Chapter 19

Multiple Wide Sense Stationary Random Processes

19.1 Introduction

In Chapters 7 and 12 we defined multiple random variables X and Y as a mapping from the sample space $\mathcal{S}$ of the experiment to a *point* (x, y) in the x-y plane. We now extend that definition to be a mapping from $\mathcal{S}$ to a point in the x-y plane that evolves with time, and denote that point as $(x[n], y[n])$ for $-\infty < n < \infty$. The mapping, denoted either by $(X[n], Y[n])$ or equivalently by $[X[n]\,Y[n]]^T$, is called a *jointly distributed random process.* An example is the mapping from a point at some geographical location, where the possible choices for the location constitute $\mathcal{S}$, to the daily temperature and pressure at that point or $(T[n], P[n])$. Instead of treating the random processes, which describe temperature and pressure, separately, it makes more sense to analyze them jointly. This is especially true if the random processes are correlated. For example, a drop in barometric pressure usually indicates the onset of a storm, which in turn will cause a drop in the temperature. Another example of great interest is the effect of a change in the Federal Reserve discount rate, which is the percentage interest charged to banks by the federal government, on the rate of job creation. It is generally assumed that by lowering the discount rate, companies can borrow money more cheaply and thus invest in new products and services, thus increasing the demand for labor. The jointly distributed random processes describing this situation are $I[n]$, the daily discount interest rate, and $J[n]$, the daily number of employed Americans. Many other examples are possible, encompassing a wide range of disciplines.

In this chapter we extend the concept of a wide sense stationary (WSS) random process to *two jointly distributed WSS random processes.* The extension to any number of WSS random processes can be found in [Bendat and Piersol 1971, Jenkins and Watts 1968, Kay 1988, Koopmans 1974, Robinson 1967]. Multiple random process theory is known by the synonymous terms *multivariate random* *processes, multichannel random processes*, and *vector random processes*. Also, the characterization of the random processes at the input and output of an LSI system is explored. We will find that the extensions are much the same as in going from a single random *variable* to two random *variables*, especially since the definitions are based on samples of the random process, which themselves are random variables. As in previous chapters our focus will be on discrete-time random processes but the analogous concepts and formulas for continuous-time random processes will be summarized later.

19.2 Summary

Two random processes are jointly WSS if they are individually WSS (satisfy (19.1)–(19.4)) and also the cross-correlation given by (19.5) does not depend on n. The sequence given by (19.5) is called the cross-correlation sequence. The cross-correlation sequence has the properties given in Property 19.1–19.4, which differ from those of the ACS. Jointly WSS random processes are defined to be uncorrelated if (19.12) holds. The cross-power spectral density is defined by (19.13) and is evaluated using (19.14). It has the properties given by Property 19.5–19.9, which differ from those of the PSD. The correlation between two jointly WSS random processes can be measured in the frequency domain using the coherence function defined in (19.20). The ACS and PSD for the sum of two jointly distributed WSS random processes is given in Section 19.5. If the random processes are uncorrelated, then the ACS and PSD of the sum random process are given by (19.25) and (19.26), respectively. For the filtering operation shown in Figure 19.2a the cross-correlation sequence is given by (19.27) and the cross-power spectral density by (19.28). For the filtering operation shown in Figure 19.2b the cross-correlation sequence is given by (19.29) and the cross-power spectral density by (19.30). The corresponding definitions and formulas for continuous-time random processes are given in Section 19.6. Estimation of the cross-correlation sequence is discussed in Section 19.7 with the estimate given by (19.46). Finally, an application of cross-correlation to brain physiology research is described in Section 19.8.

19.3 Jointly Distributed WSS Random Processes

We will denote the two discrete-time random processes by $X[n]$ and $Y[n]$ for $-\infty < n < \infty$. Of particular interest will be the extension of the concept of wide sense stationarity from one to two random processes. To do so we first assume that each random process is *individually WSS*, which is to say that

$$\mu_X[n] = E[X[n]] = \mu_X \tag{19.1}$$

$$r_X[k] = E[X[n]X[n+k]] \tag{19.2}$$

$$\mu_Y[n] = E[Y[n]] = \mu_Y \tag{19.3}$$

$$r_Y[k] = E[Y[n]Y[n+k]] \tag{19.4}$$

or the first two moments do not depend on n. For the concept of wide sense stationarity to be useful in the context of two random processes, we require a further definition. To motivate it, consider the situation in which we add the two random processes together and wish to determine the overall average power. Then, if $Z[n] = X[n] + Y[n]$, we need to find $E[Z^2[n]]$. Proceeding we have

$$\begin{aligned} E[Z^2[n]] &= E[(X[n] + Y[n])^2] \\ &= E[X^2[n]] + E[X[n]Y[n]] + E[Y[n]X[n]] + E[Y^2[n]] \\ &= r_X[0] + 2E[X[n]Y[n]] + r_Y[0]. \end{aligned}$$

To complete the calculation we require knowledge of the joint moment $E[X[n]Y[n]]$. If it does not depend on n, then $E[Z^2[n]]$ will likewise not depend on n. More generally, if we were to compute $E[Z[n]Z[n+k]]$, then we would require knowledge of $E[X[n]Y[n+k]]$ and so we will *assume* that the latter does not depend on n. Therefore, with this assumption we can now define

$$r_{X,Y}[k] = E[X[n]Y[n+k]] \qquad k = \ldots, -1, 0, 1, \ldots. \tag{19.5}$$

This new sequence is called the *cross-correlation sequence* (CCS). Returning to our average power computation we can now write that

$$E[Z^2[n]] = r_X[0] + 2r_{X,Y}[0] + r_Y[0]$$

and the average power is seen not to depend on n. Note also from the definition of the CCS, that the ACS is just $r_{X,X}[k]$.

If $X[n]$ and $Y[n]$ are WSS random processes and a CCS can be defined ($E[X[n]Y[n+k]]$ not dependent on n), then the random processes are said to be *jointly wide sense stationary*. In summary, for the two random processes to be jointly WSS we require the conditions (19.1)–(19.5) to hold. An example follows.

Example 19.1 – CCS for WSS random processes delayed with respect to each other

Let $X[n]$ be a WSS random process and let $Y[n]$ be a delayed version of $X[n]$ so that $Y[n] = X[n - n_0]$. Then, to determine if the random processes are jointly WSS we have

$$\begin{aligned} E[X[n]] &= \mu_X \\ E[Y[n]] &= E[X[n-n_0]] = \mu_X \\ E[X[n]X[n+k]] &= r_X[k] \\ E[Y[n]Y[n+k]] &= E[X[n-n_0]X[n+k-n_0]] = r_X[k] \\ E[X[n]Y[n+k]] &= E[X[n]X[n+k-n_0]] = r_X[k-n_0] \end{aligned} \tag{19.6}$$

all of which follow from our definition of $Y[n]$ and the assumption that $X[n]$ is WSS. Note that $E[X[n]Y[n+k]]$ does not depend on n and so a CCS can be defined. It is given by (19.6) as

$$r_{X,Y}[k] = r_X[k - n_0]. \tag{19.7}$$

Since all the first and second moments do not depend on n, the random processes are jointly WSS.

$\diamond$

We will henceforth assume that $X[n]$ and $Y[n]$ are jointly WSS unless stated otherwise. From the previous example it is observed that the CCS has very different properties than the ACS. Unlike the ACS, *the CCS does not necessarily have its maximum value at* $k = 0$. In the previous example, the maximum of the CCS occurs at $k = n_0$ (see (19.7)). Also, in general *we do not have* $r_{X,Y}[-k] = r_{X,Y}[k]$ or the *CCS is not symmetric about* $k = 0$. In the previous example, we have from (19.7)

$$\begin{aligned} r_{X,Y}[-k] &= r_X[-k - n_0] \\ &= r_X[k + n_0] \neq r_X[k - n_0] = r_{X,Y}[k]. \end{aligned}$$

Furthermore, even though the CCS is symmetric about $k = n_0$ in the previous example, it need not be symmetric at all.

CCS asymmetry requires vigilance.

Since the CCS is not symmetric, in contrast to the ACS, one must be careful. The cross-second moment $E[X[m]Y[n]]$, where $X[n]$ and $Y[n]$ are jointly WSS, is expressed in terms of the CCS as $r_{X,Y}[n - m]$, *not* $r_{X,Y}[m - n]$. To determine the argument k of the CCS for $r_{X,Y}[k]$, always take the index of the Y random variable and subtract the index of the X random variable. For example, $E[X[3]Y[1]] = r_{X,Y}[1 - 3] = r_{X,Y}[-2]$. This is especially important in light of the fact that the definition of the CCS is not standard. Some authors use $r_{X,Y}[k] = E[X[n]Y[n-k]]$, which will produce a CCS that is "flipped around" in k, relative to our definition.

We give one more example and then summarize the properties of the CCS.

Example 19.2 – Another calculation of the CCS

Assume that $X[n] = U[n]$ and $Y[n] = U[n] + 2U[n - 1]$, where $U[n]$ is white noise with variance $\sigma_U^2 = 1$. Thus, $X[n]$ is a white noise random process and $Y[n]$ is a general MA random process, i.e., no Gaussian assumption is made. Then, it is easily shown that $\mu_X[n] = \mu_Y[n] = 0$, $r_X[k] = \delta[k]$, and

$$r_Y[k] = \begin{cases} 5 & k = 0 \\ 2 & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

so that $X[n]$ and $Y[n]$ are individually WSS. Now computing the cross-second moment, we have

$$\begin{aligned} E[X[n]Y[n+k]] &= E[U[n](U[n+k]+2U[n+k-1])] \\ &= r_U[k]+2r_U[k-1] \\ &= \delta[k]+2\delta[k-1] \end{aligned}$$

and it is seen to be independent of n. Hence, the CCS is

$$r_{X,Y}[k]=\delta[k]+2\delta[k-1]$$

and the random processes are jointly WSS. The ACSs and the CCS are shown in Figure 19.1. We observe that $r_{X,Y}[-k] \neq r_{X,Y}[k]$ and that the maximum does not occur at $k=0$. We can assert, however, that the maximum must be less than or equal to $\sqrt{r_X[0]r_Y[0]}$ since by the Cauchy-Schwarz inequality (see Appendix 7A)

$$\begin{aligned} |r_{X,Y}[k]| &= |E[X[n]Y[n+k]]| \\ &\leq \sqrt{E[X^2[n]]E[Y^2[n+k]]} \\ &= \sqrt{r_X[0]r_Y[0]}. \end{aligned}$$

For this example we see that

$$|r_{X,Y}[k]| \leq \sqrt{1\cdot 5}=\sqrt{5}.$$

◇

We now summarize the properties (or more appropriately the *non*properties) of the CCS.

Property 19.1 – CCS is not necessarily symmetric.

$$r_{X,Y}[-k] \neq r_{X,Y}[k] \tag{19.8}$$

□

Property 19.2 – The maximum of the CCS can occur for any value of k.

□

Property 19.3 – The maximum value of the CCS is bounded.

$$|r_{X,Y}[k]| \leq \sqrt{r_X[0]r_Y[0]} \tag{19.9}$$

□

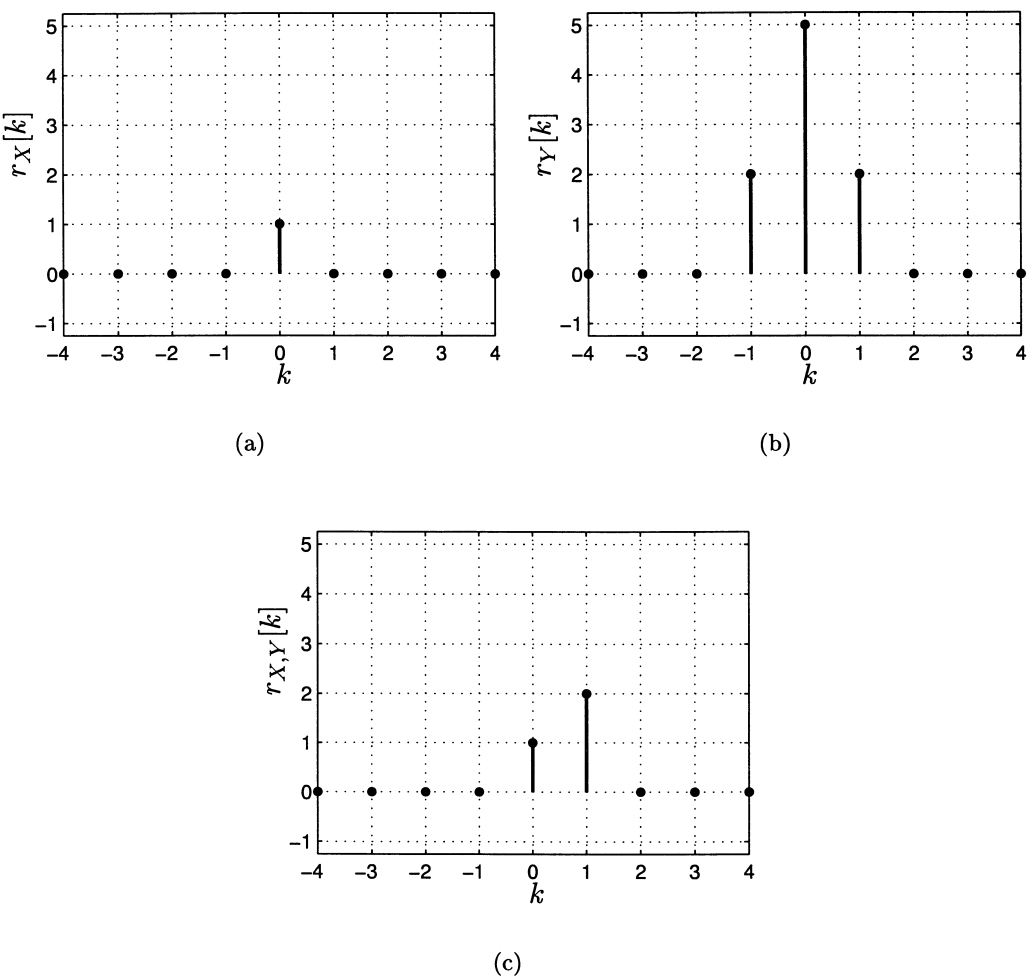


Figure 19.1: Autocorrelation and cross-correlation sequences for Example 19.2.

A fourth property that is useful arises by considering $E[Y[n]X[n+k]]$, which is the cross-second moment with $X[n]$ and $Y[n]$ interchanged. Assuming jointly WSS random processes, this moment becomes

$$\begin{aligned} E[Y[n]X[n+k]] &= E[X[n+k]Y[n]] \\ &= E[X[m]Y[m-k]] \qquad (\text{let } m = n+k) \\ &= r_{X,Y}[-k] \qquad (\text{from definition of CCS}). \end{aligned}$$

Therefore, $E[Y[n]X[n+k]]$ does not depend on n and so we can define another cross-correlation sequence as

$$r_{Y,X}[k] = E[Y[n]X[n+k]] \tag{19.10}$$

and it is seen to be equal to $r_{X,Y}[-k]$. Thus, as our last property we have

Property 19.4 – Interchanging $X[n]$ and $Y[n]$ flips the CCS about $k = 0$.

$$r_{Y,X}[k] = r_{X,Y}[-k] \tag{19.11}$$

□

Next, we define the concept of *uncorrelated jointly WSS random processes.* Two *zero mean* jointly WSS random processes are said to be uncorrelated if

$$r_{X,Y}[k] = 0 \qquad \text{for } -\infty < k < \infty \tag{19.12}$$

including $k = 0$. (For nonzero mean random processes the definition of uncorrelated random processes is that $r_{X,Y}[k] = \mu_X \mu_Y$ for $-\infty < k < \infty$.) Of course, if the random processes are independent so that $E[X[n]Y[n+k]] = 0$ does not depend on n, then they must be jointly WSS as well. It also follows from Property 19.4 that if the random processes are uncorrelated, then $r_{Y,X}[k] = 0$ for all k. An example follows.

Example 19.3 – Uncorrelated sinusoidal random processes

Let $X[n] = \cos(2\pi f_0 n + \Theta_1)$ and $Y[n] = \cos(2\pi f_0 n + \Theta_2)$, where $\Theta_1 \sim \mathcal{U}(0, 2\pi)$, $\Theta_2 \sim \mathcal{U}(0, 2\pi)$, and Θ_1 and Θ_2 are independent random variables. Then, we have seen previously that $X[n]$ and $Y[n]$ are individually WSS (see Example 17.4) and

$$\begin{aligned}
r_{X,Y}[k] &= E[X[n]Y[n+k]] \\
&= E_{\Theta_1,\Theta_2}[\cos(2\pi f_0 n + \Theta_1)\cos(2\pi f_0(n+k) + \Theta_2)] \\
&= E_{\Theta_1}[\cos(2\pi f_0 n + \Theta_1)]E_{\Theta_2}[\cos(2\pi f_0(n+k) + \Theta_2)] \quad \text{(independent random variables and (12.30))} \\
&= 0
\end{aligned}$$

since each random sinusoid has a zero mean (see Example 16.11). Thus, the random processes are uncorrelated and jointly WSS. Can you interpret this result physically?

◇

19.4 The Cross-Power Spectral Density

The PSD of a WSS random process was seen earlier to describe the distribution of average power with frequency. Also, the average power of the random process in a band of frequencies is obtained by integrating the PSD over that frequency band. In a similar vein to the definition of the PSD, we can define the *cross-power spectral density* (CPSD) of two jointly WSS random processes as

$$P_{X,Y}(f) = \lim_{M\to\infty} \frac{1}{2M+1} E\left[\left(\sum_{n=-M}^{M} X[n]\exp(-j2\pi f n)\right)^* \left(\sum_{n=-M}^{M} Y[n]\exp(-j2\pi f n)\right)\right] \tag{19.13}$$

which results in the usual PSD if $Y[n] = X[n]$. Using a similar derivation as the one that resulted in the Wiener-Khinchine theorem, it can be shown that (see Problem 19.8)

$$P_{X,Y}(f) = \sum_{k=-\infty}^{\infty} r_{X,Y}[k] \exp(-j2\pi fk). \tag{19.14}$$

It is less clear than for the PSD what the physical significance of the CPSD is. From (19.13) it appears that the CPSD will be large when the Fourier transforms of $X[n]$ and $Y[n]$ at a given frequency are large and *are in phase.* Conversely, when the Fourier transforms are either small or out of phase, the CPSD will be small. This is confirmed by the results of Example 19.3 in which the sinusoidal processes have all their power at $f = \pm f_0$ since $P_X(f) = P_Y(f) = \frac{1}{2}\delta(f + f_0) + \frac{1}{2}\delta(f - f_0)$. However, because they have phases that are independent of each other and can take on values in $(0, 2\pi)$ uniformly, $r_{X,Y}[k] = 0$ and therefore, $P_{X,Y}(f) = 0$. On the other hand, if the phase random variables were statistically dependent, say $\Theta_1 = \Theta_2$, then the CPSD would be large (see Problem 19.9). Another example follows.

Example 19.4 – CCS for WSS random processes delayed with respect to each other (continued)

We continue Example 19.1 in which $Y[n] = X[n - n_0]$ and $X[n]$ is WSS. We saw that the CCS is given by $r_{X,Y}[k] = r_X[k - n_0]$. Using (19.14) the CPSD is

$$P_{X,Y}(f) = \sum_{k=-\infty}^{\infty} r_X[k - n_0] \exp(-j2\pi fk)$$

and letting $l = k - n_0$ produces

$$\begin{aligned} P_{X,Y}(f) &= \sum_{l=-\infty}^{\infty} r_X[l] \exp[-j2\pi f(l + n_0)] \\ &= \sum_{l=-\infty}^{\infty} r_X[l] \exp(-j2\pi fl) \exp(-j2\pi f n_0) \\ &= P_X(f) \exp(-j2\pi f n_0). \end{aligned}$$

It is seen that the CPSD is a complex function and that $P_{X,Y}(-f) \neq P_{X,Y}(f)$. It does appear, however, that $P_{X,Y}(-f) = P^*_{X,Y}(f)$ so that it has the symmetry properties

$$\begin{aligned} |P_{X,Y}(-f)| &= |P_{X,Y}(f)| \\ \angle P_{X,Y}(-f) &= -\angle P_{X,Y}(f) \end{aligned} \tag{19.15}$$

or the magnitude of the CPSD is an even function and the phase of the CPSD is an odd function. This result is indeed true as we will prove in Property 19.6.

◊

One way to think about the CPSD is as a correlation between the *normalized* Fourier transforms of $X[n]$ and $Y[n]$ *at a given frequency.* From (19.13) we see that if

$$\begin{aligned} X_{2M+1}(f) &= \frac{1}{\sqrt{2M+1}} \sum_{n=-M}^{M} X[n] \exp(-j2\pi f n) \\ Y_{2M+1}(f) &= \frac{1}{\sqrt{2M+1}} \sum_{n=-M}^{M} Y[n] \exp(-j2\pi f n) \end{aligned} \tag{19.16}$$

then

$$P_{X,Y}(f) = \lim_{M\to\infty} E[X^*_{2M+1}(f) Y_{2M+1}(f)]. \tag{19.17}$$

This is a correlation between the two *complex random variables* $X_{2M+1}(f)$ and $Y_{2M+1}(f)$. In fact, a normalized version of the CPSD is a *complex correlation coefficient.* Indeed, from the Cauchy-Schwarz inequality for complex random variables (see Appendix 7A for real random variables), we have that (recall that if $X = U+jV$, then $E[X]$ is defined as $E[X] = E[U] + jE[V]$)

$$|E[X^*_{2M+1}(f) Y_{2M+1}(f)]| \leq \sqrt{E[|X_{2M+1}(f)|^2] E[|Y_{2M+1}(f)|^2]} \tag{19.18}$$

and therefore as $M \to \infty$, this becomes from (19.17) and (17.30)

$$|P_{X,Y}(f)| \leq \sqrt{P_X(f) P_Y(f)}. \tag{19.19}$$

Thus, if we normalize the CPSD to form the complex function of frequency

$$\gamma_{X,Y}(f) = \frac{P_{X,Y}(f)}{\sqrt{P_X(f)P_Y(f)}} \tag{19.20}$$

then we have that $|\gamma_{X,Y}(f)| \leq 1$. The complex function of frequency $\gamma_{X,Y}(f)$ is called the *coherence function* and it is a *complex correlation coefficient.* It measures the correlation between the Fourier transforms of two jointly WSS random processes at a given frequency. As an example, consider the random processes of Example 19.4. Then

$$\begin{aligned} \gamma_{X,Y}(f) &= \frac{P_{X,Y}(f)}{\sqrt{P_X(f)P_Y(f)}} \\ &= \frac{P_X(f)\exp(-j2\pi f n_0)}{\sqrt{P_X(f)P_X(f)}} \\ &= \exp(-j2\pi f n_0) \qquad\qquad \text{(since } P_X(f) \geq 0\text{)}. \end{aligned}$$

The magnitude of the coherence is unity for all frequencies, meaning that the Fourier transform of $Y[n]$ at a given frequency can be perfectly predicted from the Fourier transform of $X[n]$ at the same frequency since $Y[n] = X[n - n_0]$. It follows that

$Y_{2M+1}(f) = \exp(-j2\pi f n_0)X_{2M+1}(f)$ and therefore $Y_{2M+1}(f) = \gamma_{X,Y}(f)X_{2M+1}(f)$ for all f. Furthermore, since the coherence magnitude is unity for *all* frequencies, the prediction of the frequency component of $Y[n]$ is perfect for all frequencies as well. This says finally that $Y[n]$ can be perfectly predicted from $X[n]$. To do so just let $\hat{Y}[n] = X[n + n_0]$. In general, we will see later that if $Y[n]$ is the output of an LSI system whose input is $X[n]$, then the coherence magnitude is always unity. Can you interpret $Y[n] = X[n - n_0]$ as the action of an LSI system? Finally, in contrast to perfect prediction, consider the CPSD if $X[n]$ and $Y[n]$ are zero mean and uncorrelated. Then since $r_{X,Y}[k] = 0$, we have that $P_{X,Y}(f) = 0$ for f, and of course the coherence will be zero as well. We now summarize the properties of the CPSD.

Property 19.5 – CPSD is Fourier transform of the CCS.

$$P_{X,Y}(f) = \sum_{k=-\infty}^{\infty} r_{X,Y}[k] \exp(-j2\pi f k)$$

Proof: See Problem 19.8.

□

Property 19.6 – CPSD is a hermitian function.

A complex function $g(f)$ is hermitian if its real part is an even function and its imaginary part is an odd function about $f = 0$. This is equivalent to saying that $g(-f) = g^*(f)$. Thus,

$$P_{X,Y}(-f) = P^*_{X,Y}(f) \tag{19.21}$$

(see also (19.15) which is valid for a hermitian function).

Proof:

$$\begin{aligned}
P_{X,Y}(f) &= \sum_{k=-\infty}^{\infty} r_{X,Y}[k] \exp(-j2\pi f k) \\
P_{X,Y}(-f) &= \sum_{k=-\infty}^{\infty} r_{X,Y}[k] \exp(j2\pi f k) \\
&= \sum_{k=-\infty}^{\infty} r_{X,Y}[k] \cos(2\pi f k) + j \sum_{k=-\infty}^{\infty} r_{X,Y}[k] \sin(2\pi f k) \\
&= \left(\sum_{k=-\infty}^{\infty} r_{X,Y}[k] \cos(2\pi f k) - j \sum_{k=-\infty}^{\infty} r_{X,Y}[k] \sin(2\pi f k) \right)^* \\
&= \left(\sum_{k=-\infty}^{\infty} r_{X,Y}[k] \exp(-j2\pi f k) \right)^* \\
&= P^*_{X,Y}(f)
\end{aligned}$$

□

Property 19.7 – CPSD is bounded.

$$|P_{X,Y}(f)| \leq \sqrt{P_X(f)P_Y(f)} \tag{19.22}$$

Proof: See argument leading to (19.20).

□

Property 19.8 – CPSD is zero for zero mean uncorrelated random processes.

If $X[n]$ and $Y[n]$ are jointly WSS random processes that are zero mean and uncorrelated, then $P_{X,Y}(f) = 0$ for all f.

Proof: Since the random processes are zero mean and uncorrelated, $r_{X,Y}[k] = 0$ by definition. Hence, the CPSD is zero as well, being the Fourier transform of the CCS.

□

Property 19.9 – CPSD of $(Y[n], X[n])$ is the complex conjugate of the CPSD of $(X[n], Y[n])$.

$$P_{Y,X}(f) = P^*_{X,Y}(f) \tag{19.23}$$

Proof:

$$\begin{aligned}
P_{Y,X}(f) &= \sum_{k=-\infty}^{\infty} r_{Y,X}[k] \exp(-j2\pi fk) \\
&= \sum_{k=-\infty}^{\infty} r_{X,Y}[-k] \exp(-j2\pi fk) && \text{(using (19.11))} \\
&= \sum_{l=-\infty}^{\infty} r_{X,Y}[l] \exp(j2\pi fl) && \text{(let } l = -k\text{)} \\
&= P_{X,Y}(-f) \\
&= P^*_{X,Y}(f) && \text{(using (19.21))}
\end{aligned}$$

□

We conclude this section with one more example.

Example 19.5 – MA Random Process

Let $Y[n] = X[n] - bX[n-1]$, where $X[n]$ is white Gaussian noise with variance σ_X^2. We wish to determine the CPSD between the input $X[n]$ and output $Y[n]$ random processes (assuming they are jointly WSS, which will be borne out shortly). (Are $X[n]$ and $Y[n]$ individually WSS?) To do so we first find the CCS and then take the

Fourier transform of it. Proceeding we have

$$
\begin{aligned}
r_{X,Y}[k] &= E[X[n]Y[n+k]] \\
&= E[X[n](X[n+k] - bX[n+k-1])] \\
&= E[X[n]X[n+k]] - bE[X[n]X[n+k-1]] \\
&= r_X[k] - br_X[k-1]
\end{aligned}
$$

which does not depend on n and hence $X[n]$ and $Y[n]$ are jointly WSS with the CCS

$$r_{X,Y}[k] = \sigma_X^2 \delta[k] - b\sigma_X^2 \delta[k-1].$$

The CPSD is found as the Fourier transform to yield

$$
\begin{aligned}
P_{X,Y}(f) &= \sigma_X^2 - b\sigma_X^2 \exp(-j2\pi f) \\
&= \sigma_X^2 (1 - b\exp(-j2\pi f)).
\end{aligned}
$$

◇

Note that in the previous example we can view $Y[n]$ as the output of an LSI filter with frequency response $H(f) = 1 - b\exp(-j2\pi f)$. Therefore, we have the result

$$P_{X,Y}(f) = H(f)\sigma_X^2. \tag{19.24}$$

More generally, we will prove in the next section that if $X[n]$ is the input to an LSI system with $Y[n]$ as its corresponding output, then $X[n]$ and $Y[n]$ are jointly WSS and $P_{X,Y}(f) = H(f)P_X(f)$. As an application note, if the input to the LSI system is white noise with $\sigma_X^2 = 1$, then $P_{X,Y}(f) = H(f)$. To measure the frequency response of an unknown LSI system one can input white noise with a variance equal to one and then estimate the CCS from the input and observed output (see Section 19.7). Upon Fourier transforming that estimate one obtains an estimate of the frequency response. Lastly, since $P_{X,Y}(f) = H(f)$ for $P_X(f) = 1$, it is clear that the properties of the CPSD should mirror those of a frequency response, i.e., complex in general, hermitian, etc.

19.5 Transformations of Multiple Random Processes

We now consider the effect of some transformations on jointly WSS random processes. As a simple first example, we add the two random processes together. Hence, assume $X[n]$ and $Y[n]$ are jointly WSS random processes, and $Z[n] = X[n] + Y[n]$. We next compute the first two moments. Clearly, we will have $\mu_Z[n] = \mu_X[n] + \mu_Y[n] = \mu_X + \mu_Y$ and

$$
\begin{aligned}
r_Z[k] &= E[Z[n]Z[n+k]] \\
&= E[(X[n]+Y[n])(X[n+k]+Y[n+k])] \\
&= r_X[k] + r_{X,Y}[k] + r_{Y,X}[k] + r_Y[k] \qquad \text{(assumed jointly WSS)}
\end{aligned}
$$

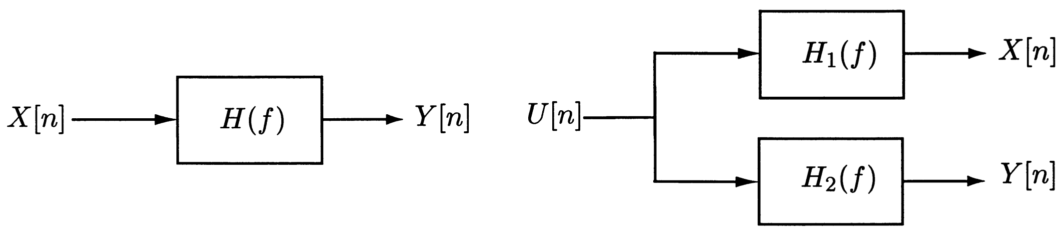
and hence $Z[n]$ is a WSS random process. Its PSD is found by taking the Fourier transform of the ACS to yield

$$P_Z(f) = P_X(f) + P_{X,Y}(f) + P_{Y,X}(f) + P_Y(f).$$

If in particular $X[n]$ and $Y[n]$ are zero mean and uncorrelated, so that $r_{X,Y}[k] = 0$ and hence $r_{Y,X}[k] = r_{X,Y}[-k] = 0$ as well, we have

$$\begin{aligned} r_Z[k] &= r_X[k] + r_Y[k] & (19.25)\\ P_X(f) &= P_X(f) + P_Y(f). & (19.26) \end{aligned}$$

Another frequently encountered transformation is that due to filtering of a WSS random process by one or two LSI filters. These transformations are shown in Figure 19.2. For the transformation shown in Figure 19.2a we already know from Chapter



(a) (b)

Figure 19.2: Common filtering operations.

18 that if $X[n]$ is WSS, then $Y[n]$ is also WSS and its mean and ACS are easily found. The question arises, however, as to whether $X[n]$ and $Y[n]$ are *jointly* WSS. To answer this we compute $E[X[n]Y[n+k]]$ to see if it depends on n. Proceeding, we have for the filtering operation shown in Figure 19.2a with $h[k]$ denoting the impulse response

$$\begin{aligned} E[X[n]Y[n+k]] &= E\left[X[n]\sum_{l=-\infty}^{\infty} h[l]X[n+k-l]\right] \\ &= \sum_{l=-\infty}^{\infty} h[l]E[X[n]X[n+k-l]] \\ &= \sum_{l=-\infty}^{\infty} h[l]r_X[k-l] \end{aligned}$$

and we see that it does not depend on n. Hence, *if $X[n]$ is the input to an LSI system and $Y[n]$ is its corresponding output, then $X[n]$ and $Y[n]$ are jointly WSS.*

Also, we have for the CCS

$$r_{X,Y}[k] = \sum_{l=-\infty}^{\infty} h[l] r_X[k-l] \tag{19.27}$$

which can be seen to be a discrete convolution or $r_{X,Y}[k] = h[k] \star r_X[k]$. As a result, by Fourier transforming the CCS we obtain the CPSD as

$$P_{X,Y}(f) = H(f) P_X(f) \tag{19.28}$$

which agrees with our earlier result of (19.24). As previously asserted, we can also now prove that if $X[n]$ is a WSS random process that is input to an LSI system and $Y[n]$ is the output random process, then the coherence magnitude is one. This says that $Y[n]$ is perfectly predictable from $X[n]$, which upon reflection just says that to predict $Y[n]$ we need only pass $X[n]$ through the same filter! To verify the assertion about the coherence magnitude

$$\begin{aligned} \gamma_{X,Y}(f) &= \frac{P_{X,Y}(f)}{\sqrt{P_X(f) P_Y(f)}} \\ &= \frac{H(f) P_X(f)}{\sqrt{P_X(f) |H(f)|^2 P_X(f)}} \qquad \text{(using (19.28) and (18.11))} \\ &= \frac{H(f)}{|H(f)|} \\ &= \exp(j\phi(f)) \end{aligned}$$

where $\phi(f)$ is the phase response of the LSI system or $\phi(f) = \angle H(f)$ Thus, $|\gamma_{X,Y}(f)| = 1$ (assuming $H(f) \neq 0$) and $Y[n]$ is perfectly predictable from $X[n]$ as

$$Y[n] = \sum_{k=-\infty}^{\infty} h[k] X[n-k] \qquad \text{for all } n$$

where $h[k]$ is the impulse response of $H(f)$. Also, $X[n]$ can be perfectly predicted from $Y[n]$ as one might expect from the analogous result of the symmetry of the correlation coefficient, which is $\rho_{X,Y} = \rho_{Y,X}$ (see Problem 19.21).

Next consider the transformation depicted in Figure 19.2b. The input random process $U[n]$ is WSS so that $X[n]$ and $Y[n]$ are individually WSS according to Theorem 18.3.1. To determine if they are jointly WSS we again compute $E[X[n]Y[n+k]]$ to see if it depends on n. Therefore, with $h_1[k]$, $h_2[k]$ denoting the impulse responses, and $H_1(f)$, $H_2(f)$ denoting the corresponding frequency responses

$$\begin{aligned} E[X[n]Y[n+k]] &= E\left[\sum_{i=-\infty}^{\infty} h_1[i]U[n-i] \sum_{j=-\infty}^{\infty} h_2[j]U[n+k-j]\right] \\ &= \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} h_1[i]h_2[j]E[U[n-i]U[n+k-j]] \\ &= \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} h_1[i]h_2[j]r_U[k+i-j] \end{aligned}$$

and does not depend on n. Hence, $X[n]$ and $Y[n]$ are jointly WSS and the CCS is

$$r_{X,Y}[k] = \sum_{i=-\infty}^{\infty} h_1[i] \underbrace{\sum_{j=-\infty}^{\infty} h_2[j]r_U[k+i-j]}_{g[k+i]}$$

where $g[n] = h_2[n] \star r_U[n]$. Continuing we have

$$\begin{aligned} r_{X,Y}[k] &= \sum_{i=-\infty}^{\infty} h_1[i]g[k+i] \\ &= \sum_{l=-\infty}^{\infty} h_1[-l]g[k-l] \qquad (\text{let } l=-i) \\ &= h_1[-k] \star g[k] \end{aligned}$$

so that

$$r_{X,Y}[k] = h_1[-k] \star h_2[k] \star r_U[k] \tag{19.29}$$

(this should be reminscent of another relationship that results if $h_1[k] = h_2[k] = h[k]$). Upon Fourier transforming both sides we have the CPSD

$$P_{X,Y}(f) = H_1^*(f)H_2(f)P_U(f). \tag{19.30}$$

An interesting observation from (19.30) is that if the two filters have nonoverlapping passbands, as shown in Figure 19.3, then

$$H_1^*(f)H_2(f) = 0 \qquad -\tfrac{1}{2} \le f \le \tfrac{1}{2}$$

and $P_{X,Y}(f) = 0$. Taking the inverse Fourier transform of the CPSD produces the CCS, which is $r_{X,Y}[k] = 0$ for all k. Hence, for nonoverlapping passband filters as shown in Figure 19.3 the $X[n]$ and $Y[n]$ random processes are *uncorrelated.* (Note that because of the nonoverlapping passbands we must have $\mu_X = 0$ or $\mu_Y = 0$.) Since this holds for any filters satisfying the nonoverlapping constraint, it also holds

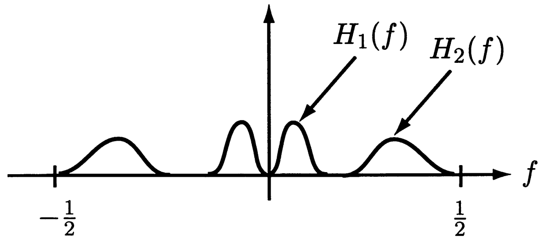


Figure 19.3: Nonoverlapping passband filters.

in particular for any narrowband filters with nonoverlapping passbands. What this says is that the *the Fourier transform of a WSS random process $U[n]$ is uncorrelated at two different frequencies.* (Actually, it is the *truncated* Fourier transform or $U_{2M+1}(f) = (1/\sqrt{2M+1})\sum_{n=-M}^{M} U[n]\exp(-j2\pi fn)$, which is required for existence, that is uncorrelated at different frequencies as $M \to \infty$.) This is because the Fourier transform can be thought of as resulting from filtering the random process with a narrowband filter and then determining the amplitude and phase of the resulting sinusoidal output. The *spectral representation* of a WSS random process is based upon this interpretation (see [Brockwell and Davis 1987] and also Problem 19.22).

⚠ **Two random processes can be individually WSS but not jointly WSS.**

All the examples thus far of individually WSS random processes have also resulted in jointly WSS random processes. To dispel the notion that this is true in general consider the following example. Let $X[n] = A$ and $Y[n] = (-1)^n A$, where A is a random variable with $E[A] = 0$ and $\text{var}(A) = 1$. Then, $\mu_X[n] = \mu_Y[n] = 0$ and it is easily shown that $r_X[k] = 1$ for all k and $r_Y[k] = (-1)^k$ for all k. Therefore, $X[n]$ and $Y[n]$ are individually WSS random processes but they are not jointly WSS since

$$E[X[n]Y[n+k]] = E[A^2(-1)^{n+k}] = (-1)^n(-1)^k$$

which depends on n. For example, since $X[0] = Y[2] = A$ and $X[1] = -Y[3] = A$, we have that

$$\begin{aligned} E[X[0]Y[2]] &= E[A^2] = 1 \\ E[X[1]Y[3]] &= E[A(-A)] = -1 \end{aligned}$$

so that the cross-correlation between two samples spaced two units apart depends on n.

⚠

19.6 Continuous-Time Definitions and Formulas

Two continuous-time random processes $X(t)$ and $Y(t)$ for $-\infty < t < \infty$ are *jointly WSS* if $X(t)$ is WSS, $Y(t)$ is WSS, and we can define the cross-correlation function (CCF) as

$$r_{X,Y}(\tau) = E[X(t)Y(t+\tau)] \qquad -\infty < \tau < \infty \tag{19.31}$$

which does not depend on t. Some properties (actually *non*properties) of the CCF are

Property 19.10 – CCF is not necessarily symmetric about $\tau = 0$.

$$r_{X,Y}(\tau) \neq r_{X,Y}(-\tau) \tag{19.32}$$

□

Property 19.11 – The maximum of the CCF can occur for any value of τ.

□

Property 19.12 – The maximum value of the CCF is bounded.

$$|r_{X,Y}(\tau)| \leq \sqrt{r_X(0) r_Y(0)} \tag{19.33}$$

□

Property 19.13 – Interchanging $X(t)$ and $Y(t)$ flips the CCF about $\tau = 0$.

$$r_{Y,X}(\tau) = r_{X,Y}(-\tau) \tag{19.34}$$

□

Two zero mean jointly WSS continuous random processes are said to be uncorrelated if $r_{X,Y}(\tau) = 0$ for $-\infty < \tau < \infty$.

The CPSD for two jointly WSS random processes is defined as

$$P_{X,Y}(F) = \lim_{T\to\infty} \frac{1}{T} E\left[\left(\int_{-T/2}^{T/2} X(t)\exp(-j2\pi Ft)dt\right)^* \left(\int_{-T/2}^{T/2} Y(t)\exp(-j2\pi Ft)dt\right)\right] \tag{19.35}$$

and is evaluated as

$$P_{X,Y}(F) = \int_{-\infty}^{\infty} r_{X,Y}(\tau)\exp(-j2\pi F\tau)d\tau. \tag{19.36}$$

Some properties of the CPSD follow. The proofs are similar to those for the discrete-time case.

Property 19.14 – CPSD is a complex and hermitian function.
The hermitian property is

$$P_{X,Y}(-F) = P^*_{X,Y}(F) \tag{19.37}$$

□

Property 19.15 – CPSD is bounded.

$$|P_{X,Y}(F)| \leq \sqrt{P_X(F)P_Y(F)} \tag{19.38}$$

□

Property 19.16 – CPSD of $(Y(t), X(t))$ is the complex conjugate of the CPSD of $(X(t), Y(t))$.

$$P_{Y,X}(F) = P^*_{X,Y}(F) \tag{19.39}$$

□

The formulas for the linear system configuration corresponding to that shown in Figure 19.2a are (continuous-time system is assumed to be LTI with impulse response $h(\tau)$ and frequency response $H(f)$)

$$\begin{aligned} r_{X,Y}(\tau) &= h(\tau) \star r_X(\tau) && (19.40) \\ P_{X,Y}(F) &= H(F)P_X(F) && (19.41) \end{aligned}$$

and for the configuration of Figure 19.2b (continuous-time systems are assumed to be LTI with impulse responses $h_1(\tau)$, $h_2(\tau)$, and corresponding frequency responses $H_1(f)$, $H_2(f)$)

$$\begin{aligned} r_{X,Y}(\tau) &= h_1(-\tau) \star h_2(\tau) \star r_U(\tau) && (19.42) \\ P_{X,Y}(F) &= H_1^*(F)H_2(F)P_U(F). && (19.43) \end{aligned}$$

An example of great practical importance is given next to illustrate the concepts and formulas.

Example 19.6 – Measurement of Channel Delay
It is frequently of interest to be able to measure the propagation time of a signal through a channel. This allows one to determine distance if the speed of propagation is known. This idea forms the basis for the global positioning system (GPS) [Hofmann-Wellenhof, Lichtenegger, Collins 1992]. See also Problem 19.28 for another application. To do so we transmit a WSS random process $X(t)$, that is bandlimited to W Hz (meaning that $P_X(F) = 0$ for $|F| > W$) through a channel and

observe the output of the channel $Y(t)$. We furthermore assume that the channel is modeled as an LTI system with frequency response

$$H(F) = \frac{\exp(-j2\pi F t_0)}{1 + j2\pi F}. \tag{19.44}$$

Note that the numerator term represents a delay of t_0 seconds, sometimes called the propagation or bulk delay, and the term $H_{\mathrm{LP}}(F) = 1/(1 + j2\pi F)$ represents a low-pass filter response since $H_{\mathrm{LP}}(0) = 1$ and $H_{\mathrm{LP}}(F) \to 0$ as $F \to \infty$. A question arises as to how to choose the transmit random process $X(t)$ so that we can accurately measure the delay t_0 through the channel. In the ideal case in which $Y(t)$ is just a delayed replica of $X(t)$ or $Y(t) = X(t - t_0)$, we know that the CCF is

$$\begin{aligned} r_{X,Y}(\tau) &= E[X(t)Y(t+\tau)] \\ &= E[X(t)X(t+\tau-t_0)] \\ &= r_X(\tau - t_0). \end{aligned}$$

Since the ACF has a maximum at lag zero, there will be maximum of $r_{X,Y}(\tau)$ at $\tau = t_0$, suggesting that the location of this maximum can be used to measure the delay. But when the channel has the frequency response given by (19.44) the maximum of the CCF may no longer be located at $\tau = t_0$. To see why, first compute the CCF as

$$\begin{aligned} r_{X,Y}(\tau) &= \int_{-\infty}^{\infty} P_{X,Y}(F) \exp(j2\pi F\tau) dF && \text{(inverse Fourier transform)} \\ &= \int_{-\infty}^{\infty} H(F) P_X(F) \exp(j2\pi F\tau) dF && \text{(from (19.41))} \\ &= \int_{-\infty}^{\infty} \frac{\exp(-j2\pi F t_0)}{1 + j2\pi F} P_X(F) \exp(j2\pi F\tau) dF && \text{(from (19.44))} \\ &= \int_{-\infty}^{\infty} \frac{1}{1 + j2\pi F} P_X(F) \exp(j2\pi F(\tau - t_0)) dF \end{aligned}$$

and since $X(t)$ is assumed to be bandlimited to W Hz, we have

$$r_{X,Y}(\tau) = \int_{-W}^{W} \frac{1}{1 + j2\pi F} P_X(F) \exp(j2\pi F(\tau - t_0)) dF. \tag{19.45}$$

If, as an example, we choose $X(t)$ to be bandlimited white noise (see Example 17.11) or $P_X(F) = N_0/2$ for $|F| \leq W$ and $P_X(F) = 0$ for $|F| > W$, then

$$r_{X,Y}(\tau) = \frac{N_0}{2} \int_{-W}^{W} \frac{1}{1 + j2\pi F} \exp(j2\pi F(\tau - t_0)) dF.$$

To evaluate this we first note that

$$\mathcal{F}^{-1}\left\{\frac{1}{1+j2\pi F}\right\} = \int_{-\infty}^{\infty} \frac{1}{1+j2\pi F} \exp(j2\pi F t)dF = \exp(-t)u(t)$$

so that if we define the *frequency window* function

$$G(F) = \begin{cases} 1 & |F| \leq W \\ 0 & |F| > W \end{cases}$$

then

$$\begin{aligned} r_{X,Y}(\tau) &= \frac{N_0}{2} \int_{-\infty}^{\infty} G(F) \frac{1}{1+j2\pi F} \exp(j2\pi F(\tau - t_0))dF \\ &= \frac{N_0}{2} g(t) \star \exp(-t)u(t)\big|_{t=\tau-t_0} \qquad \text{(convolution in time yields} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{multiplication in frequency).} \end{aligned}$$

where $g(t)$ is the inverse Fourier transform of $G(F)$. We have chosen to express the integral in the time domain since its physical significance becomes clearer. In particular, note that the convolution in time results in a wider pulse. But

$$g(t) = 2W \frac{\sin(2\pi W t)}{2\pi W t} \qquad \text{(see Example 17.11)}$$

so that using a convolution integral, we have

$$\begin{aligned} r_{X,Y}(\tau) &= N_0 W \int_{-\infty}^{\infty} \exp(-\xi)u(\xi) \frac{\sin[2\pi W((\tau - t_0) - \xi)]}{2\pi W((\tau - t_0) - \xi)} d\xi \\ &= N_0 W \int_{0}^{\infty} \exp(-\xi) \frac{\sin[2\pi W((\tau - t_0) - \xi)]}{2\pi W((\tau - t_0) - \xi)} d\xi. \end{aligned}$$

This is shown in Figure 19.4 for the case when $W = 1$ and $t_0 = 2$ as the light line and has been normalized to have a maximum value of 1. The integral has been evaluated numerically. Note that the maximum does not occur at $t_0 = 2$ because the phase response of the channel has added a time delay. To remedy this problem we can insert an *equalizing filter* at the channel output whose frequency response is

$$H_{\text{eq}}(F) = \begin{cases} 1 + j2\pi F & |F| \leq W \\ 0 & |F| > W. \end{cases}$$

Then, we have for the CPSD between the input $X(t)$ and output random process of the equalizer $Y(t)$

$$P_{X,Y}(F) = H_{\text{eq}}(F)H(F)P_X(F) = \begin{cases} \frac{N_0}{2} \exp(-j2\pi F t_0) & |F| \leq W \\ 0 & |F| > W. \end{cases}$$

The CCF is found as before by using the inverse Fourier transform

$$r_{X,Y}(\tau) = \int_{-\infty}^{\infty} P_{X,Y}(F) \exp(j2\pi F \tau)dF$$

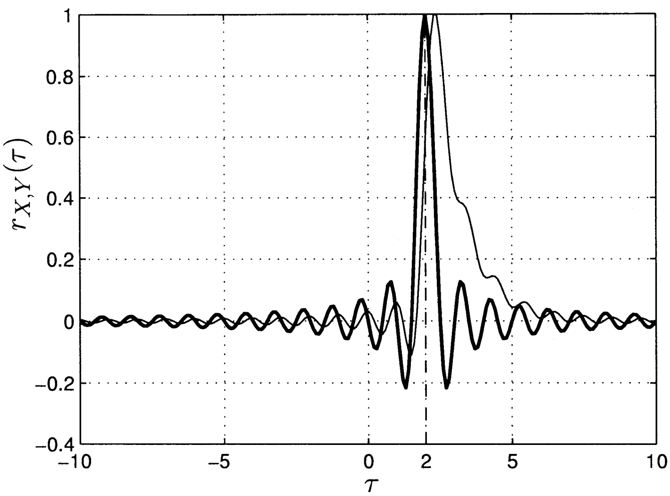


Figure 19.4: Cross-correlation functions for $W = 1$ and $t_0 = 2$. Both curves are normalized to yield one at their peak. The light line is for no equalization while the dark line incorporates equalization. The dashed line indicates $\tau = 2$, the true delay.

$$
\begin{aligned}
&= \int_{-W}^{W} \frac{N_0}{2} \exp(j2\pi F(\tau - t_0))dF \\
&= N_0 W \frac{\sin(2\pi W(\tau - t_0))}{2\pi W(\tau - t_0)}
\end{aligned}
$$

which is shown in Figure 19.4 as the dark line. As before it has been normalized to yield one at its peak. Note that the maximum now occurs at the correct location and also the width of the maximum peak is narrower. This allows a better location of the maximum in the presence of noise.

◊

19.7 Cross-correlation Sequence Estimation

The estimation of the CCS is similar to that for the ACS (see Section 17.7). The main difference between the two stems from the fact that the ACS is guaranteed to have a maximum at $k = 0$ while the maximum for the CCS can be located anywhere. Furthermore, two samples of a WSS random process tend to become less correlated as the spacing between them increases. This implies that it is only necessary to estimate the ACS $r_X[k]$ for $k = 0, 1, \ldots, M$ if we assume that $r_X[k] \approx 0$ for $k > M$. For the CCS, however, we must estimate $r_{X,Y}[k]$ for $k = -M_1, \ldots, 0, \ldots, M_2$ (recall that $r_{X,Y}[-k] \neq r_{X,Y}[k]$) for which $r_{X,Y}[k] \approx 0$ if $k < -M_1$ or $k > M_2$. In practice, it is not clear how M_1 and M_2 should be chosen. Frequently, a preliminary estimate of $r_{X,Y}[k]$ is made, followed by a search for the maximum location. Then, the data records used to estimate the CCS are *shifted* relative to each other to place the maximum at $k = 0$. This is called *time alignment* [Jenkins and Watts

1968]. We assume that this has already been done. Then, we estimate the CCS for $|k| \leq M$ assuming that we have observed the realizations for $X[n]$ and $Y[n]$, both for $n = 0, 1, \ldots, N-1$. The estimate becomes

$$\hat{r}_{X,Y}[k] = \begin{cases} \frac{1}{N-k}\sum_{n=0}^{N-1-k} x[n]y[n+k] & k = 0, 1, \ldots, M \\ \frac{1}{N-|k|}\sum_{n=|k|}^{N-1} x[n]y[n+k] & k = -M, -(M-1), \ldots, -1. \end{cases} \tag{19.46}$$

Note that the summation limits have been chosen to make sure that all the available products $x[n]y[n+k]$ are used. Similar to the estimation of the ACS, there will be a different number of products for each k. For example, if $N = 4$ so that $\{x[0], x[1], x[2], x[3]\}$ and $\{y[0], y[1], y[2], y[3]\}$ are observed, and we wish to compute the CCS estimate for $|k| \leq M = 2$, we will have

$$\begin{aligned}
\hat{r}_{X,Y}[-2] &= \frac{1}{2}\sum_{n=2}^{3} x[n]y[n-2] = \frac{1}{2}(x[2]y[0] + x[3]y[1]) \\
\hat{r}_{X,Y}[-1] &= \frac{1}{3}\sum_{n=1}^{3} x[n]y[n-1] = \frac{1}{3}(x[1]y[0] + x[2]y[1] + x[3]y[2]) \\
\hat{r}_{X,Y}[0] &= \frac{1}{4}\sum_{n=0}^{3} x[n]y[n] = \frac{1}{4}(x[0]y[0] + x[1]y[1] + x[2]y[2] + x[3]y[3]) \\
\hat{r}_{X,Y}[1] &= \frac{1}{3}\sum_{n=0}^{2} x[n]y[n+1] = \frac{1}{3}(x[0]y[1] + x[1]y[2] + x[2]y[3]) \\
\hat{r}_{X,Y}[2] &= \frac{1}{2}\sum_{n=0}^{1} x[n]y[n+2] = \frac{1}{2}(x[0]y[2] + x[1]y[3]).
\end{aligned}$$

As an example, consider the jointly WSS random processes described in Example 19.2, where $X[n] = U[n]$, $Y[n] = U[n] + 2U[n-1]$ and $U[n]$ is white noise with variance $\sigma_U^2 = 1$. We further assume that $U[n]$ has a Gaussian PDF for each n for the purpose of computer simulation (although we could use any PDF or PMF). Recall that the theoretical CCS is $r_{X,Y}[k] = \delta[k] + 2\delta[k-1]$. The estimated CCS using $N = 1000$ data samples is shown in Figure 19.5. The MATLAB code used to estimate the CCS is given below.

```
% assume realizations are x[n], y[n] for n=1,2,...,N
for k=0:M % compute zero and positive lags, see (19.46)
 % compute values for k=0,1,...,M
  rxypos(k+1,1)=(1/(N-k))*sum(x(1:N-k).*y(1+k:N));
end
for k=1:M % compute negative lags, see (19.46)
 % compute values for k=-M,-(M-1),...,-1
  rxyneg(k+1,1)=(1/(N-k))*sum(x(k+1:N).*y(1:N-k));
end
rxy=[flipud(rxyneg(2:M+1,1));rxypos]; % arrange values from k=-M to k=M
```

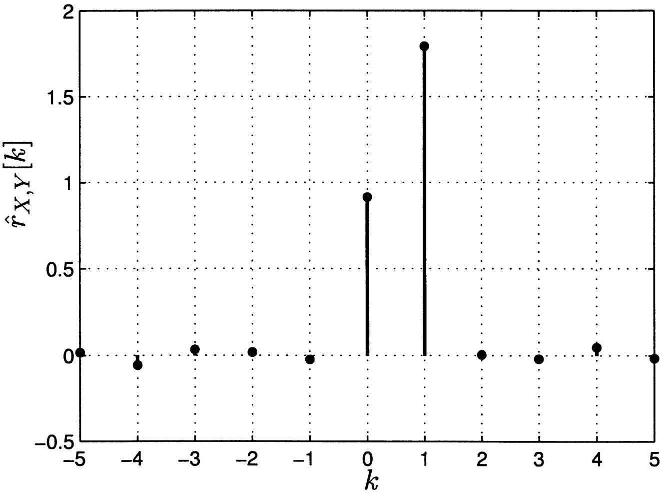
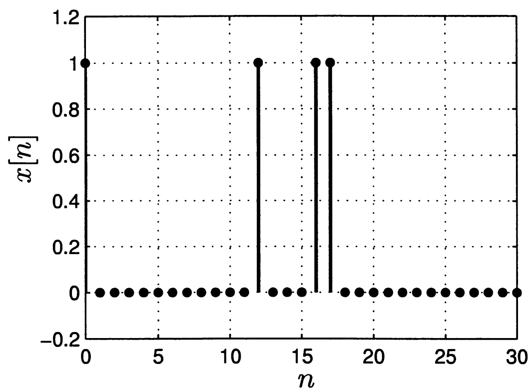


Figure 19.5: Estimated CCS using realizations for $X[n]$ and $Y[n]$ with $N = 1000$ samples. The theoretical CCS is $r_{X,Y}[k] = \delta[k] + 2\delta[k-1]$.

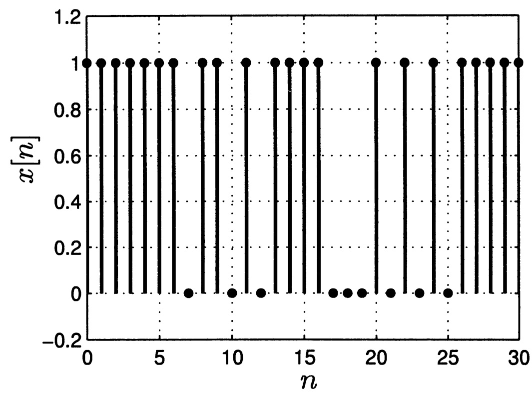
Finally, we note that estimation of the CPSD is more difficult and so we refer the interested reader to [Jenkins and Watts 1968, Kay 1988].

19.8 Real-World Example – Brain Physiology Research

Understanding the operation of the human brain is one of the most important goals of physiological research. Currently, there is an enormous effort to decipher its inner workings. At a very fundamental level is the study of its cells or *neurons*, which when working in unison form the basis for our behavior. Their electrical activity and the transmission of that activity to neighboring neurons yields clues as to the brain's operation. When an individual neuron "fires" it produces a *spike* or electrical pulse that propagates to nearby neurons. The connections between the neurons that allow this propagation to occur are called *synapses* and it is this connectivity that is the focus of much research. A typical spike train that might be recorded is shown in Figure 19.6a for a neuron at rest and in Figure 19.6b for a neuron that has been excited by some stimulus. Clearly, the firing rate increases in response to a stimulus. The model used to produce this figure is an IID Bernoulli random process with $p = p_q = 0.1$ for Figure 19.6a and $p = p_s = 0.6$ for Figure 19.6b. The subscripts "q" and "s" are meant to indicate the state of the neuron, either *quiescent* or *stimulated.* Now consider the question of whether two neurons are connected via a synapse. If they are, and a stimulus is applied to the first neuron, then the electrical pulse will propagate to the second neuron and appear some time later. Then, we would expect the second neuron electrical activity to change from that in Figure 19.6a to that in Figure 19.6b. It would be fairly simple



(a) Quiescent, $p_q = 0.1$



(b) Stimulated, $p_s = 0.6$

Figure 19.6: Typical spike trains for neurons.

then to estimate the p for each possible connected neuron and choose the neuron or neurons (there may be multiple connections with the stimulated neuron) for which p is large. Unfortunately, it is not easy to stimulate a single neuron so that when a stimulus is applied, many neurons may be activated. Thus, we need a method to associate one stimulated neuron with its connected ones. Ideally, if we record the electrical activity at two neurons under consideration, denoted by $X_1[n]$ and $X_2[n]$, then for connected neurons $X_2[n] = X_1[n - n_0]$. Since we have assumed that the spike train for the first neuron $X_1[n]$ is an IID random process, it is therefore WSS and we know from Example 19.1, the two random processes are jointly WSS. Therefore, we have as before

$$
\begin{aligned}
r_{X_1,X_2}[k] &= E[X_1[n]X_2[n+k]] \\
&= E[X_1[n]X_1[n-n_0+k]] \\
&= r_{X_1}[k-n_0]
\end{aligned}
$$

and therefore the CCS will exhibit a maximum at $k = n_0$. Otherwise, if the neurons are not connected, we would expect a much smaller value of the maximum or no discernible maximum at all. For example, for unconnected but simultaneously stimulated neurons it is reasonable to assume that $X_1[n]$ and $X_2[n]$ are uncorrelated and hence $r_{X_1,X_2}[k] = E[X_1[n]]E[X_2[n+k]] = p_s^2$, which presumably will be less than $r_{X_1}[k - n_0]$ at its peak. Note that for connected neurons

$$
r_{X_1,X_2}[k] = \operatorname{cov}(X_1[n], X_2[n+k]) + E[X_1[n]]E[X_2[n+k]] > E[X_1[n]]E[X_2[n+k]]
$$

if the covariance is positive.

Specifically, we assume that a neuron output is modeled as an IID Bernoulli random process that takes on the values 1 and 0 with probabilities p_s and $1 - p_s$, respectively. For two neurons that are connected we have that $r_{X_1,X_2}[k] = r_{X_1}[k - n_0]$. But

$$\begin{aligned} r_{X_1}[k] &= E[X_1[n]X_1[n+k]] \\ &= \begin{cases} E[X_1^2[n]] & k = 0 \\ E[X_1[n]]E[X_1[n+k]] & k \neq 0 \end{cases} \\ &= \begin{cases} p_s & k = 0 \\ p_s^2 & k \neq 0 \end{cases} \\ &= p_s(1-p_s)\delta[k] + p_s^2. \end{aligned}$$

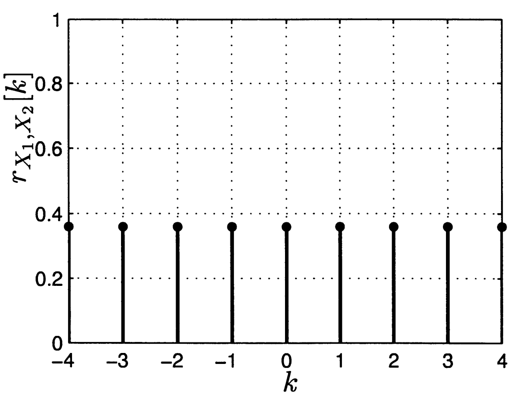
Hence, for two connected neurons the CCS is

$$r_{X_1,X_2}[k] = p_s(1-p_s)\delta[k-n_0] + p_s^2.$$

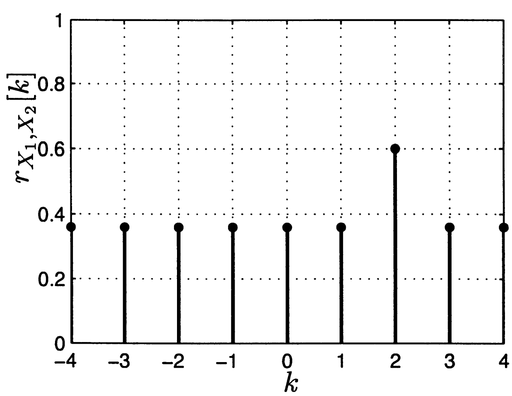
For two neurons that are not connected, so that their outputs are uncorrelated (even if both are stimulated), the CCS is

$$\begin{aligned} r_{X_1,X_2}[k] &= E[X_1[n]]X_2[n+k]] \\ &= E[X_1[n]]E[X_2[n+k]] \\ &= p_s^2 \qquad \text{for all } k. \end{aligned}$$

As a result, the maximum is p_s^2 for unconnected neurons but $p_s(1-p_s)+p_s^2 = p_s > p_s^2$ for connected neurons. The two different CCSs are shown in Figure 19.7 for $p_s = 0.6$ and $n_0 = 2$. As an example, for $p_s = 0.6$ we show realizations of three neuron



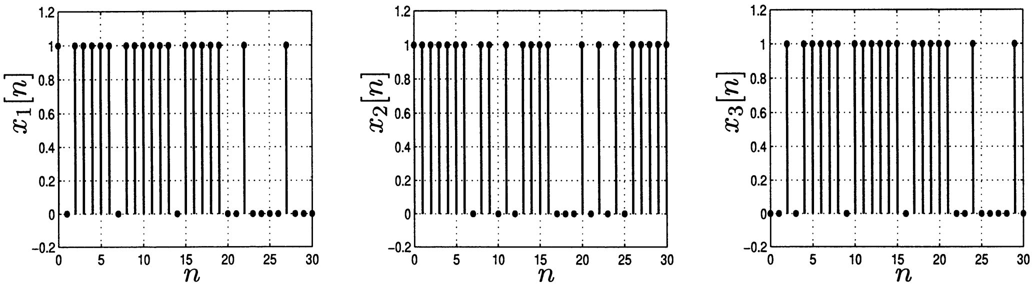
(a) Unconnected



(b) Connected with $n_0 = 2$

Figure 19.7: CCS for unconnected and connected stimulated neurons with $p_s = 0.6$.

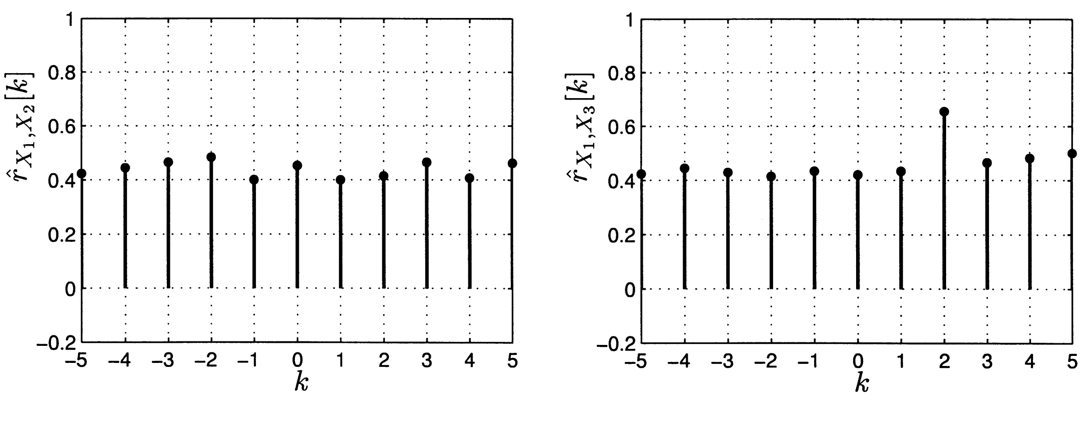
outputs in Figure 19.8, where only neuron 1 and neuron 3 are connected. There is



(a) Neuron 1 (b) Neuron 2 (c) Neuron 3

Figure 19.8: Spike trains for three neurons with neuron 1 connected to neuron 3 with a delay of two samples. The spike train of neuron 2 is uncorrelated with those for neurons 1 and 3.

a two sample delay between neurons 1 and 3. Neuron 2 is not connected to either of the other neurons and hence its spike train is uncorrelated with the others. The theoretical CCS between neurons 1 and 2 is given in Figure 19.7a while that between neurons 1 and 3 is given in Figure 19.7b. The estimated CCS for the spike trains shown in Figure 19.8 and based on the estimate of (19.46) is shown in Figure 19.9.



(a) Unconnected neurons 1 and 2 (b) Connected neurons 1 and 3 with $n_0 = 2$

Figure 19.9: Estimated CCS for unconnected and connected stimulated neurons with $p_s = 0.6$.

It is seen that as expected there is a maximum at $k = n_0 = 2$. The interested reader should consult [Univ. Pennsylvannia 2005] for further details.

References

Bendat, J.S., A.G. Piersol, *Random Data: Analysis and Measurement Procedures*, Wiley-Interscience, New York, 1971.

Brockwell, P.J., R.A. Davis, *Time Series: Theory and Methods*, Springer-Verlag, New York, 1987.

Hofmann-Wellenhof, H. Lichtenegger, J. Collins, *Global Positioning System: Theory and Practice*, Springer-Verlag, New York, 1992.

Jenkins, G.M., D.G. Watts, *Spectral Analysis and Its Applications*, Holden-Day, San Francisco, 1968.

Kay, S., *Modern Spectral Estimation: Theory and Application*, Prentice Hall, Englewood Cliffs, NJ, 1988.

Koopmans, L.H.,*The Spectral Analysis of Time Series*, Academic Press, New York, 1974.

Robinson, E.A., *Multichannel Time Series Analysis with Digital Computer Programs*, Holden-Day, San Francisco, 1967.

University of Pennsylvannia, Multiple Unit Laboratory, http://mulab.physiol.upenn.edu/analysis.html

Problems

19.1 (⌣̈) **(w)** Two discrete-time random processes are defined as $X[n] = U[n]$ and $Y[n] = (-1)^n U[n]$ for $-\infty < n < \infty$, where $U[n]$ is white noise with variance σ_U^2. Are the random processes $X[n]$ and $Y[n]$ jointly WSS?

19.2 (w) Two discrete-time random processes are defined as $X[n] = a_1U_1[n] + a_2U_2[n]$ and $Y[n] = b_1U_1[n] + b_2U_2[n]$ for $-\infty < n < \infty$, where $U_1[n]$ and $U_2[n]$ are jointly WSS and a_1, a_2, b_1, b_2 are constants. Are the random processes $X[n]$ and $Y[n]$ jointly WSS?

19.3 (f) If the CCS is given as $r_{X,Y}[k] = (1/2)^{|k-1|}$ for $-\infty < k < \infty$, plot it and describe which properties are the same or different from an ACS.

19.4 (f) If $Y[n] = X[n] + W[n]$, where $X[n]$ and $W[n]$ are jointly WSS, find $r_{X,Y}[k]$ and $P_{X,Y}(f)$.

19.5 (⌣̈) **(w)** A discrete-time random process is defined as $Y[n] = X[n]W[n]$, where $X[n]$ is WSS and $W[n]$ is an IID Bernoulli random process that takes on values ± 1 with equal probability. The random processes $X[n]$ and $W[n]$ are independent of each other, which means that $X[n_1]$ is independent of $W[n_2]$ for all n_1 and n_2. Find $r_{X,Y}[k]$ and explain your results.

19.6 (⌣) **(w)** In this problem we show that for the AR random process $X[n] = aX[n-1] + U[n]$, which was described in Example 17.5, the cross-correlation sequence $E[X[n]U[n+k]] = 0$ for $k > 0$. Do so by evaluating $E[X[n](X[n+k] - aX[n+k-1])]$. Determine and plot the CCS $r_{X,U}[k]$ for $-\infty < k < \infty$ if $a = 0.5$ and $\sigma_U^2 = 1$. Hint: Refer back to Example 17.5 for the ACS of an AR random process.

19.7 (f) If $X[n]$ and $Y[n]$ are jointly WSS with ACSs

$$\begin{aligned} r_X[k] &= 4\left(\frac{1}{2}\right)^{|k|} \\ r_Y[k] &= 3\delta[k] + 2\delta[k+1] + 2\delta[k-1] \end{aligned}$$

determine the maximum possible value of $r_{X,Y}[k]$.

19.8 (t) Derive (19.14). To do so use the relationship $\sum_{m=-M}^{M} \sum_{n=-M}^{M} g[m-n] = \sum_{k=-2M}^{2M} (2M + 1 - |k|)g[k]$.

19.9 (f) For the two sinusoidal random processes $X[n] = \cos(2\pi f_0 n + \Theta_1)$ and $Y[n] = \cos(2\pi f_0 n + \Theta_2)$, where $\Theta_1 = \Theta_2 \sim \mathcal{U}(0, 2\pi)$ find the CPSD and explain your results versus the case when Θ_1 and Θ_2 are independent random variables.

19.10 (⌣) **(f,c)** If $r_{X,Y}[k] = \delta[k] + 2\delta[k-1]$, plot the magnitude and phase of the CPSD. You will need a computer to do this.

19.11 (f) For the random processes $X[n] = U[n]$ and $Y[n] = U[n] - bU[n-1]$, where $U[n]$ is discrete white noise with variance $\sigma_U^2 = 1$, find the CPSD and explain what happens as $b \to 0$.

19.12 (⌣) **(w)** If a random process is defined as $Z[n] = X[n] - Y[n]$, where $X[n]$ and $Y[n]$ are jointly WSS, determine the ACS and PSD of $Z[n]$.

19.13 (w) For the random processes $X[n]$ and $Y[n]$ defined in Problem 19.11 find the coherence function. Explain what happens as $b \to 0$.

19.14 (f) Determine the CPSD for two jointly WSS random processes if $r_{X,Y}[k] = \delta[k] - \delta[k-1]$. Also, explain why the coherence function at $f = 0$ is zero. Hint: The random processes $X[n]$ and $Y[n]$ are those given in Problem 19.11 if $b = 1$.

19.15 (⌣) **(f)** If $Y[n] = -X[n]$ for $-\infty < n < \infty$, determine the coherence function and relate it to the predictability of $Y[n_0]$ based on observing $X[n]$ for $-\infty < n < \infty$.

19.16 (t) A *cross-spectral matrix* is defined as

$$\begin{bmatrix} P_X(f) & P_{X,Y}(f) \\ P_{Y,X}(f) & P_Y(f) \end{bmatrix}.$$

Prove that the cross-spectral matrix is positive semidefinite for all f. Hint: Show that the principal minors of the matrix are all nonnegative (see Appendix C for the definition of principal minors). To do so use the properties of the coherence function.

19.17 (w) The random processes $X[n]$ and $Y[n]$ are zero mean jointly WSS and are uncorrelated with each other. If $r_X[k] = 2\delta[k]$ and $r_Y[k] = (1/2)^{|k|}$ for $-\infty < k < \infty$, find the PSD of $X[n] + Y[n]$.

19.18 (⌣) (t) In this problem we derive an extension of the Wiener smoother (see Section 18.5.1). We consider the problem of estimating $Y[n_0]$ based on observing $X[n]$ for $-\infty < n < \infty$. To do so we use the linear estimator

$$\hat{Y}[n_0] = \sum_{k=-\infty}^{\infty} h[k]X[n_0 - k].$$

To find the optimal impulse response we employ the orthogonality principle to yield the infinite set of simultaneous linear equations

$$E\left[(Y[n_0] - \sum_{k=-\infty}^{\infty} h[k]X[n_0 - k])X[n_0 - l]\right] = 0 \qquad -\infty < l < \infty.$$

Assuming that $X[n]$ and $Y[n]$ are jointly WSS random processes, determine the frequency response of the optimal Wiener estimator. Then, show how the Wiener smoother, where $Y[n]$ represents the signal $S[n]$ and $X[n]$ represents the signal $S[n]$ plus noise $W[n]$ (recall that $S[n]$ and $W[n]$ are zero mean and uncorrelated random processes), arises as a special case of this solution.

19.19 (f) For the random processes defined in Example 19.2 determine the CPSD. Next, find the optimal Wiener smoother for $Y[n_0]$ based on the realization of $X[n]$ for $-\infty < n < \infty$.

19.20 (t) Prove that if $X[n]$ is a WSS random process that is input to an LSI system and $Y[n]$ is the corresponding random process output, then the coherence function between the input and output has a magnitude of one.

19.21 (t) Consider a WSS random process $X[n]$ that is input to an LSI system with frequency response $H(f)$, where $H(f) \neq 0$ for $|f| \leq 1/2$, and let $Y[n]$ be the corresponding random process output. It is desired to predict $X[n_0]$ based on observing $Y[n]$ for $-\infty < n < \infty$. Draw a linear filtering diagram (similar to that shown in Figure 19.2) to explain why $X[n_0]$ is perfectly predictable by passing $Y[n]$ through a filter with frequency response $1/H(f)$.

19.22 (t) In this problem we argue that a Fourier transform is actually a narrowband filtering operation. First consider the Fourier transform at $f = f_0$ for the truncated random process $X[n]$, $n = -M, \ldots, 0, \ldots, M$ which is $\hat{X}(f_0) = \sum_{k=-M}^{M} X[k]\exp(-j2\pi f_0 k)$. Next show that this may be written as

$$\hat{X}(f_0) = \sum_{k=-\infty}^{\infty} X[k]h[n-k]\bigg|_{n=0}$$

where

$$h[k] = \begin{cases} \exp(j2\pi f_0 k) & k = -M, \ldots, 0, \ldots, M \\ 0 & |k| > M. \end{cases}$$

Notice that this is a convolution sum so that $h[k]$ can be considered as the impulse response, although a complex one, of an LSI filter. Finally, find and plot the frequency response of this filter. Hint: You will need

$$\sum_{k=-M}^{M} \exp(jk\theta) = \frac{\sin((2M+1)\theta/2)}{\sin(\theta/2)}.$$

19.23 (⌣) (w) Consider the continuous-time averager

$$Y(t) = \frac{1}{T}\int_{t-T}^{t} X(\xi)d\xi$$

where the random process $X(t)$ is continuous-time white noise with PSD $P_X(F) = N_0/2$ for $-\infty < F < \infty$. Determine the CCF $r_{X,Y}(\tau)$ and show that it is zero for τ outside the interval $[0, T]$. Explain why it is zero outside this interval.

19.24 (f) If a continuous-time white noise process $X(t)$ with ACF $r_X(\tau) = (N_0/2)\delta(\tau)$ is input to an LTI system with impulse response $h(\tau) = \exp(-\tau)u(\tau)$, determine $r_{X,Y}(\tau)$.

19.25 (t) Can the CPSD ever have the same properties as the PSD in terms of being real and symmetric? If so, give an example. Hint: Consider the relationship given in (19.43).

19.26 (⌣) (f,c) Consider the random processes $X[n] = U[n]$ and $Y[n] = U[n] - bU[n-1]$, where $U[n]$ is white Gaussian noise with variance $\sigma_U^2 = 1$. Find $r_{X,Y}[k]$ and then to verify your results perform a computer simulation. To do so first generate $N = 1000$ samples of $X[n]$ and $Y[n]$. Then, estimate the CCS for $b = -0.1$ and $b = -1$. Explain your results.

19.27 (f,c) An AR random process is given by $X[n] = aX[n-1] + U[n]$, where $U[n]$ is white Gaussian noise with variance σ_U^2. Find the CCS $r_{X,U}[k]$ and then to verify your results perform a computer simulation using $a = 0.5$ and $\sigma_U^2 = 1$. To do so first generate $N = 1000$ samples of $U[n]$ and $X[n]$. Then, estimate the CCS. Hint: Remember to set the initial condition $X[-1] \sim \mathcal{N}(0, \sigma_U^2/(1-a^2))$.

19.28 (w) In this problem we explore the use of the CCF to determine the direction of arrival of a sound source. Referring to Figure 19.10, a sound source emits a pulse that propagates to a set of two receivers. Because the distance from the source to the receivers is large, it is assumed that the wavefronts are planar as shown. If the source has the angle θ with respect to the x axis as shown, it first reaches receiver 2 and then reaches receiver 1 at a time $t_0 = d\cos(\theta)/c$ seconds later, where d is the distance between receivers and c is the propagation speed. Assume that the received signal at receiver 2 is a WSS random process $X_2(t) = U(t)$ with a PSD

$$P_U(F) = \begin{cases} N_0/2 & |F| \leq W \\ 0 & |F| > W \end{cases}$$

and therefore the received signal at receiver 1 is $X_1(t) = U(t - t_0)$. Determine the CCF $r_{X_1,X_2}(\tau)$ and describe how it could be used to find the arrival angle θ.

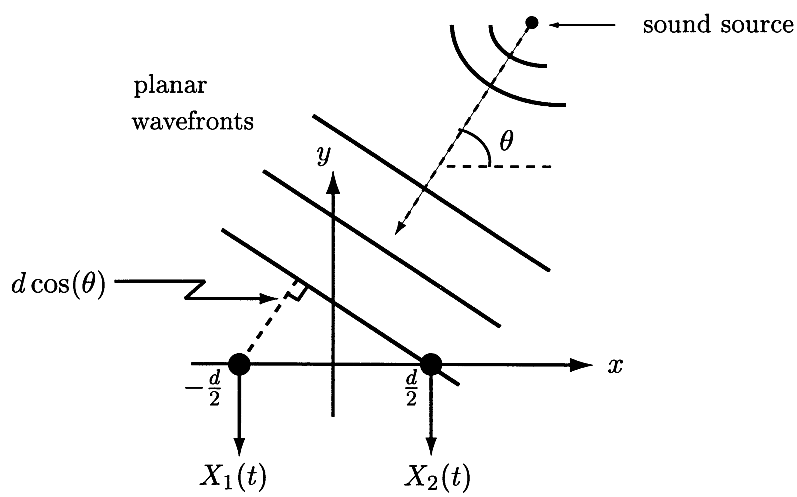


Figure 19.10: Geometry for sound source arrival angle measurement (figure for Problem 19.28).