Chapter 22

Markov Chains

22.1 Introduction

We have seen in Chapter 16 that an important random process is the IID random process. When applicable to a specific problem, it lends itself to a very simple analysis. A Bernoulli random process, which consists of independent Bernoulli trials, is the archetypical example of this. In practice, it is found, however, that there is usually some dependence between samples of a random process. In Chapters 17 and 18 we modeled this dependence using wide sense stationary random process theory, but restricted the modeling to only the first two moments. In an effort to introduce a more general dependence into the modeling of a random process, we now reconsider the Bernoulli random process but assume dependent samples. We briefly introduced this extension in Example 4.10 as a sequence of *dependent* Bernoulli trials. The dependence of the PMF that we will be interested in is dependence *on the previous trial only.* This type of dependence leads to what is generically referred to as a *Markov random process.* A special case of this for a *discrete-time/discrete-valued* (DTDV) random process is called a *Markov chain.* Specifically, it has the property that the probability of the random process $X[n]$ at time $n = n_0$ only depends upon the *outcome* or *realization* of the random process at the previous time $n = n_0 - 1$. It can then be viewed as the next logical step in extending an IID random process to a random process with statistical dependence. Recall from Chapter 8 that for discrete random variables statistical dependence is quantified using conditional probabilities. The reader should review Example 4.10 and also Chapter 8 in preparation for our discussion of Markov chains.

Although we will restrict our description to a DTDV Markov random process, i.e., the Markov chain, there are many generalizations that are important in practice. The interested reader can consult the excellent books by [Bharucha-Reid 1988], [Cox and Miller 1965], [Gallagher 1996] and [Parzen 1962] for these other random processes. Before proceeding with our discussion we present an example to illustrate typical concepts associated with a Markov chain.

In the game of golf it is very desirable to be a good putter. The best golfers in the world are able to hit a golf ball lying on the green into the hole using only a few strokes, called *putting the ball.* At times they can even "one-putt" the ball, in which they require only a single stroke to hit the ball into the hole. Of course, their chances of doing so rely heavily on how far the ball is from the hole when they first reach the green. If the ball is say 3 feet from the hole, then they will almost always one-putt. If, however, it is near the edge of the green, possibly 20 feet from the hole, then their chances are small. For our hypothetical golfer we will assume that her chance of a one-putt is 50% at the start of a round of golf, i.e., at hole one. If she one-putts on hole one, then her chances on hole two will remain at 50%. If not, she becomes somewhat discouraged which reduces her chances at hole two to only 25%. Hence, at each hole her chances of a one-putt are 50% if she has one-putted the previous hole and 25% if she has not. To model this situation we let $X[n] = 1$ for a one-putt at hole n and $X[n] = 0$ otherwise. We label hole one by $n = 0$. A round of golf, which consists of 18 holes, produces a sequence of 18 1's and 0's with a 1 indicating a one-putt. For the probabilities assumed a typical set of outcomes is shown in Figure 22.1. Note that she has played three rounds of

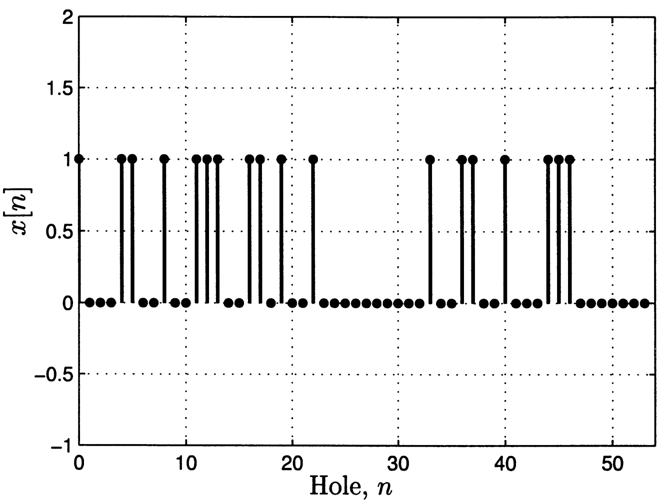

Figure 22.1: Outcomes of three rounds of golf. A 1 indicates a one-putt on hole n.

golf or 54 holes, of which 18 were one-putts. It appears that her probability of a one-putt is closer to 1/3 than either 1/2 or 1/4. Also, it is of interest to determine the average number of holes played between one-putts. The actual number varies as seen in Figure 22.1 and is $\{4, 1, 3, 3, 1, 1, 3, 1, 2, 3, 11, 3, 1, 3, 4, 1, 1\}$ for an average of $46/17 = 2.70$. It would seem that the expected number of holes played between one-putts, about 3, is the reciprocal of the probability of a one-putt, about 1/3. This suggests a geometric-type PMF, which we will confirm in Section 22.6.

Probabilistically, we are observing a sequence of *dependent Bernoulli trials.* The dependence arises (in contrast to the usual Bernoulli random process which had independent trials) due to the probability of a one-putt at hole n being dependent upon the outcome at hole $n-1$. We can model this dependence using conditional probabilities to say that

$$\begin{aligned} P[\text{one-putt at hole } n|\text{no one-putt at hole } n-1] &= \frac{1}{4} \\ P[\text{one-putt at hole } n|\text{one-putt at hole } n-1] &= \frac{1}{2} \end{aligned}$$

or

$$\begin{aligned} P[X[n]=1|X[n-1]=0] &= \frac{1}{4} \\ P[X[n]=1|X[n-1]=1] &= \frac{1}{2}. \end{aligned}$$

Completing the conditional probability description, we have

$$\begin{aligned} P[X[n]=0|X[n-1]=0] &= \frac{3}{4} \\ P[X[n]=1|X[n-1]=0] &= \frac{1}{4} \\ P[X[n]=0|X[n-1]=1] &= \frac{1}{2} \\ P[X[n]=1|X[n-1]=1] &= \frac{1}{2}. \end{aligned}$$

Note that we have assumed that the *conditional probabilities do not change with "time"* (actually hole number). Lastly, we require the initial probability of a one-putt for the first hole. We assign this to be $P[X[0]=1]=1/2$. In summary, we have two sets of conditional probabilities and one set of initial probabilities which can be arranged conveniently using a matrix and vector to be

$$\begin{aligned} \mathbf{P} &= \begin{bmatrix} P[X[n]=0|X[n-1]=0] & P[X[n]=1|X[n-1]=0] \\ P[X[n]=0|X[n-1]=1] & P[X[n]=1|X[n-1]=1] \end{bmatrix} \qquad (22.1) \\ &= \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathbf{p}[0] &= \begin{bmatrix} P[X[0]=0] \\ P[X[0]=1] \end{bmatrix} \qquad (22.2) \\ &= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}. \end{aligned}$$

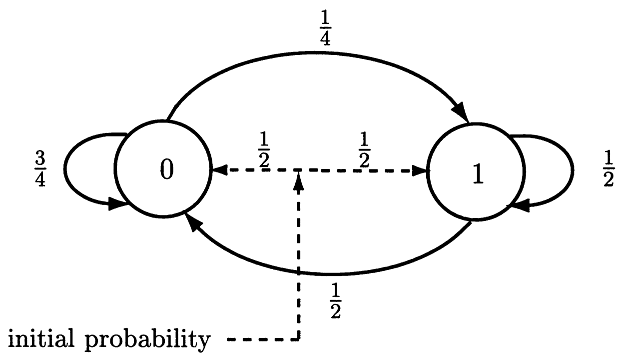

Figure 22.2: Markov state probability diagram for putting example.

The probabilities can also be summarized using the diagram shown in Figure 22.2, where for example the *conditional probability* of a one-putt on hole n given that the golfer has not one-putted on hole $n-1$ is $1/4$. We may view this diagram as one in which we are in "state" 0, which corresponds to the previous outcome of no one-putt and will move to "state" 1, which corresponds to a one-putt, with a conditional probability of $1/4$. If we do move to a new state, it means the outcome is a 1 and otherwise, the outcome is a 0. In interpreting the diagram one should visualize that a 0 or 1 is emitted as we *enter* the 0 or 1 state, respectively. Then, the current state becomes the last value emitted. Also, our initial *unconditional probabilities* of $1/2$ and $1/2$ of entering state 0 or state 1 are shown as dashed lines. The diagram is called the *Markov state probability diagram.* The use of the term "state" is derived from physics in that the future evolution (in terms of probabilities) of the process is only dependent upon the current state and not upon how the process arrived in that state. The probabilistic structure summarized in Figure 22.2 is called a *Markov chain.* As mentioned previously, it is a DTDV random process. Although we have used a dependent Bernoulli random process as an example, it easily generalizes to any finite number of states. It is common in the discussion of Markov chains to term the matrix of conditional probabilities $\mathbf{P}$ in (22.1) as the *state transition probability matrix* or more succinctly the *transition probability matrix.* The initial probability vector $\mathbf{p}[0]$ in (22.2) is called the *initial state probability vector* or more succinctly the *initial probability vector.* Note that in using the state probability diagram to summarize the Markov chain we will henceforth omit the initial probability assignment in the diagram but it should be kept in mind that it is necessary in order to complete the description.

As an example of a typical probability computation, consider the probability of $X[0]=0, X[1]=1, X[2]=1$ versus $X[0]=1, X[1]=1, X[2]=1$. Then, using the chain rule (see (4.10)) we have

$$\begin{aligned} P[X[0]=0, X[1]=1, X[2]=1] &= P[X[2]=1|X[1]=1, X[0]=0] \\ &\quad \cdot P[X[1]=1|X[0]=0]P[X[0]=0]. \end{aligned}$$

But due to the assumption that the probability of $X[n]$ only depends upon the outcome at time $n-1$, which is called the *Markov property*, we have

$$P[X[2]=1|X[1]=1,X[0]=0]=P[X[2]=1|X[1]=1]$$

and therefore

$$\begin{aligned} P[X[0]=0,X[1]=1,X[2]=1] &= P[X[2]=1|X[1]=1]P[X[1]=1|X[0]=0] \\ &\quad \cdot P[X[0]=0]. \end{aligned}$$

But from Figure 22.2 this is

$$P[X[0]=0,X[1]=1,X[2]=1]=\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)=\frac{1}{16}.$$

Similary,

$$\begin{aligned} P[X[0]=1,X[1]=1,X[2]=1] &= P[X[2]=1|X[1]=1]P[X[1]=1|X[0]=1] \\ &\quad \cdot P[X[0]=1]=\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=\frac{1}{8}. \end{aligned}$$

We see that joint probabilities are easily determined from the initial probabilities and the transition probabilities. If we are only interested in the marginal PMF at a given time say $P[X[n]=k]$ for $k=0,1$, as, for example, $P[X[2]=1]$, we need only sum over the other variables of the joint PMF. This produces

$$\begin{aligned} P[X[2]=1] &= \sum_{i=0}^{1}\sum_{j=0}^{1} P[X[0]=i,X[1]=j,X[2]=1] \\ &= \sum_{i=0}^{1}\sum_{j=0}^{1} P[X[2]=1|X[0]=i,X[1]=j]P[X[1]=j|X[0]=i] \\ &\qquad \cdot P[X[0]=i] \\ &= \sum_{i=0}^{1}\sum_{j=0}^{1} P[X[2]=1|X[1]=j]P[X[1]=j|X[0]=i]P[X[0]=i] \\ &\qquad \text{(Markov property)} \\ &= \sum_{j=0}^{1} P[X[2]=1|X[1]=j]\underbrace{\sum_{i=0}^{1} P[X[1]=j|X[0]=i]P[X[0]=i]}_{P[X[1]=j]}. \end{aligned}$$

Note that $P[X[1]=j]$ can be found and then used to find $P[X[2]=1]$. Of course, this is getting somewhat messy algebraically but as shown in the next section the use of vectors and matrices will simplify the computation.

Finally, some questions of interest to the golfer are:

1. After playing many holes, will the probability of a one-putt settle down to some constant value? Mathematically, will $P[X[n] = k]$ converge to some constant PMF as $n \to \infty$?

2. Given that the golfer has just one-putted, how many holes on the average will she have to wait until the next one-putt? Or given that she has not one-putted, how many holes on the average will she have to wait until she one-putts? In the first case, mathematically we wish to determine if given $X[n_0] = 1$ and $X[n_0 + 1] = 0, \ldots, X[n_0 + N - 1] = 0, X[n_0 + N] = 1$, what is $E[N]$?

We will answer both these questions shortly, but before doing so some definitions are necessary.

22.2 Summary

A motivating example of a Markov chain is given in Section 22.1. A Markov chain is defined by the property of (22.3). The state transition probabilities, which describe the probabilities of movements between states, is given by (22.4). When arranged in a matrix it is equivalent to (22.5) for a two-state Markov chain and is called the transition probability matrix. The probabilities of the states are defined in (22.6) and succinctly summarized by the vector of (22.7) for a two-state Markov chain. Table 22.1 summarizes the notational conventions. The state probability vector can be found for any time by using (22.9). To evaluate a power of the transition probability matrix (22.12) can be used if the eigenvalues of the matrix are distinct. For a two-state Markov chain the state probabilities are explicitly found in Section 22.4 with the general transition probability matrix given by (22.14). For an ergodic Markov chain the state probabilities approach a constant value as time increases and this value is found by solving (22.17). Also, the value of the n-step transition probability matrix approaches the steady-state value given by (22.19). In Section 22.6 the occupation time of a state for an ergodic Markov chain is shown to be given by the steady-state probabilities and also, the mean recurrence time is the inverse of the occupation time. An explicit solution for the steady-state or stationary probabilities can be found using (22.22). The MATLAB code for a computer simulation of a 3-state Markov chain is given in Section 22.8 while a concluding real-world example is given in Section 22.9.

22.3 Definitions

We restrict ourselves to a discrete-time random process $X[n]$ with K possible values or states. In the introduction $K = 2$ and the values were $0, 1$. This is a DTDV random process that starts at $n = 0$ (semi-infinite). We define $X[n]$ as a Markov chain if *given* the entire past set of outcomes, the PMF of $X[n]$ depends on only the outcome of the previous sample $X[n-1]$ so that

$$P[X[n] = j|X[n-1] = i, X[n-2] = k, \ldots, X[0] = l] = P[X[n] = j|X[n-1] = i].$$

Using the concept of a PMF this is equivalent to

$$p_{X[n]|X[n-1],\ldots,X[0]} = p_{X[n]|X[n-1]}. \tag{22.3}$$

This implies that the joint PMF only depends on the product of the *first-order* conditional PMFs and the initial probabilities, for example

$$\begin{aligned} p_{X[0],X[1],X[2]} &= p_{X[2]|X[1],X[0]} p_{X[1]|X[0]} p_{X[0]} \\ &= \underbrace{p_{X[2]|X[1]}}_{\substack{\text{conditional} \\ \text{probability}}} \underbrace{p_{X[1]|X[0]}}_{\substack{\text{conditional} \\ \text{probability}}} \underbrace{p_{X[0]}}_{\substack{\text{initial} \\ \text{probability}}} . \end{aligned}$$

As mentioned previously, this is an extension of the idea of independence in that it asserts a type of *conditional independence.* Most importantly, the *joint PMF* is obtained as the product of *first-order conditional PMFs.* An example follows.

Example 22.1 – A coin with memory

Assume that a coin is tossed three times with the outcome of a head represented by a 1 and a tail by a 0. If the coin has memory and is modeled by the state probability diagram of Figure 22.2, determine the probability of the sequence HTH. Note that the conditional probabilities are equivalent to those in Example 4.10. Writing the joint probability in the more natural order of increasing time, we have

$$\begin{aligned} P[X[0] = 1, X[1] = 0, X[2] = 1] &= P[X[0] = 1]P[X[1] = 0|X[0] = 1] \\ &\quad \cdot P[X[2] = 1|X[1] = 0] \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{16}. \end{aligned}$$

Hence, the sequence HTH is less probable than for a fair coin without memory for which 3 independent tosses would yield a probability of 1/8. Can you explain why this is less probable?

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We will now use the terminology of the introduction to refer to the conditional probabilities $P[X[n] = j|X[n-1] = i]$ as the *state transition probabilities.* Note that they are assumed not to depend on n and therefore the Markov chain is said to be *homogeneous.* To simplify the notation further and to prepare for subsequent probability calculations we denote the state transition probabilities as

$$P_{ij} = P[X[n] = j|X[n-1] = i] \qquad i = 0, 1, \ldots, K-1; j = 0, 1, \ldots, K-1. \tag{22.4}$$

This is the conditional probability of observing an outcome j given that the previous outcome was i. It is also said that P_{ij} is the probability of the chain moving from state i to state j, but keep in mind that it is a conditional probability. In the case of a two-state Markov chain or $K = 2$, we have $i = 0, 1; j = 0, 1$ and the state transition probabilities are most conveniently arranged in a matrix $\mathbf{P}$. From (22.1) we have

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \tag{22.5}$$

which as previously mentioned is the transition probability matrix. Note that the sum of the elements along each row must be one since they represent all the values of a conditional PMF. In accordance with the assumption of homogeneity $\mathbf{P}$ is a constant matrix. Finally, we define the state probabilities at time n as

$$p_i[n] = P[X[n] = i] \qquad i = 0, 1, \ldots, K-1. \tag{22.6}$$

This is the probability of observing an outcome i at time n or equivalently the PMF of $X[n]$. This notation is somewhat at odds with our previous notation, which would be $p_{X[n]}[i]$, but is a standard one. The PMF depends on n and it is this PMF that we will be most concerned. In particular, how the PMF changes with n will be of interest. Hence, a Markov chain is in general *a nonstationary random process.* For ease of notation and later computation we also define the *state probability vector* for $K = 2$ as

$$\mathbf{p}[n] = \begin{bmatrix} p_0[n] \\ p_1[n] \end{bmatrix}. \tag{22.7}$$

A summary of these definitions and notation is given in Table 22.1. An example is given next to illustrate the utility of definitions (22.4) and (22.6) and their vector/matrix representations of (22.5) and (22.7).

Example 22.2 – Two-state Markov chain

Consider the computation of $P[X[n] = j]$ for a two-state Markov chain ($K = 2$). Then,

$$\begin{aligned} P[X[n] = j] &= \sum_{i=0}^{1} P[X[n-1] = i, X[n] = j] \\ &= \sum_{i=0}^{1} P[X[n] = j | X[n-1] = i] P[X[n-1] = i] \end{aligned}$$

which can now be written as

$$p_j[n] = \sum_{i=0}^{1} P_{ij} p_i[n-1] \qquad j = 0, 1.$$

In vector/matrix notation we have

$$\underbrace{\begin{bmatrix} p_0[n] & p_1[n] \end{bmatrix}}_{\mathbf{p}^T[n]} = \underbrace{\begin{bmatrix} p_0[n-1] & p_1[n-1] \end{bmatrix}}_{\mathbf{p}^T[n-1]} \underbrace{\begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}}_{\mathbf{P}}$$

Terminology	Description	Notation
Random process	DTDV	$X[n] \quad n = 0, 1, \ldots$
State	Sample space	$k = 0, 1, \ldots, K-1$
State probability vector	PMF of $X[n]$	$\mathbf{p}[n] = [p_0[n] \ldots p_{K-1}[n]]^T$ $p_k[n] = P[X[n] = k]$
State transition probability matrix	Conditional prob.	$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0,K-1} \\ P_{10} & P_{11} & \cdots & P_{1,K-1} \\ \vdots & \vdots & \ddots & \vdots \\ P_{K-1,0} & P_{K-1,1} & \cdots & P_{K-1,K-1} \end{bmatrix}$ $P_{ij} = P[X[n] = j \mid X[n-1] = i]$
Initial state probability vector	PMF of $X[0]$	$\mathbf{p}[0]$

Table 22.1: Markov chain definitions and notation.

or

$$\mathbf{p}^T[n] = \mathbf{p}^T[n-1]\mathbf{P}. \tag{22.8}$$

The evolution of the state probability vector in time is easily found by post-multiplying the previous state probability vector (in row form) by the transition probability matrix.

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Note that we have defined $\mathbf{p}[n]$ as a column vector in accordance with our usual convention. Other textbooks may use row vectors. A numerical example follows.

Example 22.3 – Golfer one-putting

From Figure 22.2 we have the transition probability matrix and initial state probability vector as

$$\begin{aligned} \mathbf{P} &= \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ \mathbf{p}^T[0] &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \end{aligned}$$

To find $\mathbf{p}[1]$ we use (22.8) to yield

$$\begin{aligned}\mathbf{p}^T[1] &= \mathbf{p}^T[0]\mathbf{P} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \end{bmatrix}.\end{aligned}$$

As expected the elements of $\mathbf{p}[1]$ sum to one. Also, note that $p_1[1] = 3/8 < 1/2$, which means that initially the probability of a one-putt is 1/2 but after the first hole, it is reduced to 3/8. Can you explain why? We can continue in this manner to compute the state probability vector for $n = 2$ as

$$\begin{aligned}\mathbf{p}^T[2] &= \mathbf{p}^T[1]\mathbf{P} \\ &= \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{21}{32} & \frac{11}{32} \end{bmatrix}\end{aligned}$$

and so forth for all n.

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22.4 Computation of State Probabilities

We are now in a position to determine $\mathbf{p}[n]$ for all n. The key of course is the recursion of (22.8). In a slightly more general form where we wish to go from $\mathbf{p}[n_1]$ to $\mathbf{p}[n_2]$, the resulting equations are known as the *Chapman-Kolmogorov* equations. For example, if $n_2 = n_1 + 2$, then

$$\begin{aligned}\mathbf{p}^T[n_2] &= \mathbf{p}^T[n_2 - 1]\mathbf{P} \\ &= (\mathbf{p}^T[n_2 - 2]\mathbf{P})\mathbf{P} \\ &= \mathbf{p}^T[n_1]\mathbf{P}^2.\end{aligned}$$

The matrix $\mathbf{P}^2$ is known as the *two-step transition probability matrix.* It allows the state probabilities for two steps into the future to be found if we know the state probabilities at the current time. In general, then we see that

$$\mathbf{p}^T[n_1 + n] = \mathbf{p}^T[n_1]\mathbf{P}^n$$

as is easily verified, where $\mathbf{P}^n$ is the *n-step transition probability matrix.* In particular, if $n_1 = 0$, then

$$\mathbf{p}^T[n] = \mathbf{p}^T[0]\mathbf{P}^n \qquad n = 1, 2, \ldots \tag{22.9}$$

which can be used to find the state probabilities for all time. These probabilities can exhibit markedly different behaviors depending upon the entries in $\mathbf{P}$. To illustrate

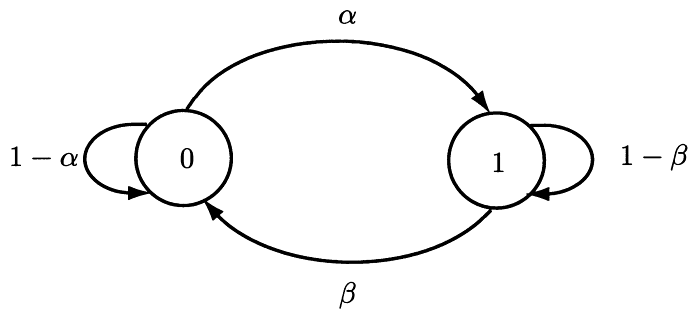

Figure 22.3: General two-state probability diagram.

this consider the two-state Markov chain with the state probability diagram shown in Figure 22.3. This corresponds to the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \tag{22.10}$$

where $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. As always the rows sum to one. We give an example and then generalize the results.

Example 22.4 – State probability vector computation for all n

Let $\alpha = \beta = 1/2$ and $\mathbf{p}^T[0] = [1\,0]$ so that we are intially in state 0 and the transition to either of the states is equally probable. Then from (22.9) we have

$$\begin{aligned} \mathbf{p}^T[n] &= \mathbf{p}^T[0]\mathbf{P}^n \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^n \\ \mathbf{p}^T[1] &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ \mathbf{p}^T[2] &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

Clearly, $\mathbf{p}^T[n] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ for all $n \geq 1$. The Markov chain is said to be in *steady-state* for $n \geq 1$. In addition, for $n \geq 1$, the PMF $\mathbf{p}^T[n] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is called the *steady-state PMF*.

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More generally, the state probabilities of a Markov chain may or may not approach a steady-state value. It depends upon the form of $\mathbf{P}$. To study the behavior more thoroughly we require a means of determining $\mathbf{P}^n$. To do so we next review the diagonalization of a matrix using an eigenanalysis (see also Appendix D).

Computing Powers of P

Assuming that the eigenvalues of $\mathbf{P}$ are distinct, it is possible to find eigenvectors $\mathbf{v}_i$ that are linearly independent. Arranging them as the columns of a matrix and assuming that $K = 2$, we have the modal matrix $\mathbf{V} = [\mathbf{v}_1\,\mathbf{v}_2]$ which is a nonsingular matrix since the eigenvectors are linearly independent. Then we can write that

$$\mathbf{V}^{-1}\mathbf{P}\mathbf{V} = \mathbf{\Lambda} \tag{22.11}$$

where $\mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2)$ and λ_i is the eigenvalue corresponding to the ith eigenvector of $\mathbf{P}$. Now from (22.11) we have that $\mathbf{P} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$ and therefore, the powers of $\mathbf{P}$ can be found as follows.

$$\begin{aligned}
\mathbf{P}^2 &= (\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1})(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}) = \mathbf{V}\mathbf{\Lambda}^2\mathbf{V}^{-1} \\
\mathbf{P}^3 &= \mathbf{P}^2\mathbf{P} = (\mathbf{V}\mathbf{\Lambda}^2\mathbf{V}^{-1})\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} = \mathbf{V}\mathbf{\Lambda}^3\mathbf{V}^{-1}
\end{aligned}$$

and in general we have that

$$\mathbf{P}^n = \mathbf{V}\mathbf{\Lambda}^n\mathbf{V}^{-1}. \tag{22.12}$$

But since $\mathbf{\Lambda}$ is a diagonal matrix its powers are easily found as

$$\mathbf{\Lambda}^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$$

and finally we have that

$$\mathbf{P}^n = \mathbf{V}\begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}\mathbf{V}^{-1}. \tag{22.13}$$

It should be observed that the eigenvectors need not be normalized to unity length for (22.13) to hold. As an example, if

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

then the eigenvalues are found from the characteristic equation as the solutions of $\det(\mathbf{P} - \lambda\mathbf{I}) = 0$. This yields the equation $(1/2 - \lambda)(1 - \lambda) = 0$ which produces $\lambda_1 = 1/2$ and $\lambda_2 = 1$. The eigenvectors are found from

$$\begin{aligned}
(\mathbf{P} - \lambda_1\mathbf{I})\mathbf{v}_1 &= \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}\mathbf{v}_1 = \mathbf{0} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
(\mathbf{P} - \lambda_2\mathbf{I})\mathbf{v}_2 &= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\mathbf{v}_2 = \mathbf{0} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

and hence the modal matrix and its inverse are

$$\begin{aligned}\mathbf{V} &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ \mathbf{V}^{-1} &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.\end{aligned}$$

Finally, for $n \geq 1$ we can easily find the powers of $\mathbf{P}$ from (22.13) as

$$\begin{aligned}\mathbf{P}^n &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1}{2}\right)^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{1}{2}\right)^n & 1-\left(\frac{1}{2}\right)^n \\ 0 & 1 \end{bmatrix}.\end{aligned}$$

This can easily be verified by direct multiplication of $\mathbf{P}$.

Δ

Now returning to the problem of determining the state probability vector for the general two-state Markov chain, we need to first find the eigenvalues of (22.10). The characteristic equation is

$$\det(\mathbf{P} - \lambda\mathbf{I}) = \det \begin{bmatrix} 1-\alpha-\lambda & \alpha \\ \beta & 1-\beta-\lambda \end{bmatrix} = 0$$

which produces $(1-\alpha-\lambda)(1-\beta-\lambda) - \alpha\beta = 0$ or

$$\lambda^2 + (\alpha+\beta-2)\lambda + (1-\alpha-\beta) = 0.$$

Letting $r = \alpha + \beta$, which is nonnegative, we have that $\lambda^2 + (r-2)\lambda + (1-r) = 0$ for which the solution is

$$\begin{aligned}\lambda &= \frac{-(r-2) \pm \sqrt{(r-2)^2 - 4(1-r)}}{2} \\ &= \frac{-(r-2) \pm r}{2} \\ &= 1 \text{ and } 1-r.\end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1-\alpha-\beta$. Next we determine the corresponding eigenvectors as

$$(\mathbf{P} - \lambda_1\mathbf{I})\mathbf{v}_1 = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix} \mathbf{v}_1 = \mathbf{0} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(\mathbf{P} - \lambda_2\mathbf{I})\mathbf{v}_2 = \begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} \mathbf{v}_2 = \mathbf{0} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ -\frac{\beta}{\alpha} \end{bmatrix}$$

and therefore the modal matrix and its inverse are

$$\begin{aligned}\mathbf{V} &= \begin{bmatrix} 1 & 1 \\ 1 & -\frac{\beta}{\alpha} \end{bmatrix} \\ \mathbf{V}^{-1} &= -\frac{1}{1+\beta/\alpha}\begin{bmatrix} -\frac{\beta}{\alpha} & -1 \\ -1 & 1 \end{bmatrix}.\end{aligned}$$

With the matrix

$$\mathbf{\Lambda}^n = \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^n \end{bmatrix}$$

we have

$$\mathbf{P}^n = -\frac{1}{1+\beta/\alpha}\begin{bmatrix} 1 & 1 \\ 1 & -\frac{\beta}{\alpha} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^n \end{bmatrix}\begin{bmatrix} -\frac{\beta}{\alpha} & -1 \\ -1 & 1 \end{bmatrix}$$

and after some algebra

$$\mathbf{P}^n = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix} + (1-\alpha-\beta)^n \begin{bmatrix} \frac{\alpha}{\alpha+\beta} & -\frac{\alpha}{\alpha+\beta} \\ -\frac{\beta}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \end{bmatrix}. \tag{22.14}$$

We now examine three cases of interest. They are distinguished by the value that $\lambda_2 = 1-\alpha-\beta$ takes on. Clearly, as seen from (22.14) this is the factor that influences the behavior of $\mathbf{P}^n$ with n. Since α and β are both conditional probabilities we must have that $0 \le \alpha+\beta \le 2$ and hence $-1 \le \lambda_2 = 1-\alpha-\beta \le 1$. The cases are delineated by whether this eigenvalue is strictly less than one in magnitude or not.

Case 1. $-1 < 1-\alpha-\beta < 1$

Here $|1-\alpha-\beta| < 1$ and therefore from (22.14) as $n \to \infty$

$$\mathbf{P}^n \to \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}. \tag{22.15}$$

As a result,

$$\begin{aligned}\mathbf{p}^T[n] &= \mathbf{p}^T[0]\mathbf{P}^n \to \begin{bmatrix} p_0[0] & p_1[0] \end{bmatrix}\begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}\end{aligned}$$

for any $\mathbf{p}[0]$. Hence, the Markov chain approaches a steady-state irregardless of the initial state probabilities. It is said to be an *ergodic Markov chain*, the reason for which we will discuss later. Also, the state probability vector approaches the *steady-state probability vector* $\mathbf{p}^T[\infty]$, which is denoted by

$$\boldsymbol{\pi}^T = \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}. \tag{22.16}$$

Finally, note that each row of $\mathbf{P}^n$ becomes the same as $n \to \infty$.

Case 2. $1 - \alpha - \beta = 1$ or $\alpha = \beta = 0$

If we draw the state probability diagram in this case, it should become clear what will happen. This is shown in Figure 22.4a, where the zero transition probability branches are omitted from the diagram. It is seen that there is no chance of leaving the initial state so that we should have $\mathbf{p}[n] = \mathbf{p}[0]$ for all n. To verify this, for $\alpha = \beta = 0$, the eigenvalues are both 1 and therefore $\mathbf{\Lambda} = \mathbf{I}$. Hence, $\mathbf{P} = \mathbf{I}$ and $\mathbf{P}^n = \mathbf{I}$. Here the Markov chain also attains steady-state and $\boldsymbol{\pi} = \mathbf{p}[0]$ but *the steady-state PMF depends upon the initial probability vector*, unlike in Case 1. Note that the only possible realizations are $0000\ldots$ and $1111\ldots$.

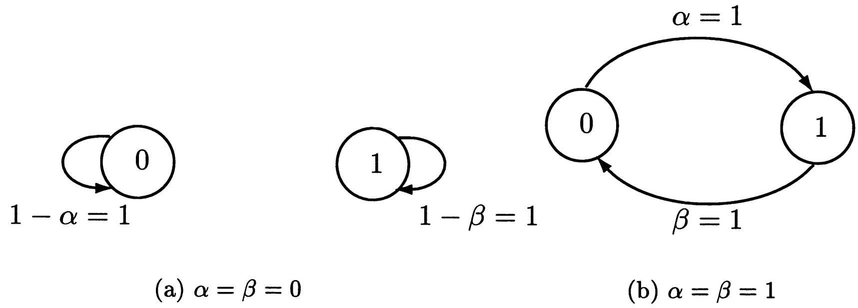


Figure 22.4: State probability diagrams for anomalous behaviors of two-state Markov chain.

Case 3. $1 - \alpha - \beta = -1$ or $\alpha = \beta = 1$

It is also easy to see what will happen in this case by referring to the state probability diagram in Figure 22.4b. The outcomes must alternate and thus the only realizations are $0101\ldots$ and $1010\ldots$, with the realization generated depending upon the initial state. Unlike the previous two cases, here there are no steady-state probabilities as we now show. From (22.14) we have

$$
\begin{aligned}
\mathbf{P}^n &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + (-1)^n \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for } n \text{ even} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{for } n \text{ odd.}
\end{aligned}
$$

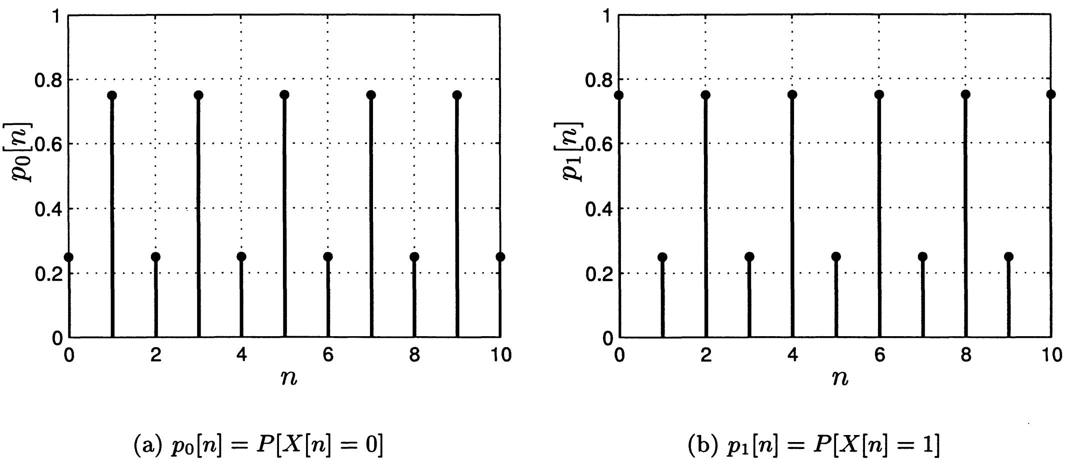
Hence, the state probability vector is

$$\begin{aligned}\mathbf{p}^T[n] &= \mathbf{p}^T[0]\mathbf{P}^n = \begin{bmatrix} p_0[0] & p_1[0] \end{bmatrix} \mathbf{P}^n \\ &= \begin{cases} \begin{bmatrix} p_0[0] & p_1[0] \end{bmatrix} & \text{for } n \text{ even} \\ \begin{bmatrix} p_1[0] & p_0[0] \end{bmatrix} & \text{for } n \text{ odd.} \end{cases}\end{aligned}$$

As an example, if $\mathbf{p}^T[0] = \begin{bmatrix} 1/4 & 3/4 \end{bmatrix}$, then

$$\mathbf{p}^T[n] = \begin{cases} \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix} & \text{for } n \text{ even} \\ \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix} & \text{for } n \text{ odd} \end{cases}$$

as shown in Figure 22.5. It is seen that the state probabilities cycle between two PMFs and hence there is no steady-state.



(a) $p_0[n] = P[X[n] = 0]$ (b) $p_1[n] = P[X[n] = 1]$

Figure 22.5: Cycling of state probability vector for Case 3.

The last two cases are of little practical importance for a two-state Markov chain since we usually have $0 < \alpha < 1$ and $0 < \beta < 1$. However, for a K-state Markov chain it frequently occurs that some of the transition probabilities are zero (corresponding to missing branches of the state probability diagram and an inability of the Markov chain to transition between certain states). Then, the dependence upon the initial state and cycling or periodic PMFs become quite important. The interested reader should consult [Gallagher 1996] and [Cox and Miller 1965] for further details. We next return to our golfing friend.

Example 22.5 – One-putting

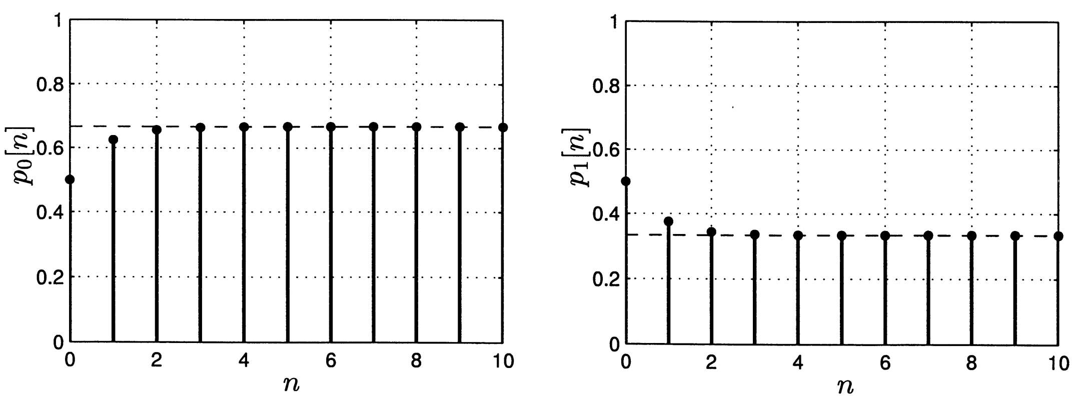
Recall that our golfer had a transition probability matrix given by

$$\mathbf{P} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

It is seen from (22.10) that $\alpha = 1/4$ and $\beta = 1/2$ and so this corresponds to Case 1 in which the same steady-state probability is reached regardless of the initial probability vector. Hence, as $n \to \infty$, $\mathbf{P}^n$ will converge to a constant matrix and therefore so will $\mathbf{p}[n]$. After many rounds of golf the probability of a one-putt or of going to state 1 is found from the second element of the stationary probability vector $\boldsymbol{\pi}$. This is from (22.16)

$$\begin{aligned} \boldsymbol{\pi}^T = \left[\begin{array}{cc} \pi_0 & \pi_1 \end{array} \right] &= \left[\begin{array}{cc} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{array} \right] \\ &= \left[\begin{array}{cc} \frac{1/2}{3/4} & \frac{1/4}{3/4} \end{array} \right] \\ &= \left[\begin{array}{cc} \frac{2}{3} & \frac{1}{3} \end{array} \right] \end{aligned}$$

so that her probability of a one-putt is now only 1/3 as we surmised by examination of Figure 22.1. At the first hole it was $p_1[0] = 1/2$. To determine how many holes she must play until this steady-state probability is attained we let this be $n = n_{ss}$ and determine from (22.14) when $(1 - \alpha - \beta)^{n_{ss}} = (1/4)^{n_{ss}} \approx 0$. This is about $n_{ss} = 10$ for which $(1/4)^{10} = 10^{-6}$. The actual state probability vector is shown in Figure 22.6 using an initial state probability of $\mathbf{p}^T[0] = [1/2 \ 1/2]$. The steady-state values of $\boldsymbol{\pi} = [2/3 \ 1/3]^T$ are also shown as dashed lines.



(a) $p_0[n] = P[X[n] = 0]$ (b) $p_1[n] = P[X[n] = 1]$

Figure 22.6: Convergence of state probability vector for Case 1 with $\alpha = 1/4$ and $\beta = 1/2$.

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22.5 Ergodic Markov Chains

We saw in the previous section that as $n \to \infty$, then for some $\mathbf{P}$ the state probability vector approaches a steady-state value irregardless of the initial state probabilities. This was Case 1 for which each element of $\mathbf{P}$ was nonzero or

$$\mathbf{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} > 0$$

where the "> 0" is meant to indicate that every element of $\mathbf{P}$ is greater than zero. Equivalently, all the branches of the state probability diagram were present. A Markov chain of this type is said to be ergodic in that a temporal average is equal to an ensemble average as we will later show. The key requirement for this to be true for any K-state Markov chain is that the $K \times K$ transition probability matrix satisfies $\mathbf{P} > 0$. The matrix $\mathbf{P}$ then has some special properties. We already have pointed out that the rows must sum to one; a matrix of this type is called a *stochastic matrix*, and for ergodicity, we must have $\mathbf{P} > 0$; a matrix satisfying this requirement is called an *irreducible stochastic matrix.* The associated Markov chain is known as an *ergodic or irreducible Markov chain.* A theorem termed the Perron-Frobenius theorem [Gallagher 1996] states that if $\mathbf{P} > 0$, then the transition probability matrix will always have one eigenvalue equal to 1 and the remaining eigenvalues will have magnitudes strictly less than 1. Such was the case for the two-state probability transition matrix of Case 1 for which $\lambda_1 = 1$ and $|\lambda_2| = |1-\alpha-\beta| < 1$. This condition on $\mathbf{P}$ assures convergence of $\mathbf{P}^n$ to a constant matrix. Convergence may also occur if some of the elements of $\mathbf{P}$ are zero but it is not guaranteed. A slightly more general condition for convergence is that $\mathbf{P}^n > 0$ for some n (not necessarily $n = 1$). An example is

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(see Problem 22.13).

We now assume that $\mathbf{P} > 0$ and determine the steady-state probabilities for a general K-state Markov chain. Since

$$\mathbf{p}^T[n] = \mathbf{p}^T[n-1]\mathbf{P}$$

and in steady-state we have that $\mathbf{p}^T[n-1] = \mathbf{p}^T[n] = \mathbf{p}^T[\infty]$, it follows that

$$\mathbf{p}^T[\infty] = \mathbf{p}^T[\infty]\mathbf{P}.$$

Letting the steady-state probability vector be $\boldsymbol{\pi} = \mathbf{p}[\infty]$, we have

$$\boldsymbol{\pi}^T = \boldsymbol{\pi}^T\mathbf{P} \tag{22.17}$$

and we need only solve for $\boldsymbol{\pi}$. An example follows.

Example 22.6 – Two-state Markov chain

We solve for the steady-state probability vector for Case 1. From (22.17) we have

$$[\ \pi_0 \quad \pi_1\ ] = [\ \pi_0 \quad \pi_1\ ] \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

so that

$$\begin{aligned} \pi_0 &= (1-\alpha)\pi_0 + \beta\pi_1 \\ \pi_1 &= \alpha\pi_0 + (1-\beta)\pi_1 \end{aligned}$$

or

$$\begin{aligned} 0 &= -\alpha\pi_0 + \beta\pi_1 \\ 0 &= \alpha\pi_0 - \beta\pi_1. \end{aligned}$$

The yields $\pi_1 = (\alpha/\beta)\pi_0$ since the two linear equations are identical. Of course, we also require that $\pi_0 + \pi_1 = 1$ and so this forms the second linear equation. The solution then is

$$\begin{aligned} \pi_0 &= \frac{\beta}{\alpha+\beta} \\ \pi_1 &= \frac{\alpha}{\alpha+\beta} \end{aligned} \tag{22.18}$$

and agrees with our previous results of (22.16).

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It can further be shown that if a steady-state probability vector exists (which will be the case if $\mathbf{P} > 0$), then the solution for $\boldsymbol{\pi}$ is unique [Gallagher 1996]. Finally, note that if we intialize the Markov chain with $\mathbf{p}[0] = \boldsymbol{\pi}$, then since $\mathbf{p}^T[1] = \mathbf{p}^T[0]\mathbf{P} = \boldsymbol{\pi}^T\mathbf{P} = \boldsymbol{\pi}^T$, the state probability vector will be $\boldsymbol{\pi}$ for $n \geq 0$. The Markov chain is then stationary since the state probability vector is the same for all n and $\boldsymbol{\pi}$ is therefore referred to as the *stationary probability vector*. We will henceforth use this terminology for $\boldsymbol{\pi}$.

Another observation of importance is that if $\mathbf{P} > 0$, then $\mathbf{P}^n$ converges, and it converges to $\mathbf{P}^\infty$, whose rows are identical. This was borne out in (22.15) and is true in general (see Problem 22.17). (Note that this is not true for Case 2 in which although $\mathbf{P}^n$ converges, it converges to $\mathbf{I}$, whose rows are not the same.) As a result of this property, the steady-state value of the state probability vector does not depend upon the initial probabilities since

$$\begin{aligned} \mathbf{p}^T[n] &= \mathbf{p}^T[0]\mathbf{P}^n \\ &= [\ p_0[0] \quad p_1[0]\ ] \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix} + \underbrace{\mathbf{p}^T[0](1-\alpha-\beta)^n \begin{bmatrix} \frac{\alpha}{\alpha+\beta} & -\frac{\alpha}{\alpha+\beta} \\ -\frac{\beta}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \end{bmatrix}}_{\to \mathbf{0}^T \text{ as } n\to\infty} \\ &\to \left[\ \frac{\beta}{\alpha+\beta} \quad \frac{\alpha}{\alpha+\beta}\ \right] = \boldsymbol{\pi}^T \end{aligned}$$

independent of $\mathbf{p}^T[0]$. Also, as previously mentioned, if $\mathbf{P} > 0$, then as $n \to \infty$

$$\mathbf{P}^n \to \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}$$

whose rows are identical. As a result, we have that

$$\lim_{n\to\infty} [\mathbf{P}^n]_{ij} = \pi_j \qquad j = 0, 1, \ldots, K-1. \tag{22.19}$$

Hence, the stationary probabilities may be obtained either by solving the set of linear equations as was done for Example 22.6 or by examining a row of $\mathbf{P}^n$ as $n \to \infty$. In Section 22.7 we give the general solution for the stationary probabilities. We next give another example.

Example 22.7 – Machine failures

A machine is in operation at the beginning of day $n = 0$. It may break during operation that day in which case repairs will begin at the beginning of the next day ($n = 1$). In this case, the machine will not be in operation at the beginning of day $n = 1$. There is a probability of 1/2 that the technician will be able to repair the machine that day. If it is repaired, then the machine will be in operation for day $n = 2$ and if not, the technician will again attempt to fix it the next day ($n = 2$). The probability that the machine will operate without a failure during the day is 7/8. After many days of operation or failure what is the probability that the machine will be working at the *beginning* of a day? Here there are two states, either $X[n] = 0$ if the machine is not in operation at the beginning of day n, or $X[n] = 1$ if the machine is in operation at the beginning of day n. The transition probabilities are given as

$$\begin{aligned} p_{01} &= P[\text{machine operational on day } n | \text{machine nonoperational on day } n-1] = \frac{1}{2} \\ p_{11} &= P[\text{machine operational on day } n | \text{machine operational on day } n-1] = \frac{7}{8} \end{aligned}$$

and so the state transition probability matrix is

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & \frac{7}{8} \end{bmatrix}$$

noting that $p_{00} = 1 - p_{01} = 1/2$ and $p_{10} = 1 - p_{11} = 1/8$. This Markov chain is shown in Figure 22.7. Since $\mathbf{P} > 0$, a steady-state is reached and the stationary probabilities are from (22.18)

$$\begin{aligned} \pi_0 &= \frac{\beta}{\alpha+\beta} = \frac{\frac{1}{8}}{\frac{1}{2}+\frac{1}{8}} = \frac{1}{5} \\ \pi_1 &= 1 - \pi_0 = \frac{4}{5}. \end{aligned}$$

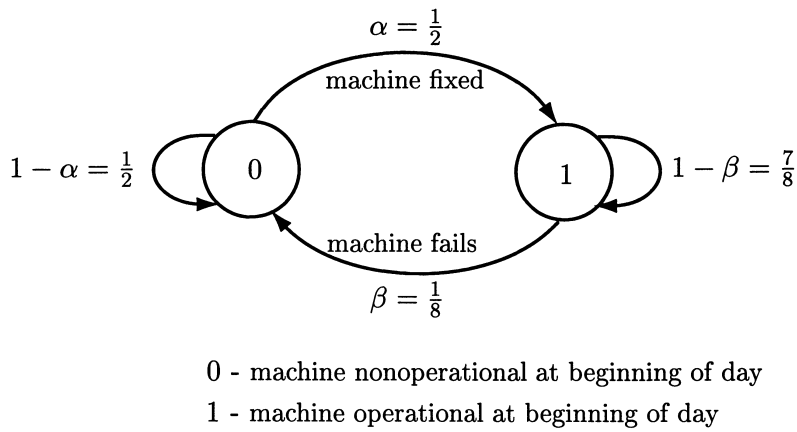

Figure 22.7: State probability diagram for Example 22.7.

The machine will be in operation at the beginning of a day with a probability of 0.8.

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Note that in the last example the states of 0 and 1 are arbitrary labels. They could just as well have been "nonoperational" and "operational". In problems such as these the state description is chosen to represent meaningful attributes of interest. One last comment concerns our apparent preoccupation with the steady-state behavior of a Markov chain. Although not always true, we are many times only interested in this because the choice of a starting time, i.e., at $n = 0$, is not easy to specify. In the previous example, it is conceivable that the machine in question has been in operation for a long time and it is only recently that a plant manager has become interested in its failure rate. Therefore, its initial starting time was probably some time in the past and we are now observing the states for some large n. We continue our discussion of steady-state characteristics in the next section.

22.6 Further Steady-State Characteristics

22.6.1 State Occupation Time

It is frequently of interest to be able to determine the percentage of time that a Markov chain is in a particular state, also called the *state occupation time.* Such was the case in Example 22.7, although a careful examination reveals that what we actually computed was the *probability* of being operational at the beginning of each day. In essence we are now asking for the *relative frequency* (or percentage of time) of the machine being operational. This is much the same as asking for the relative frequency of heads in a long sequence of independent fair coin tosses. We have proven by the law of large numbers (see Chapter 15) that this relative frequency must approach a probability of 1/2 as the number of coin tosses approaches infinity. For Markov chains the trials are not independent and so the law of large numbers does not apply directly. However, as we now show, if steady-state is attained, then the fraction of time the Markov chain spends in a particular state approaches the steady-state probability. This allows us to say that the fraction of time that the Markov chain spends in state j is just π_j.

Again consider a two-state Markov chain with states 0 and 1 and assume that $\mathbf{P} > 0$. We wish to determine the fraction of time spent in state 1. For some large n this is given by

$$\frac{1}{N}\sum_{j=n}^{n+N-1} X[j]$$

which is recognized as the sample mean of the N state outcomes for $\{X[n], X[n+1], \ldots, X[n+N-1]\}$. We first determine the expected value as

$$\begin{aligned} E\left[\frac{1}{N}\sum_{j=n}^{n+N-1} X[j]\right] &= E_{X[0]}\left[E\left[\frac{1}{N}\sum_{j=n}^{n+N-1} X[j] \middle| X[0]=i\right]\right] \\ &= E_{X[0]}\left[\frac{1}{N}\sum_{j=n}^{n+N-1} E[X[j]|X[0]=i]\right]. \end{aligned} \tag{22.20}$$

But

$$\begin{aligned} E[X[j]|X[0]=i] &= P[X[j]=1|X[0]=i] \\ &= [\mathbf{P}^j]_{i1} \rightarrow \pi_1 \end{aligned}$$

as $j \geq n \rightarrow \infty$ which follows from (22.19). The expected value does not depend upon the initial state i. Therefore, we have from (22.20) that

$$E\left[\frac{1}{N}\sum_{j=n}^{n+N-1} X[j]\right] \rightarrow E_{X[0]}\left[\frac{1}{N}\sum_{j=n}^{n+N-1} \pi_1\right] = \pi_1.$$

Thus, as $n \rightarrow \infty$, the *expected* fraction of time in state 1 is π_1. Furthermore, although it is more difficult to show, the variance of the sample mean converges to zero as $N \rightarrow \infty$ so that the fraction of time (and not just the expected value) spent in state 1 will converge to π_1 or

$$\frac{1}{N}\sum_{j=n}^{n+N-1} X[j] \rightarrow \pi_1. \tag{22.21}$$

This is the same result as for the repeated independent tossing of a fair coin. The result stated in (22.21) is that the *temporal mean* is equal to the ensemble mean which says that for large n, i.e., in steady-state, $\frac{1}{N}\sum_{j=n}^{n+N-1} X[j] \rightarrow \pi_1$ as $N \rightarrow \infty$. This is the property of *ergodicity* as previously described in Chapter 17. Thus, a Markov chain that achieves a steady-state irregardless of the initial state probabilities is called an *ergodic Markov chain.*

Returning to our golfing friend, we had previously questioned the fraction of the time she will achieve one-putts. We know that her stationary probability is $\pi_1 = 1/3$. Thus, after playing many rounds of golf, she will be one-putting about 1/3 of the time.

22.6.2 Mean Recurrence Time

Another property of the ergodic Markov chain that is of interest is the average number of steps before a state is revisited. For example, the golfer may wish to know the average number of holes she will have to play before another one-putt occurs, *given that she has just one-putted.* This is equivalent to determining the average number of steps the Markov chain will undergo before it returns to state 1. The time between visits to the same state is called the *recurrence time* and the average of this is called the *mean recurrence time.* We next determine this average.

Let T_R denote the recurrence time and note that it is an integer random variable that can take on values in the sample space $\{1, 2, \ldots\}$. For the two-state Markov chain shown in Figure 22.3 we first assume that we are in state 1 at time $n = n_0$. Then, the value of the recurrence time will be 1, or 2, or 3, etc. if $X[n_0 + 1] = 1$, or $X[n_0 + 1] = 0, X[n_0 + 2] = 1$, or $X[n_0 + 1] = 0, X[n_0 + 2] = 0, X[n_0 + 3] = 1$, etc., respectively. The probabilities of these events are $1 - \beta$, $\beta\alpha$, and $\beta(1 - \alpha)\alpha$, respectively as can be seen by referring to Figure 22.3. In general, the PMF is given as

$$P[T_R = k|\text{initially in state 1}] = \begin{cases} 1 - \beta & k = 1 \\ \beta\alpha(1 - \alpha)^{k-2} & k \geq 2 \end{cases}$$

which is a geometric-type PMF (see Chapter 5). To find the mean recurrence time we need only determine the expected value of T_R. This is

$$\begin{aligned}
E[T_R|\text{initially in state 1}] &= (1 - \beta) + \sum_{k=2}^{\infty} k\left[\beta\alpha(1 - \alpha)^{k-2}\right] \\
&= (1 - \beta) + \alpha\beta \sum_{l=1}^{\infty} (l + 1)(1 - \alpha)^{l-1} \quad \text{(let } l = k - 1\text{)} \\
&= (1 - \beta) + \left[\alpha\beta \sum_{l=1}^{\infty} (1 - \alpha)^{l-1} + \beta \sum_{l=1}^{\infty} l \underbrace{\alpha(1 - \alpha)^{l-1}}_{\text{geom}(\alpha)\text{ PMF}}\right] \\
&= (1 - \beta) + \alpha\beta \frac{1}{1 - (1 - \alpha)} + \beta\frac{1}{\alpha} \quad \text{(from Section 6.4.3)} \\
&= \frac{\alpha + \beta}{\alpha}
\end{aligned}$$

so that we have finally

$$E[T_R|\text{initially in state } 1] = \frac{1}{\pi_1}.$$

It is seen that mean recurrence time is the reciprocal of the stationary state probability. This is much the same result as for a geometric PMF and is interpreted as the number of failures (not returning to state 1) before a success (returning to state 1). For our golfer, since she has a stationary probability of one-putting of 1/3, she must wait on the average 1/(1/3)=3 holes between one-putts. This agrees with our simulation results shown in Figure 22.1.

22.7 K-State Markov Chains

Markov chains with more than two states are quite common and useful in practice but their analysis can be difficult. Most of the previous properties of a Markov chain apply to any finite number K of states. Computation of the n-step transition probability matrix is of course more difficult and requires computer evaluation. Most importantly, however, is that steady-state is still attained if $\mathbf{P} > 0$. The solution for the stationary probabilities is given next. It is derived in Appendix 22A.

The stationary probability vector for a K-state Markov chain is $\boldsymbol{\pi}^T = [\pi_0\, \pi_1 \ldots \pi_{K-1}]$. Its solution is given as

$$\boldsymbol{\pi} = (\mathbf{I} - \mathbf{P}^T + \mathbf{1}\mathbf{1}^T)^{-1}\mathbf{1} \tag{22.22}$$

where $\mathbf{I}$ is the $K \times K$ identity matrix and $\mathbf{1} = [1\, 1\, \ldots 1]^T$, which is a $K \times 1$ vector of ones. We next give an example of a 3-state Markov chain.

Example 22.8 – Weather modeling

Assume that the weather for each day can be classified as being either rainy (state 0), cloudy (state 1), or sunny (state 2). We wish to determine in the long run (steady-state) the percentage of sunny days. From the discussion in Section 22.6.1 this is the state occupation time, and is equal to the stationary probability π_2. To do so we assume the conditional probabilities

$$\begin{aligned}
\text{currently raining (state 0)} &: \; P_{00} = \frac{4}{8}, P_{01} = \frac{3}{8}, P_{02} = \frac{1}{8} \\
\text{currently cloudy (state 1)} &: \; P_{10} = \frac{3}{8}, P_{11} = \frac{2}{8}, P_{12} = \frac{3}{8} \\
\text{currently sunny (state 2)} &: \; P_{20} = \frac{1}{8}, P_{21} = \frac{3}{8}, P_{22} = \frac{4}{8}.
\end{aligned}$$

This says that if it is currently raining, then it is most probable that the next day will also have rain (4/8). The next most probable weather condition will be cloudy for the next day (3/8), and the least probable weather condition is sunny for the next day (1/8). See if you can rationalize the other entries in **P**. The complete state transition probability matrix is

$$\mathbf{P} = \begin{bmatrix} \frac{4}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{3}{8} & \frac{2}{8} & \frac{3}{8} \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} \end{bmatrix}$$

and the state probability diagram is shown in Figure 22.8. We can use this to

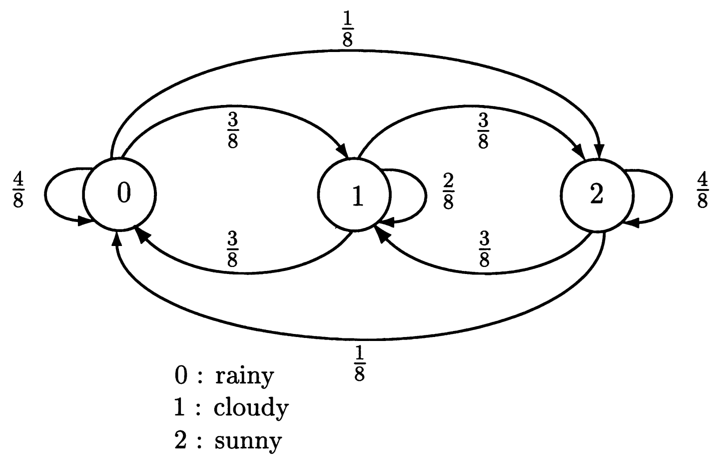

Figure 22.8: Three-state probability diagram for weather example.

determine the probability of the weather conditions on any day if we know the weather on day $n = 0$. For example, to find the probability of the weather on Saturday knowing that it is raining on Monday, we use

$$\mathbf{p}^T[n] = \mathbf{p}^T[0]\mathbf{P}^n$$

with $n = 5$ and $\mathbf{p}^T[0] = [1\,0\,0]$. Using a computer to evalute this we have that

$$\mathbf{p}[5] = \begin{bmatrix} 0.3370 \\ 0.3333 \\ 0.3296 \end{bmatrix}$$

and it appears that the possible weather conditions are nearly equiprobable. To find the stationary probabilities for the weather conditions we must solve $\boldsymbol{\pi}^T = \boldsymbol{\pi}^T\mathbf{P}$. Using the solution of (22.22), we find that

$$\boldsymbol{\pi} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

As $n \to \infty$, it is equiprobable that the weather will be rainy, cloudy, or sunny. Furthermore, because of ergodicity the fraction of days that it will be rainy, or be cloudy, or be sunny will all be 1/3.

◊

The previous result that the stationary probabilities are equal is true in general for the type of transition probability matrix given. Note that $\mathbf{P}$ not only has all its rows summing to one but also its column entries sum to one for all the columns. This is called a *doubly stochastic matrix* and always results in equal stationary probabilities (see Problem 22.27).

22.8 Computer Simulation

The computer simulation of a Markov chain is very simple. Consider the weather example of the previous section. We first need to generate a realization of a random variable taking on the values $0, 1, 2$ with the PMF $p_0[0], p_1[0], p_2[0]$. This can be done using the approach of Section 5.9. Once the realization has been obtained, say $x[0] = i$, we continue the same procedure *but* must choose the next PMF, which is actually a *conditional PMF.* If $x[0] = i = 1$ for example, then we use the PMF $p[0|1] = P_{10}, p[1|1] = P_{11}, p[2|1] = P_{12}$, which are just the entries in the second row of $\mathbf{P}$. We continue this procedure for all $n \geq 1$. Some MATLAB code to generate a realization for the weather example is given below.

```
clear all
rand('state',0)
N=1000;  % set number of samples desired
p0=[1/3 1/3 1/3]'; % set initial probability vector
P=[4/8 3/8 1/8;3/8 2/8 3/8;1/8 3/8 4/8]; % set transition prob. matrix
xi=[0 1 2]'; % set values of PMF
X0=PMFdata(1,xi,p0); % generate X[0] (see Appendix 6B for PMFdata.m
                     % function subprogram)
i=X0+1; % choose appropriate row for PMF
X(1,1)=PMFdata(1,xi,P(i,:)); % generate X[1]
i=X(1,1)+1; % choose appropriate row for PMF
for n=2:N % generate X[n]
   i=X(n-1,1)+1; % choose appropriate row for PMF
   X(n,1)=PMFdata(1,xi,P(i,:));
end
```

The reader may wish to modify and run this program to gain some insight into the effect of the conditional probabilities on the predicted weather patterns.

22.9 Real-World Example – Strange Markov Chain Dynamics

It is probably fitting that as the last real-world example, we choose one that questions what the real-world actually is. Is it a place of determinism, however complex, or one that is subject to the whims of chance events? *Random*, as defined by Webster's dictionary, means "lacking a definite plan, purpose, or *pattern*". Is this a valid definition? We do not plan to answer this question, but only to present some "food for thought". The seemingly random Markov chain provides an interesting example.

Consider a square arrangement of 101 × 101 points and define a set of states as the locations of the integer points within this square. The points are therefore denoted by the integer coordinates (i, j), where $i = 0, 1, \ldots, 100; j = 0, 1, \ldots, 100$. The number of states is $K = 101^2$. Next define a Markov chain for this set of states such that the nth outcome is a realization of the random point $\mathbf{X}[n] = [I[n]\, J[n]]^T$, where $I[n]$ and $J[n]$ are random variables taking on integer values in the interval $[0, 100]$. The initial point is chosen to be $\mathbf{X}[0] = [10\,80]^T$ and succeeding points evolve according to the random process:

1. Choose at random one of the *reference points* $(0,0), (100,0), (50,100)$.

2. Find the midpoint between the initial point and the chosen reference point and round it to the nearest integer coordinates (so that it becomes a state output).

3. Replace the initial point with the one found in step 2.

4. Go to step 1 and repeat the process, always using the previous point and one of the reference points chosen at random.

This procedure is equivalent to the formula

$$\mathbf{X}[n] = \left[\frac{1}{2}(\mathbf{X}[n-1] + \mathbf{R}[n])\right]_{\text{round}} \qquad n \geq 1 \tag{22.23}$$

where $\mathbf{R}[n] = [r_1[n] r_2[n]]^T$ is the reference point chosen at random and $[\cdot]_{\text{round}}$ denotes rounding of both elements of the vector to the nearest integer. Note that this is a Markov chain. The points generated must all lie within the square at integer coordinates due to the averaging and rounding that is ongoing. Also, the current output only depends upon the previous output $\mathbf{X}[n-1]$, i.e., justifying the claim of a Markov chain. The process is "random" due to our choice of $\mathbf{R}[n]$ from the sample space $\{(0,0), (100,0), (50,100)\}$ with equal probabilities.

The behavior of this Markov chain is shown in Figure 22.9, where the successive output points have been plotted with the first few shown with their values of n. It appears that the chain attains a steady-state and its steady-state PMF is zero over many triangular regions. It is interesting to note that the pattern consists of 3 triangles—one with vertices $(0,0), (50,0), (25,50)$, and the others with vertices

$(50, 0), (100, 0), (75, 50)$, and $(25, 50), (75, 50), (50, 100)$. Within each of these triangles resides an exact replica of the whole pattern and within each replica resides another replica, etc.! Such a figure is called a *fractal* with this particular one termed a *Sierpinski triangle*. The MATLAB code used to produce this figure is given below.

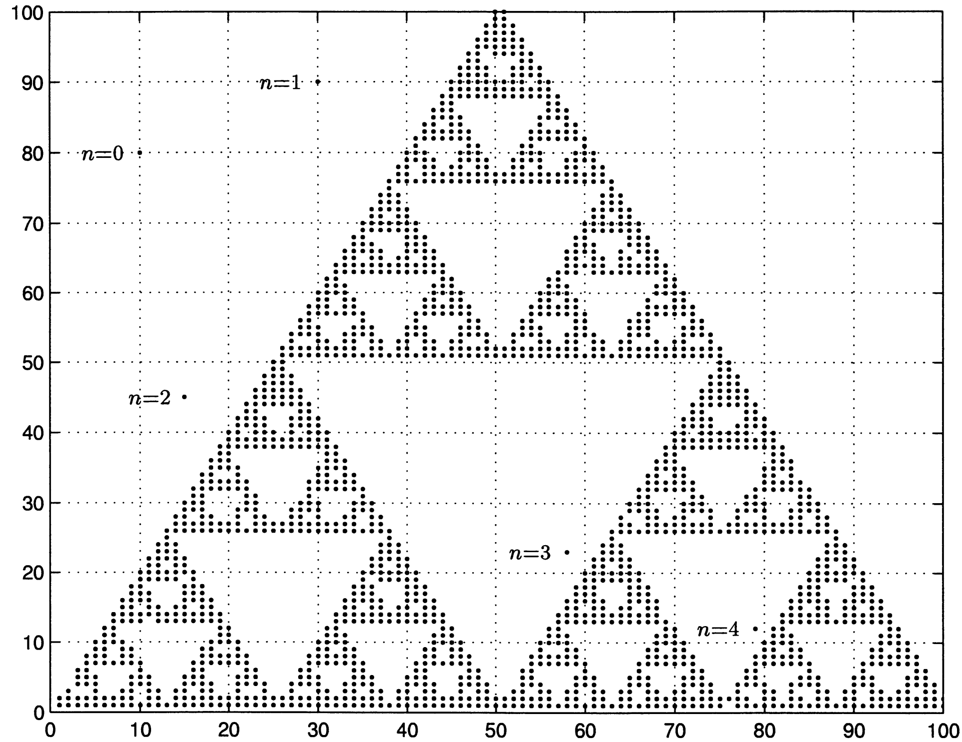

Figure 22.9: Steady-state Markov chain.

```
% sierpinski.m
%
clear all
rand('state',0)
r(:,1)=[0 0]'; % set up reference points
r(:,2)=[100 0]';
r(:,3)=[50 100]';
x0=[10 80]'; % set initial state
plot(x0(1),x0(2),'.') % plot state outcome as point
axis([0 100 0 100])
hold on
xn_1=x0;
```

```
for n=1:10000 % generate states
   j=floor(3*rand(1,1)+1); % choose at random one of three
                           % reference points
   xn=round(0.5*(r(:,j)+xn_1)); % generate new state
   plot(xn(1),xn(2),'.') % plot state outcome as point
   xn_1=xn; % make current state the previous one for
            % next transition
end
grid
hold off
```

The question arises as to whether the Markov chain is deterministic or random. We choose not to answer this question (because we don't know the answer!). Instead we refer the interested reader to the excellent book [Peitgen, Jurgens, and Saupe 1992] and also the popular layman's account [Gleick 1987] for further details. As a more practical application, it is observed that seemingly complex figures can be generated using a simple algorithm. This leads to the idea of data compression in which the only information needed to store a complex figure is the details of the algorithm. A field of sunflowers is such an example for which the reader should consult [Barnsley 1988] on how this is done.

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Problems

22.1 (w) A Markov chain has the states "A" and "B" or equivalently 0 and 1. If the conditional probabilities are $P[A|B] = 0.1$ and $P[B|A] = 0.4$, draw the state probability diagram. Also, find the transition probability matrix.

22.2 (⌣̈) **(f)** For the state probability diagram shown in Figure 22.2 find the probability of obtaining the outcomes $X[n] = 0, 1, 0, 1, 1$ for $n = 0, 1, 2, 3, 4$, respectively.

22.3 (f) For the state probability diagram shown in Figure 22.3 find the probabilities of the outcomes $X[n] = 0, 1, 0, 1, 1, 1$ for $n = 0, 1, 2, 3, 4, 5$, respectively and also for $X[n] = 1, 1, 0, 1, 1, 1$ for $n = 0, 1, 2, 3, 4, 5$, respectively. Compare the two and explain the difference.

22.4 (w) In some communication systems it is important to determine the percentage of time a person is talking. From measurements it is found that if a person is talking in a given time interval, then he will be talking in the next time interval with a probability of 0.75. If he is not talking in a time interval, then he will be talking in the next time interval with a probability of 0.5. Draw the state probability diagram using the states "talking" and "not talking".

22.5 (⌣̈) **(t)** In this problem we give an example of a random process that does *not* have the Markov property. The random process is defined as an *exclusive OR* logical function. This is $Y[n] = X[n] \oplus X[n-1]$ for $n \geq 0$, where $X[n]$ for $n \geq 0$ takes on values 0 and 1 with probabilities $1-p$ and p, respectively. The $X[n]$'s are IID. Also, for $n = 0$ we define $Y[0] = X[0]$. The definition of this operation is that $Y[n] = 0$ only if $X[n]$ and $X[n-1]$ are the same (both equal to 0 or both equal to 1), and otherwise $Y[n] = 1$. Determine $P[Y[2] = 1|Y[1] = 1, Y[0] = 0]$ and $P[Y[2] = 1|Y[1] = 1]$ to show that they are not equal in general.

22.6 (f) For the transition probability matrix given below draw the corresponding state probability diagram.

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

22.7 (w) A fair die is tossed many times in succession. The tosses are independent of each other. Let $X[n]$ denote the maximum of the first $n+1$ tosses. Determine the transition probability matrix. Hint: The maximum value cannot decrease as n increases.

22.8 (w) A particle moves along the circle shown in Figure 22.10 from one point to the other in a clockwise (CW) or counterclockwise (CCW) direction. At each step it can move either CW 1 unit or CCW 1 unit. The probabilities are $P[\text{CCW}] = p$ and $P[\text{CW}] = 1-p$ and do not depend upon the current

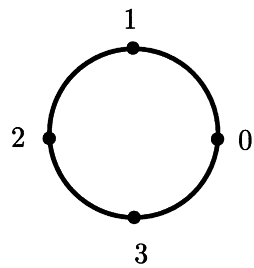

Figure 22.10: Movement of particle along a circle for Problem 22.8.

location of the particle. For the states $0, 1, 2, 3$ find the transition probability matrix.

22.9 (⌣) **(w,c)** A digital communication system transmits a 0 or a 1. After 10 miles of cable a repeater decodes the bit and declares it either a 0 or a 1. The probability of a decoding error is 0.1 as shown schematically in Figure 22.11. It is then retransmitted to the next repeater located 10 miles away. If the repeaters are all located 10 miles apart and the communication system is 50 miles in length, find the probability of an error if a 0 is initially transmitted. Hint: You will need a computer to work this problem.

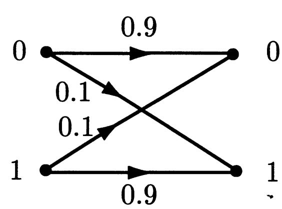

Figure 22.11: One section of a communication link.

22.10 (w,c) If $\alpha = \beta = 1/4$ for the state probability diagram shown in Figure 22.3, determine n so that the Markov chain is in steady-state. Hint: You will need a computer to work this problem.

22.11 (⌣) **(w)** There are two urns filled with red and black balls. Urn 1 has 60% red balls and 40% black balls while urn 2 has 20% red balls and 80% black balls. A ball is drawn from urn 1, its color noted, and then replaced. If it is red, the next ball is also drawn from urn 1, its color noted and then replaced. If the ball is black, then the next ball is drawn from urn 2, its color noted and then replaced. This procedure is continued indefinitely. Each time a ball is drawn the next ball is drawn from urn 1 if the ball is red and from urn 2 if it is black. After many trials of this experiment what is the probability of drawing a red ball? Hint: Define the states 1 and 2 as urns 1 and 2 chosen. Also, note that $P[\text{red drawn}] = P[\text{red drawn}|\text{urn 1 chosen}]P[\text{urn 1 chosen}] + P[\text{red drawn}|\text{urn 2 chosen}]P[\text{urn 2 chosen}]$.

22.12 (⌣̈) (w) A contestant answers questions posed to him from a game show host. If his answer is correct, the game show host gives him a harder question for which his probability of answering correctly is 0.01. If however, his answer is incorrect, the contestant is given an easy question for which his probability of answering correctly is 0.99. After answering many questions, what is the probability of answering a question correctly?

22.13 (f) For the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

will $\mathbf{P}^n$ converge as $n \to \infty$? You should be able to answer this question without the use of a computer. Hint: Determine $\mathbf{P}^2$.

22.14 (⌣̈) (w,c) For the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

does the Markov chain attain steady-state? If it does, what are the steady-state probabilities? Hint: You will need a computer to evaluate the answer.

22.15 (w,c) There are three lightbulbs that are always on in a room. At the beginning of each day the custodian checks to see if at least one lightbulb is working. If all three lightbulbs have failed, then he will replace them all. During the day each lightbulb will fail with a probability of 1/2 and the failure is independent of the other lightbulbs failing. Letting the state be the number of working lightbulbs draw the state probability diagram and determine the transition probability matrix. Show that eventually all three bulbs must fail and the custodian will then have to replace them. Hint: You will need a computer to work this problem.

22.16 (f) Find the stationary probabilities for the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

22.17 (t) In this problem we discuss the proof of the property that if $\mathbf{P} > 0$, the rows of $\mathbf{P}^n$ will all converge to the same values and that these values are the stationary probabilities. We consider the case of $K = 3$ for simplicity and assume distinct eigenvalues. Then, it is known from the Perron-Frobenius theorem that we will have the eigenvalues $\lambda_1 = 1$, $|\lambda_2| < 1$, and $|\lambda_3| < 1$. From (22.12) we have that $\mathbf{P}^n = \mathbf{V}\mathbf{\Lambda}^n\mathbf{V}^{-1}$ which for $K = 3$ is

$$\mathbf{P}^n = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{w}_1^T \\ \mathbf{w}_2^T \\ \mathbf{w}_3^T \end{bmatrix}}_{\mathbf{W}}$$

where $\mathbf{W} = \mathbf{V}^{-1}$ and $\mathbf{w}_i^T$ is the ith row of $\mathbf{W}$. Next argue that as $n \to \infty$, $\mathbf{P}^n \to \mathbf{v}_1\mathbf{w}_1^T$. Use the relation $\mathbf{P}^\infty\mathbf{1} = \mathbf{1}$ (why?) to show that $\mathbf{v}_1 = c\mathbf{1}$, where c is a constant. Next use $\boldsymbol{\pi}^T\mathbf{P}^\infty = \boldsymbol{\pi}^T$ (why?) to show that $\mathbf{w}_1 = d\boldsymbol{\pi}$, where d is a constant. Finally, use the fact that $\mathbf{w}_1^T\mathbf{v}_1 = 1$ since $\mathbf{W}\mathbf{V} = \mathbf{I}$ to show that $cd = 1$ and therefore, $\mathbf{P}^\infty = \mathbf{1}\boldsymbol{\pi}^T$. The latter is the desired result which can be verified by direct multiplication of $\mathbf{1}$ by $\boldsymbol{\pi}^T$.

22.18 (f,c) For the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

find $\mathbf{P}^{100}$ using a computer evaluation. Does the form of $\mathbf{P}^{100}$ agree with the theory?

22.19 (⌣̈) **(f,c)** Using the explicit solution for the stationary probability vector given by (22.22), determine its value for the transition probability matrix given in Problem 22.18. Hint: You will need a computer to evaluate the solution.

22.20 (w) The result of multiplying two identical matrices together produces the same matrix as shown below.

$$\begin{bmatrix} 0.2 & 0.1 & 0.7 \\ 0.2 & 0.1 & 0.7 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 0.2 & 0.1 & 0.7 \\ 0.2 & 0.1 & 0.7 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.1 & 0.7 \\ 0.2 & 0.1 & 0.7 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}.$$

Explain what this means for Markov chains.

22.21 (f) For the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0.99 & 0.01 \\ 0.01 & 0.99 \end{bmatrix}$$

solve for the stationary probabilities. Compare your probabilities to those obtained if a fair headed coin is tossed repeatedly and the tosses are independent. Do you expect the realization for this Markov chain to be similar to that of the fair coin tossing?

22.22 (c) Simulate on the computer the Markov chain described in Problem 22.21. Use $\mathbf{p}^T[0] = [1/2\ 1/2]$ for the initial probability vector. Generate a realization for $n = 0, 1, \ldots, 99$ and plot the results. What do you notice about the realization? Next generate a realization for $n = 0, 1, \ldots, 9999$ and estimate the stationary probability of observing 1 by taking the sample mean of the realization. Do you obtain the theoretical result found in Problem 22.21 (recall that this type of Markov chain is ergodic and so a temporal average is equal to an ensemble average).

22.23 (w) A person is late for work on his first day with a probability of 0.1. On succeeding days he is late for work with a probability of 0.2 if he was late the previous day and with a probability of 0.4 if he was on time the previous day. In the long run what percentage of time is he late to work?

22.24 (⌣̈) (f,c) Assume for the weather example of Example 22.8 that the transition probability matrix is

$$\mathbf{P} = \begin{bmatrix} \frac{6}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{5}{8} & \frac{2}{8} & \frac{1}{8} \\ \frac{4}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}.$$

What is the steady-state probability of rain? Compare your answer to that obtained in Example 22.8 and explain the difference. Hint: You will need a computer to find the solution.

22.25 (w,c) Three machines operate together on a manufacturing floor, and each day there is a possibility that any of the machines may fail. The probability of their failure depends upon how many other machines are still in operation. The number of machines in operation at the beginning of each day is represented by the state values of $0, 1, 2, 3$ and the corresponding state transition probability matrix is

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.1 & 0.3 & 0.6 & 0 \\ 0.4 & 0.3 & 0.2 & 0.1 \end{bmatrix}.$$

First explain why $\mathbf{P}$ has zero entries. Next determine how many days will pass before the probability of all 3 machines failing is greater than 0.8. Assume that intially all 3 machines are working. Hint: You will need a computer to find the solution.

22.26 (⌣̈) **(w,c)** A pond holds 4 fish. Each day a fisherman goes fishing and his probability of catching $k = 0, 1, 2, 3, 4$ fish that day follows a binomial PDF with $p = 1/2$. How many days should he plan on fishing so that the probability of his catching all 4 fish exceeds 0.9? Note that initially, i.e., at $n = 0$, all 4 fish are present. Hint: You will need a computer to find the solution.

22.27 (t) In this problem we prove that a doubly stochastic transition probability matrix with $\mathbf{P} > 0$ produces equal stationary probabilities. First recall that since the columns of $\mathbf{P}$ sum to one, we have that $\mathbf{P}^T\mathbf{1} = \mathbf{1}$ and therefore argue that $\mathbf{P}^{\infty^T}\mathbf{1} = \mathbf{1}$. Next use the results of Problem 22.17 that $\mathbf{P}^{\infty} = \mathbf{1}\boldsymbol{\pi}^T$ to show that $\boldsymbol{\pi} = \mathbf{1}/K$.

22.28 (⌣̈) **(c)** Use a computer simulation to generate a realization of the golf example for a large number of holes (very much greater than 18). Estimate the percentage of one-putts from your realization and compare it to the theoretical results.

22.29 (c) Repeat Problem 22.28 but now estimate the average time between one-putts. Compare your results to the theoretical value.

22.30 (c) Run the program sierpinski.m given in Section 22.9 but use instead the initial position $\mathbf{X}[0] = [50\,30]^T$. Do you obtain similar results to those shown in Figure 22.9? What is the difference, if any?

Appendix 22A

Solving for the Stationary PMF

We derive the formula of (22.22). The set of equations to be solved (after transposition) is $\mathbf{P}^T\boldsymbol{\pi} = \boldsymbol{\pi}$ or equivalently

$$(\mathbf{I} - \mathbf{P}^T)\boldsymbol{\pi} = \mathbf{0}. \tag{22A.1}$$

Since we have assumed a unique solution, it is clear that the matrix $\mathbf{I} - \mathbf{P}^T$ cannot be invertible or else we would have $\boldsymbol{\pi} = \mathbf{0}$. This is to say that the linear equations are not all independent. To make them independent we must add the constraint equation $\sum_{i=0}^{K-1} \pi_i = 1$ or in vector form this is $\mathbf{1}^T\boldsymbol{\pi} = 1$. Equivalently, the constraint equation is $\mathbf{1}\mathbf{1}^T\boldsymbol{\pi} = \mathbf{1}$. Adding this to (22A.1) produces

$$(\mathbf{I} - \mathbf{P}^T)\boldsymbol{\pi} + \mathbf{1}\mathbf{1}^T\boldsymbol{\pi} = \mathbf{1}$$

or

$$(\mathbf{I} - \mathbf{P}^T + \mathbf{1}\mathbf{1}^T)\boldsymbol{\pi} = \mathbf{1}.$$

It can be shown that the matrix $\mathbf{I} - \mathbf{P}^T + \mathbf{1}\mathbf{1}^T$ is now invertible and so the solution is

$$\boldsymbol{\pi} = (\mathbf{I} - \mathbf{P}^T + \mathbf{1}\mathbf{1}^T)^{-1}\mathbf{1}.$$