required to specify the entire PDF.

2. If all the random variables are uncorrelated so that $[\mathbf{C}]_{ij} = 0$ for $i \neq j$, then they are also independent.

3. A linear transformation of $\mathbf{X}$ produces another Gaussian random vector. Specifically, if $\mathbf{Y} = \mathbf{G}\mathbf{X}$, where $\mathbf{G}$ is an $M \times N$ matrix with $M \leq N$, then $\mathbf{Y} \sim \mathcal{N}(\mathbf{G}\boldsymbol{\mu}, \mathbf{G}\mathbf{C}\mathbf{G}^T)$.

Now we consider a discrete-time random process $X[n]$, where $n \geq 0$ for a semi-infinite random process and $-\infty < n < \infty$ for an infinite random process. *The random process is defined to be a Gaussian random process if all finite sets of samples have a multivariate Gaussian PDF as per (20.1).* Mathematically, if $\mathbf{X} = [X[n_1]\,X[n_2]\ldots X[n_K]]^T$ has a multivariate Gaussian PDF (given in (20.1) with N replaced by K) for all $\{n_1, n_2, \ldots, n_K\}$ and all K, then $X[n]$ is said to be a Gaussian random process. Some examples follow.

Example 20.1 – White Gaussian noise

White Gaussian noise was first introduced in Example 16.6. We revisit that example in light of our formal definition of a Gaussian random process. First recall that discrete-time white noise is a WSS random process $X[n]$ for which $E[X[n]] = \mu = 0$ for $-\infty < n < \infty$ and $r_X[k] = \sigma^2\delta[k]$. This says that all the samples are zero mean, uncorrelated with each other, and have the same variance σ^2. If we now furthermore assume that the samples are also *independent* and each sample has a *Gaussian*

PDF, then $X[n]$ is a Gaussian random process. It is referred to as *white Gaussian noise* (WGN). To verify this we need to show that any set of samples has a multivariate Gaussian PDF. Let $\mathbf{X} = [X[n_1]\, X[n_2] \dots X[n_K]]^T$ and note that the joint K-dimensional PDF is the product of the marginal PDFs due to the independence assumption. Also, each marginal PDF is $X[n_i] \sim \mathcal{N}(0, \sigma^2)$ by assumption. This produces the joint PDF

$$\begin{aligned} p_{\mathbf{X}}(\mathbf{x}) &= \prod_{i=1}^{K} p_{X[n_i]}(x[n_i]) \\ &= \prod_{i=1}^{K} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} x^2[n_i]\right) \\ &= \frac{1}{(2\pi\sigma^2)^{K/2}} \exp\left(-\frac{1}{2\sigma^2}\mathbf{x}^T\mathbf{x}\right) \\ &= \frac{1}{(2\pi)^{K/2} \det^{1/2}(\sigma^2\mathbf{I})} \exp\left(-\tfrac{1}{2}\mathbf{x}^T(\sigma^2\mathbf{I})^{-1}\mathbf{x}\right) \end{aligned}$$

or $\mathbf{X} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$, where $\mathbf{I}$ is the $K \times K$ identity matrix. Note also that since WGN is an IID random process, it is also stationary (see Example 16.3).

◇

Example 20.2 – Moving average random process

Consider the MA random process $X[n] = (U[n] + U[n-1])/2$, where $U[n]$ is WGN with variance σ_U^2. Then, $X[n]$ is a Gaussian random process. This is because $U[n]$ is a Gaussian random process (from previous example) and $X[n]$ is just a linear transformation of $U[n]$. For instance, if $K = 2$, and $n_1 = 0$, $n_2 = 1$, then

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \end{bmatrix}}_{\mathbf{X}} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} U[-1] \\ U[0] \\ U[1] \end{bmatrix}}_{\mathbf{U}}$$

and thus $\mathbf{X} \sim \mathcal{N}(\mathbf{0}, \mathbf{G}\mathbf{C}_U\mathbf{G}^T) = \mathcal{N}(\mathbf{0}, \sigma_U^2\mathbf{G}\mathbf{G}^T)$. The same argument applies to any number of samples K and any samples times $n_1, n_2, \dots, n_K$. Note here that the MA random process is also stationary. If we were to change the two samples to $n_1 = n_0$ and $n_2 = n_0 + 1$, then

$$\begin{bmatrix} X[n_0] \\ X[n_0+1] \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} U[n_0-1] \\ U[n_0] \\ U[n_0+1] \end{bmatrix}$$

and the joint PDF will be the same since the $\mathbf{U}$ vector has the same PDF. Again this result remains the same for any number of samples and sample times. We will see shortly that a Gaussian random process that is WSS is also stationary. Here, the $U[n]$ random process is WSS and hence $X[n]$ is WSS, being the output of an LSI filter (see Theorem 18.3.1).

As a typical probability calculation let $\sigma_U^2 = 1$ and determine $P[X[1] - X[0] > 1]$. We would expect this to be less than $P[U[1] - U[0] > 1] = Q(1/\sqrt{2})$ (since $U[1] - U[0] \sim \mathcal{N}(0, 2)$) due to the smoothing effect of the filter ($\mathcal{H}(z) = \frac{1}{2} + \frac{1}{2}z^{-1}$). Thus, let $Y = X[1] - X[0]$ or

$$Y = \underbrace{\begin{bmatrix} -1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} X[0] \\ X[1] \end{bmatrix}}_{\mathbf{X}}.$$

Then, since Y is a linear transformation of $\mathbf{X}$, we have $Y \sim \mathcal{N}(0, \operatorname{var}(Y))$, where $\operatorname{var}(Y) = \mathbf{A}\mathbf{C}\mathbf{A}^T$. Thus,

$$\begin{aligned}
\operatorname{var}(Y) &= \begin{bmatrix} -1 & 1 \end{bmatrix} \mathbf{C} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 1 \end{bmatrix} \mathbf{G}\mathbf{G}^T \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad (\mathbf{C} = \sigma_U^2 \mathbf{G}\mathbf{G}^T = \mathbf{G}\mathbf{G}^T) \\
&= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= \frac{1}{2}
\end{aligned}$$

so that $Y \sim \mathcal{N}(0, 1/2)$. Therefore,

$$P[X[1] - X[0] > 1] = Q\left(\frac{1}{\sqrt{1/2}}\right) = Q(\sqrt{2}) = 0.0786 < Q\left(\frac{1}{\sqrt{2}}\right) = 0.2398$$

and is consistent with our notion of smoothing.

$\diamond$

Example 20.3 – Discrete-time Wiener random process or Brownian motion

This random process is basically a *random walk* with Gaussian "steps" or more specifically the sum process (see also Example 16.4)

$$X[n] = \sum_{i=0}^{n} U[i] \qquad n \geq 0$$

where $U[n]$ is WGN with variance σ_U^2. Note that the increments $X[n_2] - X[n_1]$ are independent and stationary (why?). As in the previous example, any set of samples of $X[n]$ is a linear transformation of the $U[i]$'s and hence has a multivariate Gaussian PDF. For example,

$$\begin{bmatrix} X[0] \\ X[1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\mathbf{G}} \begin{bmatrix} U[0] \\ U[1] \end{bmatrix}$$

and therefore the Wiener random process is a Gaussian random process. It is clearly nonstationary, since, for example, the variance increases with n (recall from Example 16.4 that $\operatorname{var}(X[n]) = (n+1)\sigma_U^2$).

$\diamondsuit$

In Example 20.1 we saw that if the samples are uncorrelated, and the random process is Gaussian and hence the multivariate Gaussian PDF applies, then the samples are also *independent*. In Examples 20.1 and 20.2, the random processes were WSS but due to the fact that they are also Gaussian random processes, they are also *stationary*. We summarize and then prove these two properties next.

Property 20.1 – A Gaussian random process with uncorrelated samples has independent samples.

Proof:

Since the random process is Gaussian, the PDF of (20.1) applies for any set of samples. But for uncorrelated samples, the covariance matrix is diagonal and hence the joint PDF factors into the product of its marginal PDFs. Hence, the samples are independent.

$\square$

Property 20.2 – A WSS Gaussian random process is also stationary.

Proof:

Since the random process is Gaussian, the PDF of (20.1) applies for any set of samples. But if $X[n]$ is also WSS, then for any n_0

$$E[X[n_i + n_0]] = \mu_X[n_i + n_0] = \mu \qquad i = 1, 2, \ldots, K$$

and

$$\begin{aligned} [\mathbf{C}]_{ij} &= \operatorname{cov}(X[n_i + n_0], X[n_j + n_0]) \\ &= E[X[n_i + n_0]X[n_j + n_0]] - E[X[n_i + n_0]]E[X[n_j + n_0]] \\ &= r_X[n_j - n_i] - \mu^2 \qquad \text{(due to WSS)} \end{aligned}$$

for $i = 1, 2, \ldots, K$ and $j = 1, 2, \ldots, K$. Since the mean vector and the covariance matrix do not depend on n_0, the joint PDF also does not depend on n_0. Hence, the WSS Gaussian random process is also stationary.

$\square$

20.4 Linear Transformations

Any linear transformation of a Gaussian random process produces another Gaussian random process. In Example 20.2 the white noise random process $U[n]$ was Gaussian, and the MA random process $X[n]$, which was the result of a linear transformation, is another Gaussian random process. The MA random process in that example can be viewed as the output of the LSI filter with system function $\mathcal{H}(z) = 1/2 + (1/2)z^{-1}$ whose input is $U[n]$. This result, that if the input to an LSI filter is a Gaussian random process, then the output is also a Gaussian random process, is true in general. The random processes described by the linear difference equations

$$
\begin{aligned}
X[n] &= aX[n-1] + U[n] && \text{AR random process (see Example 17.5)} \\
X[n] &= U[n] - bU[n-1] && \text{MA random process (see Example 18.6)} \\
X[n] &= aX[n-1] + U[n] - bU[n-1] && \text{ARMA random process} \\
& && \text{(This is the definition.)}
\end{aligned}
$$

can also be viewed as the outputs of LSI filters with respective system functions

$$
\begin{aligned}
\mathcal{H}(z) &= \frac{1}{1-az^{-1}} \\
\mathcal{H}(z) &= 1-bz^{-1} \\
\mathcal{H}(z) &= \frac{1-bz^{-1}}{1-az^{-1}}.
\end{aligned}
$$

As a result, since the input $U[n]$ is a Gaussian random process, they are all Gaussian random processes. Furthermore, since it is only necessary to know the first two moments to specify the joint PDF of a set of samples of a Gaussian random process, the PDF for the output random process of an LSI filter is easily found. In particular, assume we are interested in the filtering of a WSS Gaussian random process by an LSI filter with frequency response $H(f)$. *Then, if the input to the filter is the WSS Gaussian random process $X[n]$, which has a mean of μ_X and an ACS of $r_X[k]$, then we know from Theorem 18.3.1 that the output random process $Y[n]$ is also WSS and its mean and ACS are*

$$
\begin{aligned}
\mu_Y &= \mu_X H(0) && (20.4) \\
P_Y(f) &= |H(f)|^2 P_X(f) && (20.5)
\end{aligned}
$$

and furthermore $Y[n]$ is a Gaussian random process (and is stationary according to Property 20.2). (See also Problem 20.7.) The joint PDF for any set of samples of $Y[n]$ is found from (20.1) by using (20.4) and (20.5). An example follows.

Example 20.4 – A differencer

A WSS Gaussian random process $X[n]$ with mean μ_X and ACS $r_X[k]$ is input to a differencer. The output random process is defined to be $Y[n] = X[n] - X[n-1]$.

What is the PDF of two successive output samples? To solve this we first note that the output random process is Gaussian and also WSS since a differencer is just an LSI filter whose system function is $\mathcal{H}(z) = 1 - z^{-1}$. We need only find the first two moments of $Y[n]$. The mean is

$$E[Y[n]] = E[X[n]] - E[X[n-1]] = \mu_X - \mu_X = 0$$

and the ACS can be found as the inverse Fourier transform of $P_Y(f)$. But from (20.5) with $H(f) = \mathcal{H}(\exp(j2\pi f)) = 1 - \exp(-j2\pi f)$, we have

$$\begin{aligned} P_Y(f) &= H(f)H^*(f)P_X(f) \\ &= [1 - \exp(-j2\pi f)][1 - \exp(j2\pi f)]P_X(f) \\ &= 2P_X(f) - \exp(j2\pi f)P_X(f) - \exp(-j2\pi f)P_X(f). \end{aligned}$$

Taking the inverse Fourier transform produces

$$r_Y[k] = 2r_X[k] - r_X[k+1] - r_X[k-1]. \tag{20.6}$$

For two successive samples, say $Y[0]$ and $Y[1]$, we require the covariance matrix of $\mathbf{Y} = [Y[0]\, Y[1]]^T$. Since $Y[n]$ has a zero mean, this is just

$$\mathbf{C}_Y = \begin{bmatrix} r_Y[0] & r_Y[1] \\ r_Y[1] & r_Y[0] \end{bmatrix}$$

and thus using (20.6), it becomes

$$\mathbf{C}_Y = \begin{bmatrix} 2(r_X[0] - r_X[1]) & 2r_X[1] - r_X[2] - r_X[0] \\ 2r_X[1] - r_X[2] - r_X[0] & 2(r_X[0] - r_X[1]) \end{bmatrix}.$$

The joint PDF is then

$$p_{Y[0],Y[1]}(y[0], y[1]) = \frac{1}{2\pi \det^{1/2}(\mathbf{C}_Y)} \exp(-\tfrac{1}{2}\mathbf{y}^T\mathbf{C}_Y^{-1}\mathbf{y})$$

where $\mathbf{y} = [y[0]\, y[1]]^T$. See also Problem 20.5.

◇

We now summarize the foregoing results in a theorem.

Theorem 20.4.1 (Linear filtering of a WSS Gaussian random process) *Suppose that $X[n]$ is a WSS Gaussian random process with mean μ_X and ACS $r_X[k]$ that is input to an LSI filter with frequency response $H(f)$. Then, the PDF of N successive output samples $\mathbf{Y} = [Y[0]\, Y[1] \dots Y[N-1]]^T$ is given by*

$$p_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi)^{N/2} \det^{1/2}(\mathbf{C}_Y)} \exp\left[-\tfrac{1}{2}(\mathbf{y} - \boldsymbol{\mu}_Y)^T\mathbf{C}_Y^{-1}(\mathbf{y} - \boldsymbol{\mu}_Y)\right] \tag{20.7}$$

where

$$\boldsymbol{\mu}_Y = \begin{bmatrix} \mu_X H(0) \\ \vdots \\ \mu_X H(0) \end{bmatrix} \tag{20.8}$$

$$\begin{aligned} [\mathbf{C}_Y]_{mn} &= r_Y[m-n] - (\mu_X H(0))^2 & (20.9) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} |H(f)|^2 P_X(f) \exp(j2\pi f(m-n))df - (\mu_X H(0))^2 & (20.10) \end{aligned}$$

for $m = 1, 2, \ldots, N; n = 1, 2, \ldots, N$. *The same PDF is obtained for any shifted set of successive samples since* $Y[n]$ *is stationary.*

Note that in the preceding theorem the covariance matrix is a symmetric *Toeplitz* matrix (all elements along each northwest-southeast diagonal are the same) due to the assumption of successive samples (see also Section 17.4).

Another transformation that occurs quite frequently is the sum of two independent Gaussian random processes. If $X[n]$ is a Gaussian random process and $Y[n]$ is another Gaussian random process, *and* $X[n]$ and $Y[n]$ are independent, then $Z[n] = X[n] + Y[n]$ is a Gaussian random process (see Problem 20.9). By independence of two random processes we mean that all sets of samples of $X[n]$ or $\{X[n_1], X[n_2], \ldots, X[n_K]\}$ and of $Y[n]$ or $\{Y[m_1], Y[m_2], \ldots, Y[m_L]\}$ are independent of each other. This must hold for all $n_1, \ldots, n_K$, $m_1, \ldots, m_L$ and for all K and L. If this is the case then the PDF of the entire set of samples can be written as the product of the PDFs of each set of samples.

20.5 Nonlinear Transformations

The Gaussian random process is one of the few random processes for which the moments at the output of a nonlinear transformation can easily be found. In particular, a polynomial transformation lends itself to output moment evaluation. This is because the *higher-order* joint moments of a multivariate Gaussian PDF can be expressed in terms of *first- and second-order* moments. In fact, this is not surprising in that the multivariate Gaussian PDF is characterized by its first- and second-order moments. As a result, in computing the joint moments, any integral of the form $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{l_1} \ldots x_N^{l_N} p_{X_1,\ldots,X_N}(x_1, \ldots, x_N)dx_1 \ldots dx_N$ must be a function of the mean vector and covariance matrix. Hence, the joint moments must be a function of the first- and second-order moments. As a particular case of interest, consider the fourth-order moment $E[X_1X_2X_3X_4]$ for $\mathbf{X} = [X_1\, X_2\, X_3\, X_4]^T$ a zero mean Gaussian random vector. Then, it can be shown that (see Problem 20.12)

$$E[X_1X_2X_3X_4] = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4]E[X_2X_3] \tag{20.11}$$

and this holds even if some of the random variables are the same (try $X_1 = X_2 = X_3 = X_4$ and compare it to $E[X^4]$ for $X \sim \mathcal{N}(0,1)$). It is seen that the fourth-order moment is expressible as the sum of products of the second-order moments, which are found from the covariance matrix. Now if $X[n]$ is a Gaussian random process with zero mean, then we have for any four samples (which by the definition of a Gaussian random process has a fourth-order Gaussian PDF)

$$\begin{aligned} E[X[n_1]X[n_2]X[n_3]X[n_4]] &= E[X[n_1]X[n_2]]E[X[n_3]X[n_4]] \\ &\quad + E[X[n_1]X[n_3]]E[X[n_2]X[n_4]] \\ &\quad + E[X[n_1]X[n_4]]E[X[n_2]X[n_3]] \end{aligned} \tag{20.12}$$

and if furthermore, $X[n]$ is WSS, then this reduces to

$$\begin{aligned} E[X[n_1]X[n_2]X[n_3]X[n_4]] &= r_X[n_2 - n_1]r_X[n_4 - n_3] + r_X[n_3 - n_1]r_X[n_4 - n_2] \\ &\quad + r_X[n_4 - n_1]r_X[n_3 - n_2]. \end{aligned} \tag{20.13}$$

This formula allows us to easily calculate the effect of a polynomial transformation on the moments of a WSS Gaussian random process. An example follows.

Example 20.5 – Effect of squaring WSS Gaussian random process

Assuming that $X[n]$ is a zero mean WSS Gaussian random process, we wish to determine the effect of squaring it to form $Y[n] = X^2[n]$. Clearly, $Y[n]$ will no longer be a Gaussian random process since it can only take on nonnegative values (see also Example 10.8). We can, however, show that $Y[n]$ is still WSS. To do so we calculate the mean as

$$E[Y[n]] = E[X^2[n]] = r_X[0]$$

which does not depend on n, and the covariance sequence as

$$\begin{aligned} E[Y[n]Y[n+k]] &= E[X^2[n]X^2[n+k]] \\ &= r_X^2[0] + 2r_X^2[k] \qquad \text{(using } n_1 = n_2 = n \\ &\qquad\qquad\qquad\qquad \text{and } n_3 = n_4 = n + k \text{ in (20.13))} \end{aligned}$$

which also does not depend on n. Thus, at the output of the squarer the random process is WSS with

$$\begin{aligned} \mu_Y &= r_X[0] \\ r_Y[k] &= r_X^2[0] + 2r_X^2[k]. \end{aligned} \tag{20.14}$$

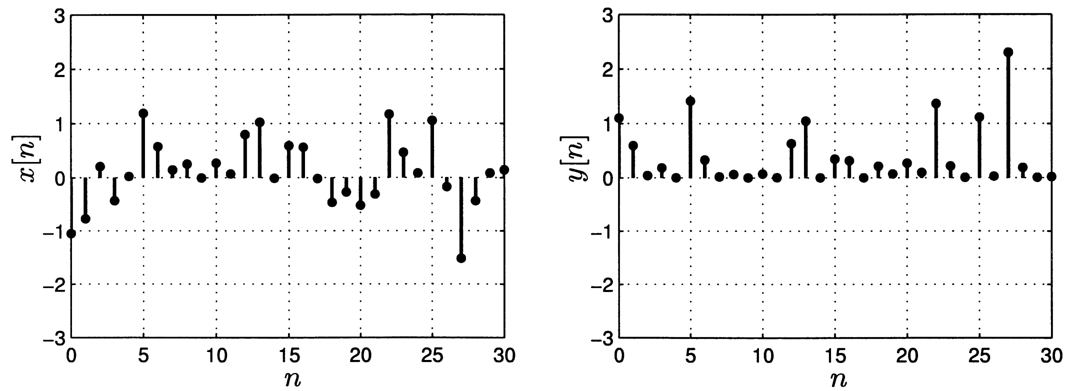
Note that if the PSD at the input to the squarer is $P_X(f)$, then the output PSD is obtained by taking the Fourier transform of (20.14) to yield

$$P_Y(f) = r_X^2[0]\delta(f) + 2P_X(f) \star P_X(f) \tag{20.15}$$

where

$$P_X(f) \star P_X(f) = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_X(\nu)P_X(f-\nu)d\nu$$

is a convolution integral. As a specific example, consider the MA random process $X[n] = (U[n]+U[n-1])/2$, where $U[n]$ is WGN with variance $\sigma_U^2 = 1$. Then, typical realizations of $X[n]$ and $Y[n]$ are shown in Figure 20.4. The MA random process



(a) MA random process

(b) Squared MA random process

Figure 20.4: Typical realization of a Gaussian MA random process and its squared realization.

has a zero mean and ACS $r_X[k] = (1/2)\delta[k] + (1/4)\delta[k+1] + (1/4)\delta[k-1]$ (see Example 17.3). Because of the squaring, the output mean is $E[Y[n]] = r_X[0] = 1/2$. The PSD of $X[n]$ can easily be shown to be $P_X(f) = (1+\cos(2\pi f))/2$ and the PSD of $Y[n]$ follows most easily by taking the Fourier transform of $r_Y[k]$. From (20.14) we have

$$\begin{aligned} r_Y[k] &= r_X^2[0] + 2r_X^2[k] \\ &= \frac{1}{4} + 2\left(\frac{1}{2}\delta[k] + \frac{1}{4}\delta[k+1] + \frac{1}{4}\delta[k-1]\right)^2 \\ &= \frac{1}{4} + 2\left(\frac{1}{4}\delta[k] + \frac{1}{16}\delta[k+1] + \frac{1}{16}\delta[k-1]\right) \end{aligned}$$

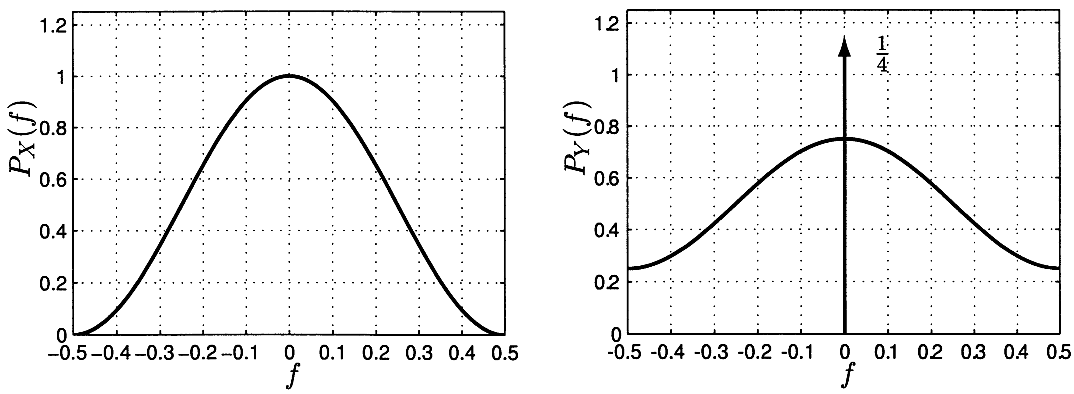
since all the cross-terms must be zero and $\delta^2[k-k_0] = \delta[k-k_0]$. Thus, we have

$$r_Y[k] = \frac{1}{4} + \frac{1}{2}\delta[k] + \frac{1}{8}\delta[k+1] + \frac{1}{8}\delta[k-1]$$

and taking the Fourier transform produces the PSD as

$$P_Y(f) = \frac{1}{4}\delta(f) + \frac{1}{2} + \frac{1}{4}\cos(2\pi f).$$

The PSDs are shown in Figure 20.5. Note that the squaring has produced an impulse



(a) MA random process

(b) Squared MA random process

Figure 20.5: PSDs of Gaussian MA random process and the squared random process.

at $f = 0$ of strength $1/4$ that is due to the nonzero mean of the $Y[n]$ random process. Also, the squaring has "widened" the PSD, the usual consequence of a convolution in frequency.

$\diamond$

20.6 Continuous-Time Definitions and Formulas

A continuous-time random process is defined to be a Gaussian random process if the random vector $\mathbf{X} = [X(t_1)\, X(t_2) \ldots X(t_K)]^T$ has a multivariate Gaussian PDF for all $\{t_1, t_2, \ldots, t_K\}$ and all K. The properties of a continuous-time Gaussian random process are identical to those for the discrete-time random process as summarized in Properties 20.1 and 20.2. Therefore, we will proceed directly to some examples of interest.

Example 20.6 – Continuous-time WGN

The continuous-time version of discrete-time WGN as defined in Example 20.1 is a *continuous-time* Gaussian random process $X(t)$ that has a zero mean and an ACF $r_X(\tau) = (N_0/2)\delta(\tau)$. The factor of $N_0/2$ is customarily used, since it is the level of the corresponding PSD (see Example 17.11). The random process is called *continuous-time white Gaussian noise (WGN)*. This was previously described in

Example 17.11. Note that in addition to the samples being uncorrelated (since $r_X(\tau) = 0$ for $\tau \neq 0$), they are also *independent* because of the Gaussian assumption. Unfortunately, for continuous-time WGN, it is not possible to explicitly write down the multivariate Gaussian PDF since $r_X(0) \to \infty$. Instead, as explained in Example 17.11 we use continuous-time WGN only as a model, reserving any probability calculations for the random process at the output of some filter, whose input is WGN. This is illustrated next.

◇

Example 20.7 – Continuous-time Wiener random process or Brownian motion

Let $U(t)$ be WGN and define the semi-infinite random process

$$X(t) = \int_0^t U(\xi)d\xi \qquad t \geq 0.$$

This random process is called the *Wiener random process* and is often used as a model for Brownian motion. It is the continuous-time equivalent of the discrete-time random process of Example 20.3. A typical realization of the Wiener random process is shown in Figure 20.6 (see Problem 20.18 on how this was done). Note that

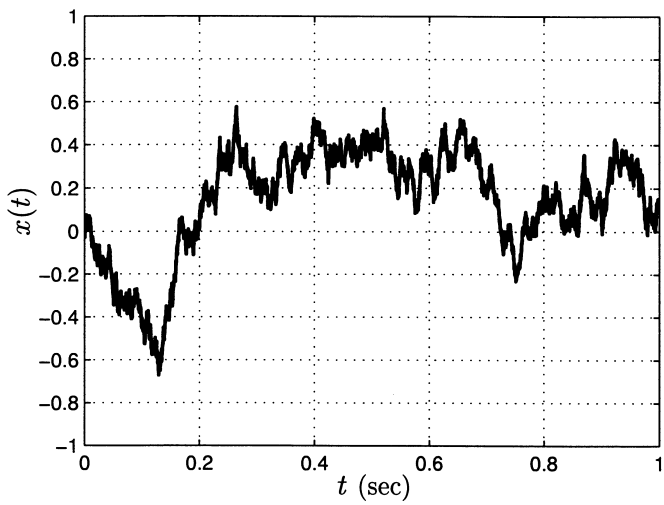


Figure 20.6: Typical realization of the Wiener random process.

because of its construction as the "sum" of independent and identically distributed random variables (the $U(t)$'s), the increments are also independent and stationary. To prove that $X(t)$ is a Gaussian random process is somewhat tricky in that it is an uncountable "sum" of independent random variables $U(\xi)$ for $0 \leq \xi \leq t$. We will take it on faith that any integral, which is a linear transformation, of a continuous-time Gaussian random process produces another continuous-time Gaussian random process (see also Problem 20.16 for a heuristic proof). As such, we need only determine the mean and covariance functions. These are found as

$$\begin{aligned} E[X(t)] &= E\left[\int_0^t U(\xi)d\xi\right] \\ &= \int_0^t E[U(\xi)]d\xi = 0 \end{aligned}$$

$$\begin{aligned} E[X(t_1)X(t_2)] &= E\left[\int_0^{t_1} U(\xi_1)d\xi_1 \int_0^{t_2} U(\xi_2)d\xi_2\right] \\ &= \int_0^{t_1}\int_0^{t_2} \underbrace{E[U(\xi_1)U(\xi_2)]}_{r_U(\xi_2-\xi_1)=(N_0/2)\delta(\xi_2-\xi_1)} d\xi_1 d\xi_2 \\ &= \frac{N_0}{2}\int_0^{t_1}\left(\int_0^{t_2}\delta(\xi_2-\xi_1)d\xi_2\right)d\xi_1. \end{aligned}$$

To evaluate the double integral we first examine the inner integral and assume that $t_2 > t_1$. Then, the function $\delta(\xi_2 - \xi_1)$ with ξ_1 *fixed* is integrated over the interval $0 \leq \xi_2 \leq t_2$ as shown in Figure 20.7. It is clear from the figure that if we fix ξ_1

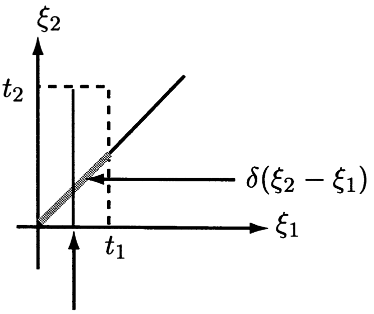


Figure 20.7: Evaluation of double integral of Dirac delta function for the case of $t_2 > t_1$.

and integrate along ξ_2, then we will include the impulse in the inner integral for all ξ_1. (This would not be the case if $t_2 < t_1$ as one can easily verify by redrawing the rectangle for this condition.) As a result, if $t_2 > t_1$, then

$$\int_0^{t_2} \delta(\xi_2 - \xi_1)d\xi_2 = 1 \qquad \text{for all } 0 \leq \xi_1 \leq t_1$$

and therefore

$$E[X(t_1)X(t_2)] = \frac{N_0}{2}\int_0^{t_1} d\xi_1 = \frac{N_0}{2}t_1$$

and similarly if $t_2 < t_1$, we will have $E[X(t_1)X(t_2)] = (N_0/2)t_2$. Combining the two results produces

$$E[X(t_1)X(t_2)] = \frac{N_0}{2}\min(t_1, t_2) \tag{20.16}$$

which should be compared to the discrete-time result obtained in Problem 16.26. Hence, the joint PDF of the samples of a Wiener random process is a multivariate Gaussian PDF with mean vector equal to zero and covariance matrix having as its (i,j)th element

$$[\mathbf{C}]_{ij} = \frac{N_0}{2}\min(t_i, t_j).$$

Note that from (20.16) with $t_1 = t_2 = t$, the PDF of $X(t)$ is $\mathcal{N}(0, (N_0/2)t)$. Clearly, *the Wiener random process is a nonstationary correlated random process whose mean is zero, variance increases with time, and marginal PDF is Gaussian.*

◇

In the next section we explore some other important continuous-time Gaussian random processes often used as models in practice.

20.7 Special Continuous-Time Gaussian Random Processes

20.7.1 Rayleigh Fading Sinusoid

In Example 16.11 we studied a discrete-time randomly phased sinusoid. Here we consider the continuous-time equivalent for that random process, which is given by $X(t) = A\cos(2\pi F_0 t + \Theta)$, where $A > 0$ is the amplitude, F_0 is the frequency in Hz, and Θ is the random phase with PDF $\mathcal{U}(0, 2\pi)$. We now further assume that the *amplitude* is also a random variable. This is frequently a good model for a sinusoidal signal that is subject to *multipath fading.* It occurs when a sinusoidal signal propagates through a medium, e.g., an electromagnetic pulse in the atmosphere or a sound pulse in the ocean, and reaches its destination by several different paths. The constructive and destructive interference of several overlapping sinusoids causes the received waveform to exhibit amplitude fluctuations or fading. An example of this was given in Figure 20.2. However, over any short period of time, say $5 \le t \le 5.5$ seconds, the waveform will have approximately a constant amplitude and a constant phase as shown in Figure 20.8. Because the amplitude and phase are not known in advance, we model them as realizations of random variables. That the waveform does not maintain the constant amplitude level and phase outside of the small interval will be of no consequence to us if we are only privy to observing the waveform over a small time interval. Hence, a reasonable model for the random process (over the small time interval) is to assume a random amplitude and random phase so that $X(t) = A\cos(2\pi F_0 t + \Theta)$, where A and Θ are random variables. A more convenient

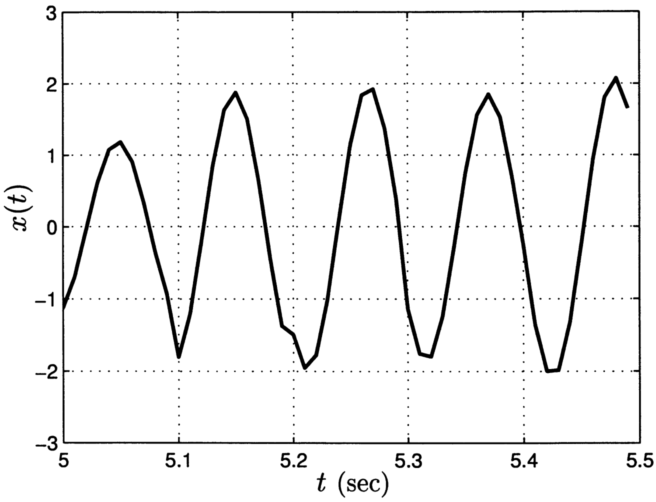


Figure 20.8: Segment of waveform shown in Figure 20.2 for $5 \leq t \leq 5.5$ seconds.

form is obtained by expanding the sinusoid as

$$\begin{aligned} X(t) &= A\cos(2\pi F_0 t + \Theta) \\ &= A\cos(\Theta)\cos(2\pi F_0 t) - A\sin(\Theta)\sin(2\pi F_0 t) \\ &= U\cos(2\pi F_0 t) - V\sin(2\pi F_0 t) \end{aligned}$$

where we have let $A\cos(\Theta) = U$, $A\sin(\Theta) = V$. Clearly, since A and Θ are random variables, so are U and V. Since the physical waveform is due to the sum of many sinusoids, we once again use a central limit theorem argument to assume that U and V are Gaussian. Furthermore, if we assume that they are independent and have the same PDF of $\mathcal{N}(0, \sigma^2)$, we will obtain PDFs for the amplitude and phase which are found to be valid in practice. With the Gaussian assumptions for U and V, the random amplitude becomes a Rayleigh distributed random variable, the random phase becomes a uniformly distributed random variable, and the amplitude and phase random variables are independent of each other. To see this note that since $U = A\cos(\Theta)$, $V = A\sin(\Theta)$, we have $A = \sqrt{U^2 + V^2}$ and $\Theta = \arctan(V/U)$. It was shown in Example 12.12 that if $X \sim \mathcal{N}(0, \sigma^2)$, $Y \sim \mathcal{N}(0, \sigma^2)$, and X and Y are independent, then $R = \sqrt{X^2 + Y^2}$ is a Rayleigh random variable, $\Theta = \arctan(Y/X)$ is a uniformly distributed random variable, and R and Θ are independent. Hence, we have that for the random amplitude/random phase sinusoid $X(t) = A\cos(2\pi F_0 t + \Theta)$, the amplitude has the PDF

$$p_A(a) = \begin{cases} \frac{a}{\sigma^2}\exp\left(-\frac{1}{2}\frac{a^2}{\sigma^2}\right) & a \geq 0 \\ 0 & a < 0 \end{cases}$$

and the phase has the PDF $\Theta \sim \mathcal{U}(0, 2\pi)$, and A and Θ are independent. This model is usually referred to as the *Rayleigh fading sinusoidal* model. It is also a

Gaussian random process since all sets of K samples can be written as

$$\begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_K) \end{bmatrix} = \begin{bmatrix} \cos(2\pi F_0 t_1) & -\sin(2\pi F_0 t_1) \\ \cos(2\pi F_0 t_2) & -\sin(2\pi F_0 t_2) \\ \vdots & \vdots \\ \cos(2\pi F_0 t_K) & -\sin(2\pi F_0 t_K) \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$$

which is a linear transformation of the Gaussian random vector $[U\, V]^T$, and so has a multivariate Gaussian PDF. (For $K > 2$ the covariance matrix will be singular, so that to be more rigorous we would need to modify our definition of the Gaussian random process. This would involve the characteristic function which exists even for a singular covariance matrix.) Furthermore, $X(t)$ is WSS, as we now show. Its mean is zero since $E[U] = E[V] = 0$ and its ACF is

$$\begin{aligned} r_X(\tau) \\ &= E[X(t)X(t+\tau)] \\ &= E[[U\cos(2\pi F_0 t) - V\sin(2\pi F_0 t)][U\cos(2\pi F_0(t+\tau)) - V\sin(2\pi F_0(t+\tau))]] \\ &= E[U^2]\cos(2\pi F_0 t)\cos(2\pi F_0(t+\tau)) + E[V^2]\sin(2\pi F_0 t)\sin(2\pi F_0(t+\tau)) \\ &= \sigma^2[\cos(2\pi F_0 t)\cos(2\pi F_0(t+\tau)) + \sin(2\pi F_0 t)\sin(2\pi F_0(t+\tau))] \\ &= \sigma^2\cos(2\pi F_0 \tau) \end{aligned} \tag{20.17}$$

where we have used $E[UV] = E[U]E[V] = 0$ due to independence. Its PSD is obtained by taking the Fourier transform to yield

$$P_X(F) = \frac{\sigma^2}{2}\delta(F + F_0) + \frac{\sigma^2}{2}\delta(F - F_0) \tag{20.18}$$

and it is seen that all its power is concentrated at $F = F_0$ as expected.

20.7.2 Bandpass Random Process

The Rayleigh fading sinusoid model assumed that our observation time was short. Within that time window, the sinusoid exhibits approximately constant amplitude and phase. If we observe a longer time segment of the random process whose typical realization is shown in Figure 20.2, then the constant in time (but random amplitude/random phase) sinusoid is not a good model. A more realistic but more complicated model is to let both the amplitude and phase be random processes so that they vary in time. As such, the random process will be made up of many frequencies, although they will be concentrated about $F = F_0$. Such a random process is usually called a *narrowband random process.* Our model, however, will actually be valid for a bandpass random process whose PSD is shown in Figure 20.9. Hence, we will assume that the bandpass random process can be represented as

$$\begin{aligned} X(t) &= A(t)\cos(2\pi F_0 t + \Theta(t)) \\ &= A(t)\cos(\Theta(t))\cos(2\pi F_0 t) - A(t)\sin(\Theta(t))\sin(2\pi F_0 t) \end{aligned}$$

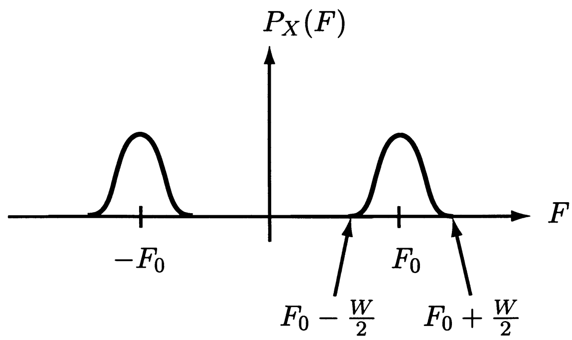


Figure 20.9: Typical PSD for bandpass random process. The PSD is assumed to be symmetric about $F = F_0$ and also that $F_0 > W/2$.

where $A(t)$ and $\Theta(t)$ are now random processes. As before we let

$$\begin{aligned} U(t) &= A(t)\cos(\Theta(t)) \\ V(t) &= A(t)\sin(\Theta(t)) \end{aligned}$$

so that we have as our model for a bandpass random process

$$X(t) = U(t)\cos(2\pi F_0 t) - V(t)\sin(2\pi F_0 t). \tag{20.19}$$

The $X(t)$ random process is seen to be a modulated version of $U(t)$ and $V(t)$ (modulation meaning that $U(t)$ and $V(t)$ are multiplied by $\cos(2\pi F_0 t)$ and $\sin(2\pi F_0 t)$, respectively). This modulation shifts the PSD of $U(t)$ and $V(t)$ to be centered about $F = F_0$. Therefore, $U(t)$ and $V(t)$ must be slowly varying or *lowpass* random processes. As a suitable description of $U(t)$ and $V(t)$ we assume that *they are each zero mean lowpass Gaussian random processes, independent of each other, and jointly WSS (see Chapter 19) with the same ACF,* $r_U(\tau) = r_V(\tau)$. Then, as before $X(t)$ is a zero mean Gaussian random process, which as we now show is also WSS. Clearly, since both $U(t)$ and $V(t)$ are zero mean, from (20.19) so is $X(t)$, and the ACF is

$$\begin{aligned} r_X(\tau) &= E[X(t)X(t+\tau)] && (20.20) \\ &= E[[U(t)\cos(2\pi F_0 t) - V(t)\sin(2\pi F_0 t)] \\ &\quad \cdot[U(t+\tau)\cos(2\pi F_0(t+\tau)) - V(t+\tau)\sin(2\pi F_0(t+\tau))] \\ &= r_U(\tau)\cos(2\pi F_0 t)\cos(2\pi F_0(t+\tau)) + r_V(\tau)\sin(2\pi F_0 t)\sin(2\pi F_0(t+\tau)) \\ &= r_U(\tau)\cos(2\pi F_0 \tau) && (20.21) \end{aligned}$$

since $E[U(t_1)V(t_2)] = 0$ for all t_1 and t_2 due to the independence assumption, and $r_U(\tau) = r_V(\tau)$ by assumption. Note that this extends the previous case in which $U(t) = U$ and $V(t) = V$ and $r_U(\tau) = \sigma^2$ (see (20.17)). The PSD is found by taking the Fourier transform of the ACF to yield

$$P_X(F) = \frac{1}{2}P_U(F+F_0) + \frac{1}{2}P_U(F-F_0). \tag{20.22}$$

If $U(t)$ and $V(t)$ have the lowpass PSD shown in Figure 20.10, then in accordance with (20.22) $P_X(F)$ is given by the dashed curve. As desired, we now have a repre-

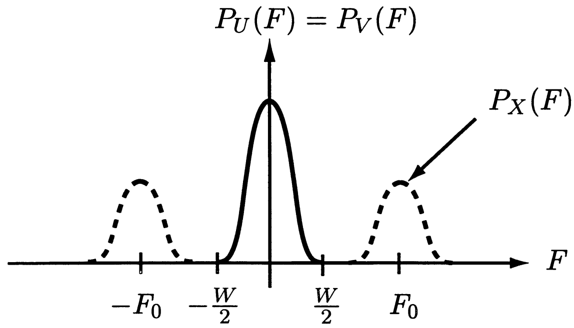


Figure 20.10: PSD for lowpass random processes $U(t)$ and $V(t)$. The PSD for the bandpass random process $X(t)$ is shown as the dashed curve.

sentation for a bandpass random process. It is obtained by *modulating* two lowpass random processes $U(t)$ and $V(t)$ up to a center frequency of F_0 Hz. Hence, (20.19) is called the *bandpass random process representation* and since the random process may either represent a signal or noise, it is also referred to as the bandpass signal representation or the bandpass noise representation. Note that because $P_U(F)$ is symmetric about $F = 0$, $P_X(F)$ must be symmetric about $F = F_0$. To represent bandpass PSDs that are not symmetric requires the assumption that $U(t)$ and $V(t)$ are correlated [Van Trees 1971].

In summary, to model a WSS Gaussian random process $X(t)$ that has a zero mean and a bandpass PSD given by

$$P_X(F) = \frac{1}{2}P_U(F + F_0) + \frac{1}{2}P_U(F - F_0)$$

where $P_U(F) = 0$ for $|F| > W/2$ as shown in Figure 20.10 by the dashed curve, we use

$$X(t) = U(t)\cos(2\pi F_0 t) - V(t)\sin(2\pi F_0 t).$$

The assumptions are that $U(t), V(t)$ are each Gaussian random processes with zero mean, independent of each other and each is WSS with PSD $P_U(F)$. The random processes $U(t), V(t)$ are lowpass random processes and are sometimes referred to as the *in phase* and *quadrature* components of $X(t)$. This is because the "carrier" sinusoid $\cos(2\pi F_0 t)$ is in phase with the sinusoidal carrier in $U(t)\cos(2\pi F_0 t)$ and 90° out of phase with the sinusoidal carrier in $V(t)\sin(2\pi F_0 t)$. See Problem 20.24 on how to extract the lowpass random processes from $X(t)$. In addition, the amplitude of $X(t)$, which is $\sqrt{U^2(t) + V^2(t)}$ is called the *envelope* of $X(t)$. This is because if $X(t)$ is written as $X(t) = \sqrt{U^2(t) + V^2(t)}\cos(2\pi F_0 t + \arctan(V(t)/U(t)))$ (see Problem 12.42) the envelope consists of the maximums of the waveform. An example of a *deterministic* bandpass signal, given for sake of illustration, is $s(t) = 3t\cos(2\pi 20t) - 4t\sin(2\pi 20t)$ for $0 \le t \le 1$, and is shown in Figure 20.11. Note that the envelope is $\sqrt{(3t)^2 + (4t)^2} = 5|t|$. For a bandpass *random process* the envelope will also be a

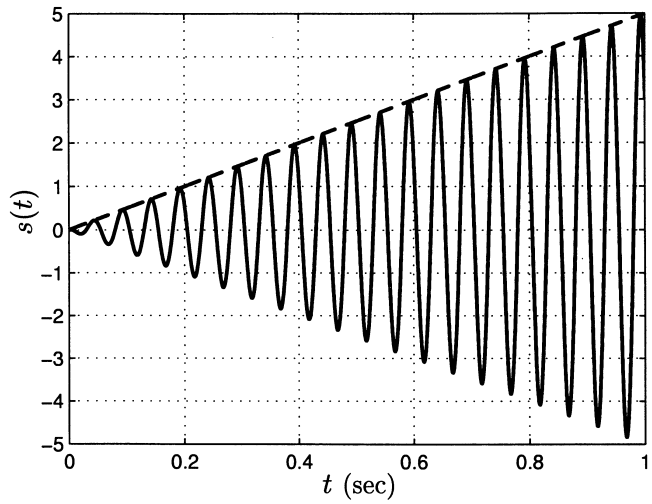


Figure 20.11: Plot of the deterministic bandpass signal $s(t) = 3t\cos(2\pi 20t) - 4t\sin(2\pi 20t)$ for $0 \le t \le 1$. The envelope is shown as the dashed line.

random process. Since $U(t)$ and $V(t)$ both have the same ACF, the characteristics of the envelope depend directly on $r_U(\tau)$. An illustration is given in the next example.

Example 20.8 – Bandpass random process envelope

Consider the bandpass Gaussian random process whose PSD is shown in Figure 20.12. This is often used as a model for bandpass "white" Gaussian noise. It results from having filtered WGN with a bandpass filter. Note that from (20.22) the PSD

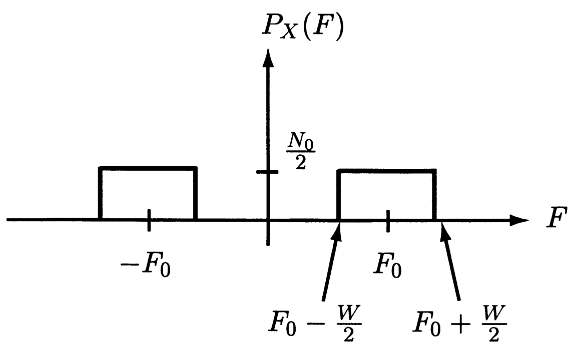


Figure 20.12: PSD for bandpass "white" Gaussian noise.

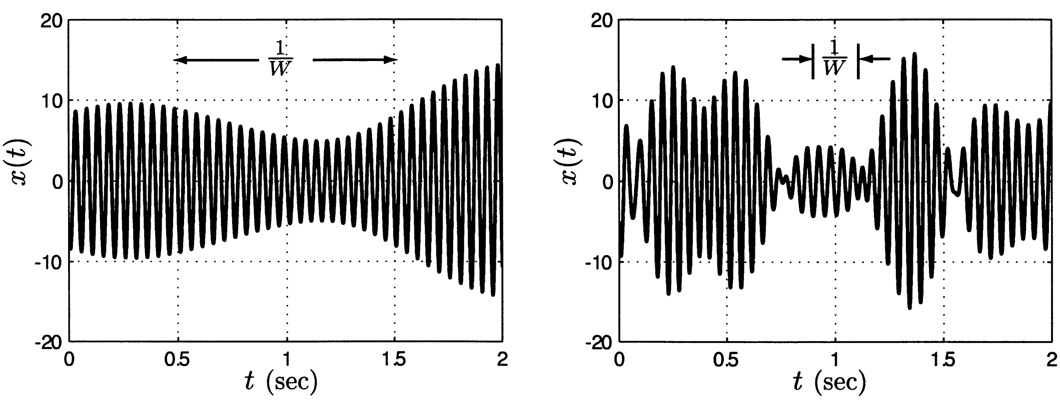
of $U(t)$ and $V(t)$ must be

$$P_U(F) = P_V(F) = \begin{cases} N_0 & |F| \leq \frac{W}{2} \\ 0 & |F| > \frac{W}{2} \end{cases}$$

and therefore by taking the inverse Fourier transform, the ACF becomes (see also (17.55) for a similar calculation)

$$r_U(\tau) = r_V(\tau) = N_0 W \frac{\sin(\pi W \tau)}{\pi W \tau}. \tag{20.23}$$

The correlation between two samples *of the envelope* will be approximately zero when $\tau > 1/W$ since then $r_U(\tau) = r_V(\tau) \approx 0$. Examples of some bandpass realizations are shown for $F_0 = 20$ Hz, $W = 1$ Hz in Figure 20.13a and for $F_0 = 20$ Hz, $W = 4$ Hz in Figure 20.13b. The time for which two samples must be separated before they become uncorrelated is called the *correlation time* τ_c. It is defined by $r_X(\tau) \approx 0$ for $\tau > \tau_c$. Here it is $\tau_c \approx 1/W$, and is shown in Figure 20.13.



(a) $F_0 = 20$ Hz, $W = 1$ Hz

(b) $F_0 = 20$ Hz, $W = 4$ Hz

Figure 20.13: Typical realizations of bandpass "white" Gaussian noise. The PSD is given in Figure 20.12.

A typical probability calculation might be to determine the probability that the envelope at $t = t_0$ exceeds some threshold γ. Thus, we wish to find $P[A(t_0) > \gamma]$, where $A(t_0) = \sqrt{U^2(t_0) + V^2(t_0)}$. Since the $U(t)$ and $V(t)$ are independent Gaussian random processes with $U(t) \sim \mathcal{N}(0, \sigma^2)$ and $V(t) \sim \mathcal{N}(0, \sigma^2)$, it follows that $A(t_0)$ is a Rayleigh random variable. Hence, we have that

$$\begin{aligned} P[A(t_0) > \gamma] &= \int_\gamma^\infty \frac{a}{\sigma^2} \exp\left(-\frac{1}{2}\frac{a^2}{\sigma^2}\right) da \\ &= \exp\left(-\frac{1}{2}\frac{\gamma^2}{\sigma^2}\right). \end{aligned}$$

To complete the calculation we need to determine σ^2. But $\sigma^2 = E[U^2(t_0)] = r_U[0] = N_0W$ from (20.23). Therefore, we have that

$$P[A(t_0) > \gamma] = \exp\left(-\frac{1}{2}\frac{\gamma^2}{N_0W}\right).$$

$\diamondsuit$

20.8 Computer Simulation

We now discuss the generation of a realization of a discrete-time Gaussian random process. The generation of a continuous-time random process realization can be accomplished by approximating it by a discrete-time realization with a sufficiently small time interval between samples. We have done this to produce Figure 20.6 (see also Problem 20.18). In particular, we wish to generate a realization of a *WSS* Gaussian random process with mean zero and ACS $r_X[k]$ or equivalently a PSD $P_X(f)$. For nonzero mean random processes we need only add the mean to the realization. The method is based on Theorem 20.4.1, where we use a WGN random process $U[n]$ as the input to an LSI filter with frequency response $H(f)$. Then, we know that the output random process will be WSS and Gaussian with a PSD $P_X(f) = |H(f)|^2 P_U(f)$. Now assuming that $P_U(f) = \sigma_U^2 = 1$, so that $P_X(f) = |H(f)|^2$, we see that a filter whose frequency response magnitude is $|H(f)| = \sqrt{P_X(f)}$ and whose phase response is arbitrary (but must be an odd function) will be required. Finding the filter frequency response from the PSD is known as *spectral factorization* [Priestley 1981]. As special cases of this problem, if we wish to generate either the AR, MA, or ARMA Gaussian random processes described in Section 20.4, then the filters are already known and have been implemented as difference equations. For example, the MA random process is generated by filtering $U[n]$ with the LSI filter whose frequency response is $H(f) = 1 - b\exp(-j2\pi f)$. This is equivalent to the implementation using the difference equation $X[n] = U[n] - bU[n-1]$. For higher-order (more coefficients) AR, MA, and ARMA random processes, the reader should consult [Kay 1988] for how the appropriate coefficients can be obtained from the PSD. Also, note that the problem of designing a filter whose frequency response magnitude *approximates* a given one is called *digital filter design.* Many techniques are available to do this [Jackson 1996]. We next give a simple example of how to generate a realization of a WSS Gaussian random process with a given PSD.

Example 20.9 – Filter determination to produce Gaussian random process with given PSD

Assume we wish to generate a realization of a WSS Gaussian random process with zero mean and PSD $P_X(f) = (1+\cos(4\pi f))/2$. Then, for $P_U(f) = 1$ the magnitude of the frequency response should be

$$|H(f)| = \sqrt{\tfrac{1}{2}(1+\cos(4\pi f))}.$$

We will choose the phase response or $\angle H(f) = \theta(f)$ to be any convenient function. Thus, we wish to determine the impulse response $h[k]$ of the filter whose frequency response is

$$H(f) = \sqrt{\tfrac{1}{2}(1+\cos(4\pi f))}\exp(j\theta(f))$$

since then we can generate the random process using a convolution sum as

$$X[n] = \sum_{k=-\infty}^{\infty} h[k]U[n-k]. \tag{20.24}$$

The impulse response is found as the inverse Fourier transform of the frequency response

$$\begin{aligned} h[n] &= \int_{-\frac{1}{2}}^{\frac{1}{2}} H(f)\exp(j2\pi f n)df \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\tfrac{1}{2}(1+\cos(4\pi f))}\exp(j\theta(f))\exp(j2\pi f n)df \qquad -\infty < n < \infty. \end{aligned}$$

This can be evaluated by noting that $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$ and therefore

$$\begin{aligned} \sqrt{\tfrac{1}{2}(1+\cos(4\pi f))} &= \sqrt{\tfrac{1}{2}(1+\cos^2(2\pi f) - \sin^2(2\pi f))} \\ &= \sqrt{\cos^2(2\pi f)} \\ &= |\cos(2\pi f)|. \end{aligned}$$

Thus,

$$h[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\cos(2\pi f)|\exp(j\theta(f))\exp(j2\pi f n)df$$

and we choose $\exp(j\theta(f)) = 1$ if $\cos(2\pi f) > 0$ and $\exp(j\theta(f)) = -1$ if $\cos(2\pi f) < 0$. This produces

$$h[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2\pi f)\exp(j2\pi f n)df$$

which is easily shown to evaluate to

$$h[n] = \begin{cases} \frac{1}{2} & n = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, from (20.24) we have that

$$X[n] = \frac{1}{2}U[n+1] + \frac{1}{2}U[n-1].$$

Note that the filter is noncausal. We could also use $X[n] = \frac{1}{2}U[n] + \frac{1}{2}U[n-2]$ if a causal filter is desired and still obtain the same PSD (see Problem 20.28).

$\diamondsuit$

Finally, it should be pointed out that an alternative means of generating successive samples of a zero mean Gaussian WSS random process is by applying a matrix transformation to a vector of independent $\mathcal{N}(0,1)$ samples. If a realization of $\mathbf{X} = [X[0]\,X[1]\ldots X[N-1]]^T$, where $\mathbf{X} \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_X)$ and $\mathbf{R}_X$ is the $N \times N$ Toeplitz autocorrelation matrix given in (17.23) is desired, then the method described in Section 14.9 can be used. We need only replace $\mathbf{C}$ by $\mathbf{R}_X$. For a nonzero mean WSS Gaussian random process, we add the mean μ to each sample after this procedure is employed. The only drawback is that the realization is assumed to consist of a fixed number of samples N, and so for each value of N the procedure must be repeated. Filtering, as previously described, allows any number of samples to be easily generated.

20.9 Real-World Example – Estimating Fish Populations

Of concern to biologists, and to us all, is the fish population. Traditionally, the population has been estimated using a count produced by a net catch. However, this is expensive, time consuming, and relatively inaccurate. A better approach is therefore needed. In the introduction we briefly indicated how an echo sonar would produce a Gaussian random process as the reflected waveform from a school of fish. We now examine this in more detail and explain how estimation of the fish population might be done. The discussion is oversimplified so that the interested reader may consult [Ehrenberg and Lytle 1972, Stanton 1983, Stanton and Chu 1998] for more detail. Referring to Figure 20.14 a sound pulse, which is assumed to be sinusoidal, is transmitted from a ship. As it encounters a school of fish, it will be reflected from each fish and the entire waveform, which is the sum of all the reflections, will be received at the ship. The received waveform will be examined for the time interval from $t = 2R_{\text{min}}/c$ to $t = 2R_{\text{max}}/c$, where R_{min} and R_{max} are the minimum and maximum ranges of interest, respectively, and c is the speed of sound in the water. This corresponds to the time interval over which the reflections from the desired ranges will be present. Based on the received waveform we wish to estimate the number of fish in the vertical direction in the desired *range window* from R_{min} to R_{max}. Note that only the fish within the nearly dashed vertical lines, which indicate the width of the transmitted sound energy, will produce reflections. For different angular regions other pulses must be transmitted. As discussed in the introduction, if there are a large number of fish producing reflections, then by the central limit theorem, the received waveform can be modeled as a Gaussian random process. As shown in Figure 20.14 the sinusoidal pulse first encounters the fish nearest in range, producing a reflection, while the fish farthest in range produces the last reflection. As a result, the many reflected pulses will overlap in time, with two of the reflected pulses shown in the figure. Hence, each reflected pulse can be

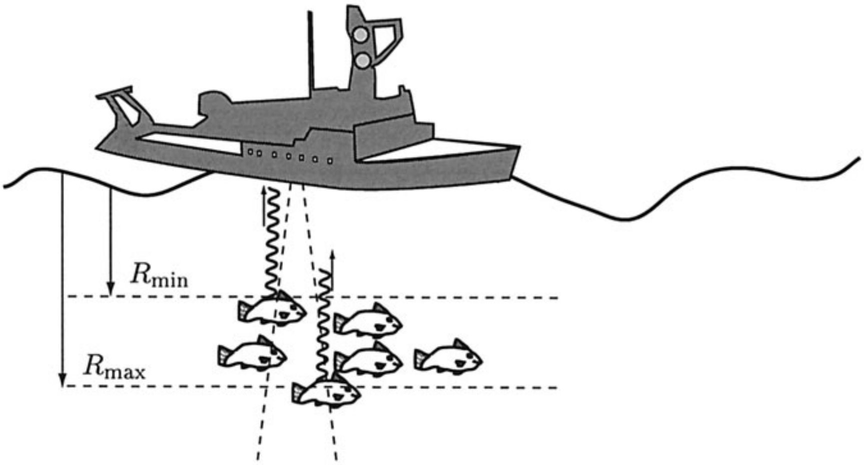


Figure 20.14: Fish counting by echo sonar.

represented as

$$X_i(t) = A_i \cos(2\pi F_0(t - \tau_i) + \Theta_i) \tag{20.25}$$

where F_0 is the transmit frequency in Hz and $\tau_i = 2R_i/c$ is the time delay of the pulse reflected from the ith fish. As explained in the introduction, since A_i, Θ_i will depend upon the fish's position, orientation, and motion, which are not known a priori, we assume that they are realizations of random variables. Futhermore, since the ranges of the individual fish are unknown, we also do not know τ_i. Hence, we replace (20.25) by

$$X_i(t) = A_i \cos(2\pi F_0 t + \Theta_i')$$

where $\Theta_i' = \Theta_i - 2\pi F_0 \tau_i$ (which is reduced by multiples of 2π until it lies within the interval $(0, 2\pi)$), and model Θ_i' as a new random variable. Hence, for N reflections we have as our model

$$\begin{aligned} X(t) &= \sum_{i=1}^{N} X_i(t) \\ &= \sum_{i=1}^{N} A_i \cos(2\pi F_0 t + \Theta_i') \end{aligned}$$

and letting $U_i = A_i \cos(\Theta_i')$ and $V_i = A_i \sin(\Theta_i')$, we have

$$\begin{aligned}
X(t) &= \sum_{i=1}^{N}(U_i \cos(2\pi F_0 t) - V_i \sin(2\pi F_0 t)) \\
&= \left(\sum_{i=1}^{N} U_i\right)\cos(2\pi F_0 t) - \left(\sum_{i=1}^{N} V_i\right)\sin(2\pi F_0 t) \\
&= U\cos(2\pi F_0 t) - V\sin(2\pi F_0 t)
\end{aligned}$$

where $U = \sum_{i=1}^{N} U_i$ and $V = \sum_{i=1}^{N} V_i$. We assume that all the fish are about the same size and hence the echo amplitudes are about the same. Then, since U and V are the sums of random variables that we assume are independent (reflection from one fish does not affect reflection from any of the others) and identically distributed (fish are same size), we use a central limit theorem argument to postulate a Gaussian PDF for U and V. We furthermore assume that U and V are independent so that if $E[U_i] = E[V_i] = 0$ and $\text{var}(U_i) = \text{var}(V_i) = \sigma^2$, then we have that $U \sim \mathcal{N}(0, N\sigma^2)$, $V \sim \mathcal{N}(0, N\sigma^2)$, and U and V are independent. This is the Rayleigh fading sinusoid model discussed in Section 20.7. As a result, the envelope of the received waveform $X(t)$, which is given by $A = \sqrt{U^2 + V^2}$ has a Rayleigh PDF. Specifically, it is

$$p_A(a) = \begin{cases} \frac{a}{N\sigma^2}\exp\left(-\frac{1}{2}\frac{a^2}{N\sigma^2}\right) & a \geq 0 \\ 0 & a < 0. \end{cases}$$

Hence, if we have previously measured the reflection characteristics of a single fish, then we will know σ^2. To estimate N we recall that the mean of the Rayleigh random variable is

$$E[A] = \sqrt{\frac{\pi}{2}N\sigma^2}$$

so that upon solving for N, we have

$$N = \frac{2}{\pi\sigma^2}E^2[A].$$

To estimate the mean we can transmit a series of M pulses and measure the envelope for each received waveform $X_m(t)$ for $m = 1, 2\ldots, M$. Calling the envelope measurement for the mth pulse $\hat{A}_m$, we can form the estimator for the number of fish as

$$\hat{N} = \frac{2}{\pi\sigma^2}\left(\frac{1}{M}\sum_{m=1}^{M}\hat{A}_m\right)^2. \tag{20.26}$$

See Problem 20.20 on how to obtain $\hat{A}_m = \sqrt{U_m^2 + V_m^2}$ from $X_m(t)$. It is shown there that $U_m = [2X_m(t)\cos(2\pi F_0 t)]_{\text{LPF}}$ and $V_m = [-2X_m(t)\sin(2\pi F_0 t)]_{\text{LPF}}$, where the designation "LPF" indicates that the time waveform has been lowpass filtered.

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Problems

20.1. (w) Determine the probability that 5 successive samples $\{X[0], X[1], X[2], X[3], X[4]\}$ of discrete-time WGN with $\sigma_U^2 = 1$ will all exceed zero. Then, repeat the problem if the samples are $\{X[10], X[11], X[12], X[13], X[14]\}$.

20.2 (⌣) **(w)** If $X[n]$ is the random process described in Example 20.2, find $P[X[0] > 0, X[3] > 0]$ if $\sigma_U^2 = 1$.

20.3 (w) If $X[n]$ is a discrete-time Wiener random process with $\text{var}(X[n]) = 2(n+1)$, determine $P[-3 \leq X[5] \leq 3]$.

20.4 (w) A discrete-time Wiener random process $X[n]$ is input to a differencer to generate the output random process $Y[n] = X[n] - X[n-1]$. Describe the characteristics of the output random process.

20.5 (⌣) **(w)** If discrete-time WGN $X[n]$ with $\sigma_X^2 = 1$ is input to a differencer to generate the output random process $Y[n] = X[n] - X[n-1]$, find the PDF of the samples $Y[0], Y[1]$. Are the samples independent?

20.6 (w) If in Example 20.4 the input random process to the differencer is an AR random process with parameters a and $\sigma_U^2 = 1$, determine the PDF of $Y[0], Y[1]$. What happens as $a \to 1$? Explain your results.

20.7 (t) In this problem we argue that if $X[n]$ is a Gaussian random process that is input to an LSI filter so that the output random process is $Y[n] = \sum_{i=-\infty}^{\infty} h[i]X[n-i]$, then $Y[n]$ is also a Gaussian random process. To do so consider a finite impulse response filter so that $Y[n] = \sum_{i=0}^{I-1} h[i]X[n-i]$ with $I = 4$ (the infinite impulse response filter argument is a bit more complicated but is similar in nature) and choose to test the set of output samples $n_1 = 0, n_2 = 1, n_3 = 2$ so that $K = 3$ (again the more general case proceeds similarly). Now prove that the output samples have a 3-dimensional Gaussian PDF. Hint: Show that the samples of $Y[n]$ are obtained as a linear transformation of $X[n]$.

20.8 (w) A discrete-time WGN random process is input to an LSI filter with system function $\mathcal{H}(z) = z - z^{-1}$. Determine the PDF of the output samples $Y[n]$ for $n = 0, 1, \ldots, N-1$. Are any of these samples independent of each other?

20.9 (t) In this problem we prove that if $X[n]$ and $Y[n]$ are both Gaussian random processes that are independent of each other, then $Z[n] = X[n] + Y[n]$ is also a Gaussian random process. To do so we prove that the characteristic function of $\mathbf{Z} = [Z[n_1]\, Z[n_2] \ldots Z[n_K]]^T$ is that of a Gaussian random vector. First note that since $\mathbf{X} = [X[n_1]\, X[n_2] \ldots X[n_K]]^T$ and $\mathbf{Y} = [Y[n_1]\, Y[n_2] \ldots Y[n_K]]^T$ are both Gaussian random vectors (by definition of a Gaussian random process), then each one has the characteristic function

$$\phi(\boldsymbol{\omega}) = \exp\left(j\boldsymbol{\omega}^T\boldsymbol{\mu} - \tfrac{1}{2}\boldsymbol{\omega}^T\mathbf{C}\boldsymbol{\omega}\right)$$

where $\boldsymbol{\omega} = [\omega_1\, \omega_2 \ldots \omega_K]^T$. Next use the property that the characteristic function of a sum of independent random vectors is the product of the characteristic functions to show that $\mathbf{Z}$ has a K-dimensional Gaussian PDF.

20.10 (⌣) **(w)** Let $X[n]$ and $Y[n]$ be WSS Gaussian random processes with zero mean and independent of each other. It is known that $Z[n] = X[n]Y[n]$ is not a Gaussian random process. However, can we say that $Z[n]$ is a WSS random process, and if so, what is its mean and PSD?

20.11 (w) An AR random process is described by $X[n] = \frac{1}{2}X[n-1] + U[n]$, where $U[n]$ is WGN with $\sigma_U^2 = 1$. This random process is input to an LSI filter with system function $\mathcal{H}(z) = 1 - \frac{1}{2}z^{-1}$ to generate the output random process $Y[n]$. Find $P[Y^2[0] + Y^2[1] > 1]$. Hint: Consider $X[n]$ as the output of an LSI filter.

20.12 (t) We prove (20.11) in this problem by using the method of characteristic functions. Recall that for a multivariate zero mean Gaussian PDF the characteristic function is

$$\phi_{\mathbf{X}}(\boldsymbol{\omega}) = \exp\left(-\tfrac{1}{2}\boldsymbol{\omega}^T \mathbf{C}\boldsymbol{\omega}\right)$$

and the fourth-order moment can be found using (see Section 14.6)

$$E[X_1X_2X_3X_4] = \left.\frac{\partial^4 \phi_{\mathbf{X}}(\boldsymbol{\omega})}{\partial\omega_1\partial\omega_2\partial\omega_3\partial\omega_4}\right|_{\boldsymbol{\omega}=\mathbf{0}}.$$

Although straightforward, the algebra is tedious (see also Example 14.5 for the second-order moment calculations). To avoid frustration (with $P[\text{frustration}] = 1$) note that

$$\boldsymbol{\omega}^T\mathbf{C}\boldsymbol{\omega} = \sum_{i=1}^{4}\sum_{j=1}^{4}\omega_i\omega_j E[X_iX_j]$$

and let $L_i = \sum_{j=1}^{4}\omega_j E[X_iX_j]$. Next show that

$$\begin{aligned}\frac{\partial \phi_{\mathbf{X}}(\boldsymbol{\omega})}{\partial\omega_k} &= -\phi_{\mathbf{X}}(\boldsymbol{\omega})L_k \\ \frac{\partial L_i}{\partial\omega_k} &= E[X_iX_k]\end{aligned}$$

and finally note that $L_i|_{\boldsymbol{\omega}=\mathbf{0}} = 0$ to avoid some algebra in the last differentiation.

20.13 (w) It is desired to estimate $r_X[0]$ for $X[n]$ being WGN. If we use the estimator, $\hat{r}_X[0] = (1/N)\sum_{n=0}^{N-1}X^2[n]$, determine the mean and variance of $\hat{r}_X[0]$. Hint: Use (20.13).

20.14 (⌣) (f) If $X[n] = U[n] + U[n-1]$, where $U[n]$ is a WGN random process with $\sigma_U^2 = 1$, find $E[X[0]X[1]X[2]X[3]]$.

20.15 (f) Find the PSD of $X^2[n]$ if $X[n]$ is WGN with $\sigma_X^2 = 2$.

20.16 (t) To argue that the continuous-time Wiener random process is a Gaussian random process, we replace $X(t) = \int_0^t U(\xi)d\xi$, where $U(\xi)$ is continuous-time WGN, by the approximation

$$\hat{X}(t) = \sum_{n=0}^{[t/\Delta t]} Z(n\Delta t)\Delta t$$

where $[x]$ indicates the largest integer less than or equal to x and $Z(t)$ is a zero mean WSS Gaussian random process. The PSD of $Z(t)$ is given by

$$P_Z(F) = \begin{cases}\frac{N_0}{2} & |F| \le W \\ 0 & |F| > W\end{cases}$$

where $W = 1/(2\Delta t)$. Explain why $\bar{X}(t)$ is a Gaussian random process. Next let $\Delta t \to 0$ and explain why $\bar{X}(t)$ becomes a Wiener random process.

20.17 (⌣̈) (w) To extract A from a realization of the random process $X(t) = A + U(t)$, where $U(t)$ is WGN with PSD $P_U(F) = 1$ for all F, it is proposed to use

$$\hat{A} = \frac{1}{T}\int_0^T X(\xi)d\xi.$$

How large should T be chosen to ensure that $P[|\hat{A} - A| \leq 0.01] = 0.99$?

20.18 (w) To generate a realization of a continuous-time Wiener random process on a computer we must replace the continuous-time random process by a sampled approximation. To do so note that we can first describe the Wiener random process by breaking up the integral into integrals over smaller time intervals. This yields

$$\begin{aligned} X(t) &= \int_0^t U(\xi)d\xi \\ &= \sum_{i=1}^{n} \underbrace{\int_{t_{i-1}}^{t_i} U(\xi)d\xi}_{X_i} \end{aligned}$$

where $t_i = i\Delta t$ with Δt very small, and $t_n = n\Delta t = t$. It is assumed that $t/\Delta t$ is an integer. Thus, the samples of $X(t)$ are conveniently found as

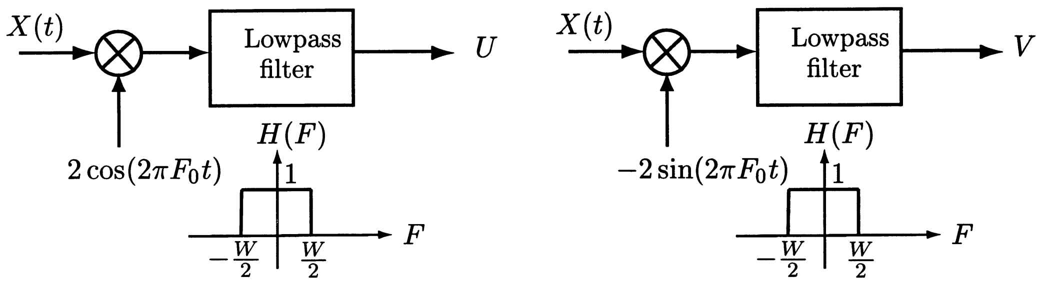
$$X(t_n) = X(n\Delta t) = \sum_{i=1}^{n} X_i$$

and the approximation is completed by connecting the samples $X(t_n)$ by straight lines. Find the PDF of the X_i's to determine how they should be generated. Hint: The X_i's are increments of $X(t)$.

20.19 (⌣̈) (f) For a continuous-time Wiener random process with $\text{var}(X(t)) = t$, determine $P[|X(t)| > 1]$. Explain what happens as $t \to \infty$ and why.

20.20 (w) Show that if $X(t)$ is a Rayleigh fading sinusoid, the "demodulation" and lowpass filtering shown in Figure 20.15 will yield U and V, respectively. What should the bandwidth of the lowpass filter be?

20.21 (c) Generate 10 realizations of a Rayleigh fading sinusoid for $0 \leq t \leq 1$. Use $F_0 = 10$ Hz and $\sigma^2 = 1$ to do so. Overlay your realizations. Hint: Replace $X(t) = U\cos(2\pi F_0 t) - V\sin(2\pi F_0 t)$ by $X[n] = X(n\Delta t) = U\cos(2\pi F_0 n\Delta t) - V\sin(2\pi F_0 n\Delta t)$ for $n = 0, 1, \ldots, N\Delta t$, where $\Delta t = 1/N$ and N is large.



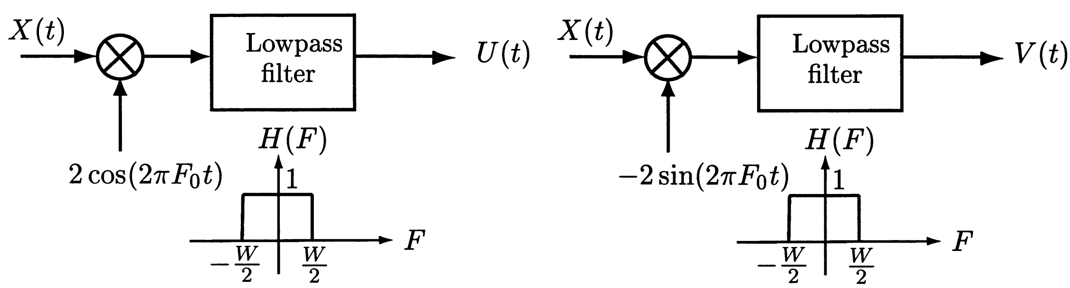
(a) (b)

Figure 20.15: Extraction of Rayleigh fading sinusoid lowpass components for Problem 20.20.

20.22 (⌣̈) **(w)** Consider $X_1(t)$ and $X_2(t)$, which are both Rayleigh fading sinusoids with frequency $F_0 = 1/2$ and which are independent of each other. Each random process has the total average power $\sigma^2 = 1$. If $Y(t) = X_1(t) + X_2(t)$, find the joint PDF of $Y(0)$ and $Y(1/4)$.

20.23 (f) A Rayleigh fading sinusoid has the PSD $P_X(F) = \delta(F+10) + \delta(F-10)$. Find the PSDs of $U(t)$ and $V(t)$ and plot them.

20.24 (w) Show that if $X(t)$ is a bandpass random process, the "demodulation" and lowpass filtering given in Figure 20.16 will yield $U(t)$ and $V(t)$, respectively.



(a) (b)

Figure 20.16: Extraction of bandpass random process lowpass components for Problem 20.24.

20.25 (⌣̈) **(f)** If a bandpass random process has the PSD shown in Figure 20.17, find the PSD of $U(t)$ and $V(t)$.

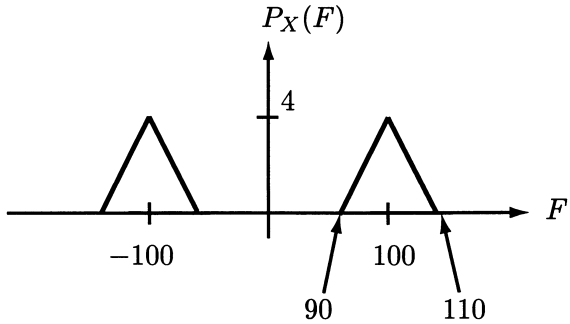


Figure 20.17: PSD for bandpass random process for Problem 20.25.

20.26 (c) The random process whose realization is shown in Figure 20.2 appears to be similar in nature to the bandpass random processes shown in Figure 20.13b. We have already seen that the marginal PDF appears to be Gaussian (see Figure 20.3). To see if it is reasonable to model it as a bandpass random process we estimate the PSD. First run the code given in Appendix 20A to produce the realization shown in Figure 20.2. Then, run the code given below to estimate the PSD using an averaged periodogram (see also Section 17.7 for a description of this). Does the estimated PSD indicate that the random process is a bandpass random process? If so, explain how you can give a *complete* probabilistic model for this random process.

```
Fs=100;  % set sampling rate for later plotting
L=50;I=20; % L = length of block, I = number of blocks
n=[0:I*L-1]'; % set up time indices
Nfft=1024; % set FFT length for Fourier transform
Pav=zeros(Nfft,1);
f=[0:Nfft-1]'/Nfft-0.5; % set discrete-time frequencies
for i=0:I-1
   nstart=1+i*L;nend=L+i*L; % set start and end time indices
                            % of block
   y=x(nstart:nend); % extract block of data
Pav=Pav+(1/(I*L))*abs(fftshift(fft(y,Nfft))).^2;
                                          % compute periodogram
                                          % and add to average
                                          % of periodograms
end
F=f*Fs; % convert to continuous-time (analog) frequency in Hz
Pest=Pav/Fs; % convert discrete-time PSD to continuous-time PSD
plot(F,Pest)
```

20.27 (f) For the Gaussian random process with mean zero and PSD

$$P_X(F) = \begin{cases} 4 & 90 \leq |F| \leq 110 \\ 0 & \text{otherwise} \end{cases}$$

find the probability that its envelope will be less than or equal to 10 at $t = 10$ seconds. Repeat the calculation if $t = 20$ seconds.

20.28 (w) Prove that $X_1[n] = \frac{1}{2}U[n+1] + \frac{1}{2}U[n-1]$ and $X_2[n] = \frac{1}{2}U[n] + \frac{1}{2}U[n-2]$, where $U[n]$ is WGN with $\sigma_U^2 = 1$, both have the same PSD given by $P_X(F) = \frac{1}{2}(1 + \cos(4\pi f))$.

20.29 (w) It is desired to generate a realization of a WSS Gaussian random process by filtering WGN with an LSI filter. If the desired PSD is $P_X(f) = |1 - \frac{1}{2}\exp(-j2\pi f)|^2$, explain how to do this.

20.30 (⌣̈) (w) It is desired to generate a realization of a WSS Gaussian random process by filtering WGN with an LSI filter. If the desired PSD is $P_X(f) = 2 - 2\cos(2\pi f)$, explain how to do this.

20.31 (⌣̈) (c) Using the results of Problem 20.30, generate a realization of $X[n]$. To verify that your data generation appears correct, estimate the ACS for $k = 0, 1, \ldots, 9$ and compare it to the theoretical ACS.

Appendix 20A

MATLAB Listing for Figure 20.2

```
clear all
rand('state',0)
t=[0:0.01:0.99]'; %  set up transmit pulse time interval
F0=10;
s=cos(2*pi*F0*t); %  transmit pulse
ss=[s;zeros(1000-length(s),1)];  % put transmit pulse in receive window
tt=[0:0.01:9.99]';  % set up receive window time interval
x=zeros(1000,1);
for i=1:100  % add up all echos, one for each 0.1 sec interval
tau=round(10*i+10*(rand(1,1)-0.5));  % time delay for each 0.1 sec interval
                                     % is uniformly distributed - round
                                     % time delay to integer
  x=x+rand(1,1)*shift(ss,tau);
end
```

shift.m subprogram

```
% shift.m
%
function y=shift(x,Ns)
%
%  This function subprogram shifts the given sequence by Ns points.
%  Zeros are shifted in either from the left or right.
%
%  Input parameters:
%    x  - array of dimension Lx1
%    Ns - integer number of shifts where Ns>0 means a shift to the
```

```
%         right and Ns<0 means a shift to the left and if Ns=0, then
%         the sequence is not shifted
%
%   Output parameters:
%    y  - array of dimension Lx1 containing the
%         shifted sequence
L=length(x);
if abs(Ns)>L
   y=zeros(L,1);
   else
if Ns>0
y(1:Ns,1)=0;
y(Ns+1:L,1)=x(1:L-Ns);
elseif Ns<0
   y(L-abs(Ns)+1:L,1)=0;
   y(1:L-abs(Ns),1)=x(abs(Ns)+1:L);
else
   y=x;
end
end
```