Chapter 21

Poisson Random Processes

21.1 Introduction

A random process that is useful for modeling events occurring in time is the *Poisson random process*. A typical realization is shown in Figure 21.1 in which the events, indicated by the "x"s, occur randomly in time. The random process, whose real-
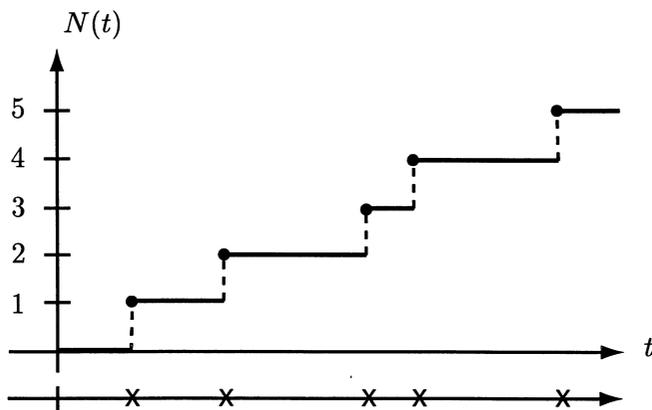
Figure 21.1: Poisson process events and the Poisson counting random process $N(t)$.

ization is a set of times, is called the *Poisson random process*. The random process that counts the number of events in the time interval $[0, t]$, and which is denoted by $N(t)$, is called the *Poisson counting random process*. It is clear from Figure 21.1 that the two random processes are equivalent descriptions of the same random phenomenon. Note that $N(t)$ is a continuous-time/discrete-valued (CTDV) random process. Also, because $N(t)$ counts the number of events from the initial time $t = 0$ up to and *including* the time t, the value of $N(t)$ at a jump is $N(t^+)$. Thus, $N(t)$ is *right-continuous* (the same property as for the CDF of a discrete random variable). The motivation for the widespread use of the Poisson random process is its ability

to model a wide range of physical and man-made random phenomena. Some of these are the distribution in time of radioactive counts, the arrivals of customers at a cashier, requests for service in computer networks, and calls made to a central location, to name just a few. In Chapter 5 we gave an example of the application of the Poisson PMF to the servicing of customers at a supermarket checkout. Here we examine the characteristics of a Poisson random process in more detail, paying particular attention not only to the probability of a given number of events in a time interval but also to the probability for the arrival times of those events. In order to avoid confusing the probabilistic notion of an event with the common usage, we will refer to the events shown in Figure 21.1 as *arrivals*.

The Poisson random process is a natural extension of a sequence of independent and identically distributed Bernoulli trials (see Example 16.1). The Poisson counting random process $N(t)$ then becomes the extension of the binomial counting random process discussed in Example 16.5. To make this identification, consider a Bernoulli random process, which is defined as a sequence of IID Bernoulli trials, with $U[n] = 1$ with probability p and $U[n] = 0$ with probability $1 - p$. Now envision a Bernoulli trial for each small time slot of width Δt in the interval $[0, t]$ as shown in Figure 21.2. Thus, we will observe either a 1 with probability p or a 0 with probability
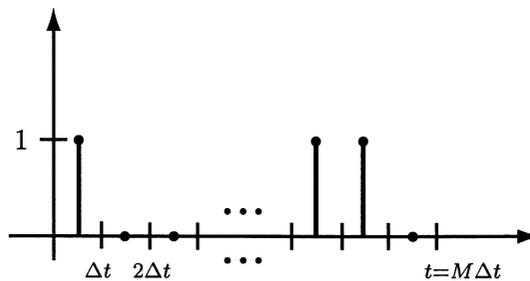

Figure 21.2: IID Bernoulli random process with one trial per time slot.

$1 - p$ for each of the $M = t/\Delta t$ time slots. Recall that on the average we will observe Mp ones. Now if $\Delta t \to 0$ and $M \to \infty$ with $t = M\Delta t$ held constant, we will obtain the Poisson random process as the limiting form of the Bernoulli random process. Also, recall that the number of ones in M IID Bernoulli trials is a binomial random variable. Hence, it seems reasonable that the number of arrivals in a Poisson random process should be a Poisson random variable in accordance with our results in Section 5.6. We next argue that this is indeed the case. For the binomial counting random process, thought of as one trial per time slot, we have that the number of ones in the interval $[0, t]$ has the PMF

$$P[N(t) = k] = \binom{M}{k} p^k (1 - p)^{M-k} \qquad k = 0, 1, \ldots, M.$$

But as $M \to \infty$ and $p \to 0$ with $E[N(t)] = Mp$ being fixed, the binomial PMF

becomes the Poisson PMF or $N(t) \sim \text{Pois}(\lambda')$, where $\lambda' = E[N(t)] = Mp$. (Note that as the number of time slots M increases, we need to let $p \to 0$ in order to maintain an average number of arrivals in $[0, t]$.) Thus, replacing λ' by $E[N(t)]$, we write the Poisson PMF as

$$P[N(t) = k] = \exp(-E[N(t)]) \frac{E^k[N(t)]}{k!} \qquad k = 0, 1, \ldots . \tag{21.1}$$

To determine $E[N(t)]$ for use in (21.1), where t may be arbitrary, we examine Mp in the limit. Thus,

$$
\begin{aligned}
E[N(t)] &= \lim_{\substack{M \to \infty \\ p \to 0}} Mp \\
&= \lim_{\substack{\Delta t \to 0 \\ p \to 0}} \frac{t}{\Delta t} p = t \lim_{\substack{\Delta t \to 0 \\ p \to 0}} \frac{p}{\Delta t} \\
&= \lambda t
\end{aligned}
$$

where we define λ as the limit of $p/\Delta t$. Since $\lambda = E[N(t)]/t$, we can interpret λ as the *average number of arrivals per second* or the *rate of the Poisson random process*. This is a parameter that is easily specified in practice. Using this definition we have that

$$P[N(t) = k] = \exp(-\lambda t) \frac{(\lambda t)^k}{k!} \qquad k = 0, 1, \ldots . \tag{21.2}$$

As mentioned previously, $N(t)$ is the Poisson counting random process and the probability of k arrivals from $t = 0$ up to and including t is given by (21.2). It is a semi-infinite random process with $N(0) = 0$ by definition.

It is possible to derive all the properties of a Poisson counting random process by employing the previous device of viewing it as the limiting form of a binomial counting random process as $\Delta t \to 0$. However, it is cumbersome to do so and therefore, we present an alternative derivation that is consistent with the same basic assumptions. One advantage of viewing the Poisson random process as a limiting form is that many of its properties become more obvious by consideration of a sequence of IID Bernoulli trials. These properties are inherited from the binomial, such as, for example, the increments $N(t_2) - N(t_1)$ must be independent. (Can you explain why this must be true for the binomial counting random process?)

21.2 Summary

The Poisson counting random process is introduced in Section 21.1. The probability of k arrivals in the time interval $[0, t]$ is given by (21.2). This probability is also derived in Section 21.3 based on a set of axioms that the Poisson random process should adhere to. Some examples of typical problems for which this probability is useful are also described in that section. The times between arrivals or interarrival

times is shown in Section 21.4 to be independent and exponentially distributed as given by (21.6). The arrival times of a Poisson random process are described by an Erlang PDF given in (21.8). An extension of the Poisson random process that is useful is the compound Poisson random process described in Section 21.6. Moments of the random process can be found from the characteristic function of (21.12). In particular, the mean is given by (21.13). A Poisson random process is easily simulated on a computer using the MATLAB code listed in Section 21.7. Finally, an application of the compound Poisson random process to automobile traffic signal planning is the subject of Section 21.8.

21.3   Derivation of Poisson Counting Random Process

We next derive the Poisson counting random process by appealing to a set of axioms that are consistent with our previous assumptions. Clearly, since the random process starts at $t = 0$, we assume that $N(0) = 0$. Next, since the binomial counting random process has increments that are independent and stationary (Bernoulli trials are IID), we assume the same for the Poisson counting random process. Thus, for two increments we assume that the random variables $I_1 = N(t_2) - N(t_1)$ and $I_2 = N(t_4) - N(t_3)$ are independent if $t_4 > t_3 > t_2 > t_1$ and also have the same PDF if additionally $t_4 - t_3 = t_2 - t_1$. Likewise, we assume this is true for *all* possible sets of increments. Note that $t_4 > t_3 > t_2 > t_1$ corresponds to *nonoverlapping* time intervals. The increments will still be independent if $t_2 = t_3$ or the time intervals have a single point in common since the probability of $N(t)$ changing at a point is zero as we will see shortly. As for the Bernoulli random process, there can be at most one arrival in each time slot. Similarly, for the Poisson counting random process we allow at most one arrival for each time slot so that

$$P[N(t + \Delta t) - N(t) = k] = \begin{cases} 1 - p & k = 0 \\ p & k = 1 \end{cases}$$

and recall that

$$\lim_{\substack{\Delta t \to 0 \\ p \to 0}} \frac{p}{\Delta t} = \lambda$$

so that for Δt small, $p = \lambda \Delta t$ and

$$P[N(t + \Delta t) - N(t) = k] = \begin{cases} 1 - \lambda \Delta t & k = 0 \\ \lambda \Delta t & k = 1 \\ 0 & k \geq 2. \end{cases}$$

Therefore, our axioms become

Axiom 1 $N(0) = 0$.

Axiom 2 $N(t)$ has independent and stationary increments.

Axiom 3 $P[N(t + \Delta t) - N(t) = k] = \begin{cases} 1 - \lambda \Delta t & k = 0 \\ \lambda \Delta t & k = 1 \end{cases}$

for all t.

With these axioms we wish to prove that (21.2) follows. The derivation is indicative of an approach commonly used for analyzing *continuous-time Markov random processes* [Cox and Miller 1965] and so is of interest in its own right.

21.3.1 Derivation

To begin, consider the determination of $P[N(t) = 0]$ for an arbitrary $t > 0$. Then referring to Figure 21.3a we see that for no arrivals in $[0, t]$, there must be no arrivals in $[0, t - \Delta t]$ and also no arrivals in $(t - \Delta t, t]$. Therefore,
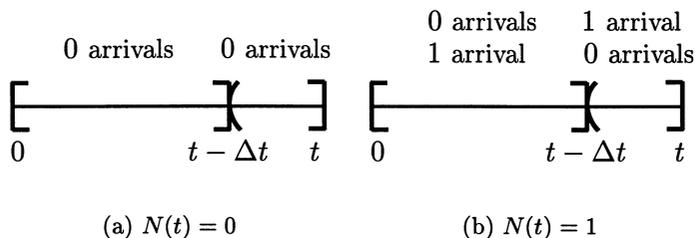

Figure 21.3: Possible number of arrivals in indicated time intervals.

$$\begin{aligned} P[N(t) = 0] &= P[N(t - \Delta t) = 0, N(t) - N(t - \Delta t) = 0] \\ &= P[N(t - \Delta t) = 0]P[N(t) - N(t - \Delta t) = 0] \quad \text{(Axiom 2 - independence)} \\ &= P[N(t - \Delta t) = 0]P[N(t + \Delta t) - N(t) = 0] \quad \text{(Axiom 2 - stationarity)} \\ &= P[N(t - \Delta t) = 0](1 - \lambda \Delta t) \quad \text{(Axiom 3).} \end{aligned}$$

If we let $P_0(t) = P[N(t) = 0]$, then

$$P_0(t) = P_0(t - \Delta t)(1 - \lambda \Delta t)$$

or

$$\frac{P_0(t) - P_0(t - \Delta t)}{\Delta t} = -\lambda P_0(t - \Delta t).$$

Now letting $\Delta t \to 0$, we arrive at the linear differential equation

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t)$$

for which the solution is $P_0(t) = c \exp(-\lambda t)$, where c is an arbitrary constant. To evaluate the constant we invoke the initial condition that $P_0(0) = P[N(0) = 0] = 1$ by Axiom 1 to yield $c = 1$. Thus, we have finally that

$$P[N(t) = 0] = P_0(t) = \exp(-\lambda t).$$

Next we use the same argument to find a differential equation for $P_1(t) = P[N(t) = 1]$ by referring to Figure 21.3b. We can either have no arrivals in $[0, t - \Delta t]$ and one arrival in $(t - \Delta t, t]$ *or* one arrival in $[0, t - \Delta t]$ and no arrivals in $(t - \Delta t, t]$. These are the only possibilities since there can be at most one arrival in a time interval of length Δt. The two events are mutually exclusive so that

$$
\begin{aligned}
P[N(t) = 1] &= P[N(t - \Delta t) = 0, N(t) - N(t - \Delta t) = 1] \\
&\quad + P[N(t - \Delta t) = 1, N(t) - N(t - \Delta t) = 0] \\
&= P[N(t - \Delta t) = 0]P[N(t) - N(t - \Delta t) = 1] \\
&\quad + P[N(t - \Delta t) = 1]P[N(t) - N(t - \Delta t) = 0] \quad \text{(independence)} \\
&= P[N(t - \Delta t) = 0]P[N(t + \Delta t) - N(t) = 1] \\
&\quad + P[N(t - \Delta t) = 1]P[N(t + \Delta t) - N(t) = 0]. \quad \text{(stationarity)}
\end{aligned}
$$

Using the definition of $P_1(t)$ and Axiom 3,

$$
P_1(t) = P_0(t - \Delta t)\lambda \Delta t + P_1(t - \Delta t)(1 - \lambda \Delta t)
$$

or

$$
\frac{P_1(t) - P_1(t - \Delta t)}{\Delta t} = -\lambda P_1(t - \Delta t) + \lambda P_0(t - \Delta t)
$$

and as $\Delta t \to 0$, we have the differential equation

$$
\frac{dP_1(t)}{dt} + \lambda P_1(t) = \lambda P_0(t).
$$

In like fashion we can show (see Problem 21.1) that if $P_k(t) = P[N(t) = k]$, then

$$
\frac{dP_k(t)}{dt} + \lambda P_k(t) = \lambda P_{k-1}(t) \quad k = 1, 2, \ldots \tag{21.3}
$$

where we know that $P_0(t) = \exp(-\lambda t)$. This is a set of simultaneous linear differential equations that fortunately can be solved recursively. Since $P_0(t)$ is known, we can solve for $P_1(t)$. Once $P_1(t)$ has been found, then $P_2(t)$ can be solved for, etc. It is shown in Problem 21.2 that by using Laplace transforms, we can easily solve these equations. The result is

$$
P_k(t) = \exp(-\lambda t)\frac{(\lambda t)^k}{k!} \quad k = 0, 1, \ldots
$$

so that finally we have the desired result

$$
P[N(t) = k] = \exp(-\lambda t)\frac{(\lambda t)^k}{k!} \quad k = 0, 1, \ldots \tag{21.4}
$$

which is the usual Poisson PMF. The only difference from that described in Section 5.5.4 is that here λ represents an *arrival rate*. Since if $X \sim \text{Pois}(\lambda')$, then $E[X] = \lambda'$, we have $\lambda' = \lambda t$. Hence, $\lambda = \lambda'/t = E[N(t)]/t$, which is seen to be the *average number of arrivals per second*.

21.3.2 Some Examples

Before proceeding with some examples it should be pointed out that the Poisson counting random process is *not* stationary or even WSS. This is evident from the PMF of $N(t)$ since $E[N(t_2)] = \lambda t_2 \neq \lambda t_1 = E[N(t_1)]$ for $t_2 \neq t_1$. As its properties are inherited from the binomial counting random process, it exhibits the properties of a sum random process (see Section 16.4). Also, in determining probabilities of events, the fact that the *increments* are independent and stationary will greatly simplify our calculations.

Example 21.1 – Customer arrivals

Customers arrive at a checkout lane at the rate of 0.1 customers per second according to a Poisson random process. Determine the probability that 5 customers will arrive during the first minute the lane is open and also 5 customers will arrive the second minute it is open. During the time interval $[0, 60]$ the probability of 5 arrivals is from (21.4)

$$P[N(60) = 5] = \exp[-0.1(60)]\frac{[(0.1)(60)]^5}{5!} = 0.1606.$$

This will also be the probability of 5 customers arriving during the second minute interval or *for any one minute interval* $[t, t + 60]$ since

$$
\begin{aligned}
P[N(t + 60) - N(t) = 5] &= P[N(60) - N(0) = 5] \quad &\text{(increment stationarity)} \\
&= P[N(60) = 5] \quad &(N(0) = 0)
\end{aligned}
$$

which is not dependent on t. Hence, the probability of 5 customers arriving in the first minute and 5 more arriving in the second minute is

$$P[N(60) - N(0) = 5, N(120) - N(60) = 5]$$
$$
\begin{aligned}
&= P[N(60) - N(0) = 5]P[N(120) - N(60) = 5] \quad &\text{(increment independence)} \\
&= P[N(60) - N(0) = 5]P[N(60) - N(0) = 5] \quad &\text{(increment stationarity)} \\
&= P^2[N(60) = 5] = 0.0258 \quad &(N(0) = 0)
\end{aligned}
$$

$\Diamond$

Example 21.2 – Traffic bursts

Consider the arrival of cars at an intersection. It is known that for any 5 minute interval 50 cars arrive on the average. For any 5 minute interval what is the probability of 20 cars in the first minute and 30 cars in the next 4 minutes? Since the probabilities of the increments do not change with the time origin due to stationarity, we can assume that the 5 minute interval in question starts at $t = 0$ and ends at $t = 300$ seconds. Thus, we wish to determine the probability of a traffic burst P_B, which is

$$P_B = P[N(60) = 20, N(300) - N(60) = 30].$$

Since the increments are independent, we have

$$P_B = P[N(60) = 20]P[N(300) - N(60) = 30]$$

and because they are also stationary

$$
\begin{aligned}
P_B &= P[N(60) = 20]P[N(240) - N(0) = 30] \\
&= P[N(60) = 20]P[N(240) = 30] \\
&= \exp(-60\lambda)\frac{(60\lambda)^{20}}{20!}\exp(-240\lambda)\frac{(240\lambda)^{30}}{30!}.
\end{aligned}
$$

Finally, since the arrival rate is given by $\lambda = 50/300 = 1/6$, the probability of a traffic burst is

$$P_B = \exp(-10)\frac{(10)^{20}}{20!}\exp(-40)\frac{(40)^{30}}{30!} = 3.4458 \times 10^{-5}.$$

$\diamondsuit$

In many applications it is important to assess not only the probability of a number of arrivals within a given time interval but also the distribution of these arrival times. Are they evenly spaced or can they bunch up as in the last example? In the next section we answer these questions.

21.4 Interarrival Times

Consider a typical realization of a Poisson random process as shown in Figure 21.4. The times $t_1, t_2, t_3, \ldots$ are called the *arrival times* while the time intervals
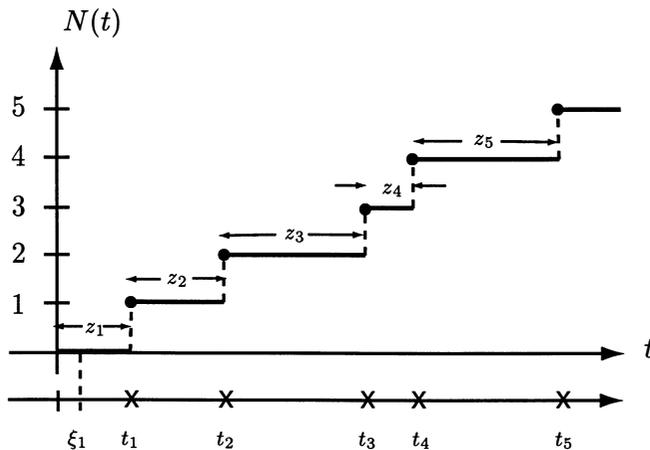

Figure 21.4: Definition of arrival times t_i's and interarrival times z_i's.

$z_1, z_2, z_3, \ldots$ are called the *interarrival times*. The interarrival times shown in Figure 21.4 are realizations of the random variables $Z_1, Z_2, Z_3, \ldots$. We wish to be

able to compute probabilities for a finite set, say $Z_1, Z_2, \ldots, Z_K$. Since $N(t)$ is a continuous-time random process, the time between arrivals is also continuous and so a joint PDF is sought. To begin we first determine $p_{Z_1}(z_1)$. Note that $Z_1 = T_1$, where T_1 is the random variable denoting the first arrival. By the definition of the first arrival if $Z_1 > \xi_1$, then $N(\xi_1) = 0$ as shown in Figure 21.4. Conversely, if $N(\xi_1) = 0$, then the first arrival has not occurred as of time ξ_1 and so $Z_1 > \xi_1$. This argument shows that the events $\{Z_1 > \xi_1\}$ and $\{N(\xi_1) = 0\}$ are equivalent and therefore

$$
\begin{aligned}
P[Z_1 > \xi_1] &= P[N(\xi_1) = 0] \\
&= \exp(-\lambda\xi_1) \qquad \xi_1 \geq 0
\end{aligned}
\tag{21.5}
$$

where we have used (21.4). As a result, the PDF is for $z_1 \geq 0$

$$
\begin{aligned}
p_{Z_1}(z_1) &= \frac{d}{dz_1} F_{Z_1}(z_1) \\
&= \frac{d}{dz_1} \left(1 - P[Z_1 > z_1]\right) \\
&= \frac{d}{dz_1}[1 - \exp(-\lambda z_1)] \\
&= \lambda\exp(-\lambda z_1)
\end{aligned}
$$

and finally the PDF of the first arrival is

$$
p_{Z_1}(z_1) = \begin{cases} \lambda\exp(-\lambda z_1) & z_1 \geq 0 \\ 0 & z_1 < 0 \end{cases}
\tag{21.6}
$$

or $Z_1 \sim \exp(\lambda)$. An example follows.

Example 21.3 – Waiting for an arrival

Assume that at $t = 0$ we start to wait for an arrival. Then we know from (21.6) that the time we will have to wait is a random variable with $Z_1 \sim \exp(\lambda)$. On the average we will have to wait $E[Z_1] = 1/\lambda$ seconds. This is reasonable in that λ is average arrivals per second and therefore $1/\lambda$ is seconds per arrival. However, say we have already waited ξ_1 seconds—what is the probability that we will have to wait more than an additional ξ_2 seconds? In probabilistic terms we wish to compute the conditional probability $P[Z_1 > \xi_1 + \xi_2 | Z_1 > \xi_1]$. This is found as follows.

$$
\begin{aligned}
P[Z_1 > \xi_1 + \xi_2 | Z_1 > \xi_1] &= \frac{P[Z_1 > \xi_1 + \xi_2, Z_1 > \xi_1]}{P[Z_1 > \xi_1]} \\
&= \frac{P[Z_1 > \xi_1 + \xi_2]}{P[Z_1 > \xi_1]}
\end{aligned}
$$

since the arrival time will be greater than both $\xi_1 + \xi_2$ and ξ_1 only if it is greater

than the former. Now using (21.5) we have that

$$
\begin{aligned}
P[Z_1 > \xi_1 + \xi_2 | Z_1 > \xi_1] &= \frac{\exp[-\lambda(\xi_1 + \xi_2)]}{\exp(-\lambda\xi_1)} \\
&= \exp(-\lambda\xi_2) \\
&= P[Z_1 > \xi_2]. \tag{21.7}
\end{aligned}
$$

Hence, the conditional probability that we will have to wait more than an additional ξ_2 seconds given that we have already waited ξ_1 seconds is just the probability that we will have to wait more than ξ_2 seconds. The fact that we have already waited does not in any way affect the probability of the first arrival. Once we have waited and observed that no arrival has occured up to time ξ_1, then the random process in essence starts over as if it were at time $t = 0$. This property of the Poisson random process is referred to as the *memoryless property*. It is somewhat disconcerting to know that the chances your bus will arrive in the next 5 minutes, given that it is already 5 minutes late, is not any better than your chances it will be late by 5 minutes. However, this conclusion is consistent with the Poisson random process model. It is also evident by examining the similar result of waiting for a fair coin to comes up heads given that it has already exhibited 10 tails in a row. In Problem 21.12 an alternative derivation of the memoryless property is given which makes use of the geometric random variable.

$\diamond$

We next give the joint PDF for two or more interarrival times. It is shown in Appendix 21A that the interarrival times $Z_1, Z_2, \ldots, Z_K$ are IID random variables with each one having $Z_1 \sim \exp(\lambda)$. This result may also be reconciled in light of the Poisson random process being the limiting form of a Bernoulli random process. Consider a Bernoulli random process $\{X[0] = 0, X[1], X[2], \ldots\}$, where $X[0] = 0$ by definition, and assume *interarrival* times of k_1 and k_2, where $k_1 \geq 1$, $k_2 \geq 1$. For example, if $X[1] = 0, X[2] = 1, X[3] = 0, X[4] = 0$, and $X[5] = 1$, then we would have $k_1 = 2$ and $k_2 = 3$. In general,

$$
\begin{aligned}
&P[\text{first interarrival time} = k_1, \text{second interarrival time} = k_2] \\
&= P[X[n] = 0 \text{ for } 1 \leq n \leq k_1 - 1, X[k_1] = 1, X[n] = 0 \\
&\quad \text{for } k_1 + 1 \leq n \leq k_1 + k_2 - 1, X[k_1 + k_2] = 1] \\
&= [(1-p)^{k_1-1}p][(1-p)^{k_2-1}p].
\end{aligned}
$$

Hence, the joint PMF factors so that the interarrival times are independent and furthermore they are identically distributed (let $k_1 = k_2$). An example follows.

Example 21.4 – Expected time for calls

A customer call service center opens at 9 A.M. The calls received follow a Poisson random process at the average rate of 600 calls per hour. The 20th call comes in at 9:01 A.M. At what time can we expect the next call to come in? Let Z_{21} be

the elapsed time from 9:01 A.M. until the next call comes in. Since the interarrival times are independent, they do not depend upon the past history of arrivals. Hence, $Z_{21} = T_{21} - T_{20} \sim \exp(\lambda)$. Since the mean of an exponential random variable Z is just $1/\lambda$ and from the information given $\lambda = 600/3600 = 1/6$ calls per second, we have that $E[Z_{21}] = 1/(1/6) = 6$ seconds. Hence, we can expect the next call to come in at 9:01:06 A.M.

$\diamond$

21.5   Arrival Times

The kth arrival time T_k is defined as the time from $t = 0$ until the kth arrival occurs. The arrival times are illustrated in Figure 21.4, where T_k is also referred to as the *waiting time* until the kth arrival. In this section we will determine the PDF of T_k. It is seen from Figure 21.4 that $t_k = \sum_{i=1}^{k} z_i$ so that the random variable of interest is

$$T_k = \sum_{i=1}^{k} Z_i.$$

But we saw in the last section that the Z_i's are IID with $Z_1 \sim \exp(\lambda)$. Hence, the PDF of T_k is obtained by determining the PDF for a sum of IID random variables. This is a problem that has been studied in Section 14.6, and is solved most readily by the use of the characteristic function. Recall that if $X_1, X_2, \ldots, X_k$ are IID random variables, then the characteristic function for $Y = \sum_{i=1}^{k} X_i$ is $\phi_Y(\omega) = \phi_X^k(\omega)$. Thus, the PDF for Y, assuming that Y is a continuous random variable, is found from the continuous-time inverse Fourier transform (defined to correspond to the Fourier transform used in the characteristic function definition, and uses a $-j$ and radian frequency ω) as

$$p_Y(y) = \int_{-\infty}^{\infty} \phi_X^k(\omega) \exp(-j\omega y) \frac{d\omega}{2\pi}.$$

From Table 11.1 we have that $\phi_{Z_1}(\omega) = \lambda/(\lambda - j\omega)$ and therefore

$$\phi_{T_k}(\omega) = \left(\frac{\lambda}{\lambda - j\omega} \right)^k = \left(\frac{1}{1 - j\omega/\lambda} \right)^k.$$

Again referring to Table 11.1, we see that this is the characteristic function of a Gamma random variable with $\alpha = k$ so that $T_k \sim \Gamma(k, \lambda)$. Specifically, this is the Erlang random variable described in Section 10.5.6 . Hence, we have that

$$p_{T_k}(t) = \frac{\lambda^k}{(k-1)!} t^{k-1} \exp(-\lambda t). \tag{21.8}$$

(See also Problem 21.15 for the derivation for $k = 2$ using a convolution integral and Problem 21.16 for an alternative derivation for the general case.) Note that for a

$\Gamma(\alpha, \lambda)$ random variable the mean is α/λ so that with $\alpha = k$, we have the expected time for the kth arrival as

$$E[T_k] = \frac{k}{\lambda} \tag{21.9}$$

or equivalently

$$E[T_k] = kE[T_1]. \tag{21.10}$$

On the average the time to the kth arrival is just k times the time to the first arrival, a somewhat pleasing result. An example follows.

Example 21.5 – Computer servers

A computer server is designed to provide downloaded software when requested. It can honor a total of 80 requests in each hour before it becomes overloaded. If the requests are made in accordance with a Poisson random process at an average rate of 60 requests per hour, what is the probability that it will be overloaded in the first hour? We need to determine the probability that the 81st request will occur at a time $t \leq 3600$ seconds. Thus, from (21.8) with $k = 81$

$$
\begin{aligned}
P[\text{overloaded in first hour}] &= P[T_{81} \leq 3600] \\
&= \int_0^{3600} \frac{\lambda^{81}}{80!} t^{80} \exp(-\lambda t) dt.
\end{aligned}
$$

Here the arrival rate of the requests is $\lambda = 60/3600 = 1/60$ per second and therefore

$$P[\text{overloaded in first hour}] = \frac{1}{60} \int_0^{3600} \frac{1}{80!} \left(\frac{t}{60}\right)^{80} \exp(-t/60) dt$$

Using the result

$$\int \frac{(at)^n}{n!} \exp(-at) dt = -\frac{\exp(-at)}{a} \sum_{i=0}^{n} \frac{(at)^i}{i!}$$

it follows that

$$
\begin{aligned}
P[\text{overloaded in first hour}] &= \frac{1}{60} \left[-\frac{\exp(-t/60)}{1/60} \sum_{i=0}^{80} \frac{(t/60)^i}{i!} \Bigg|_0^{3600} \right] \\
&= -\left[\exp(-60) \sum_{i=0}^{80} \frac{(60)^i}{i!} - 1 \right] \\
&= 1 - \exp(-60) \sum_{i=0}^{80} \frac{(60)^i}{i!} = 0.0056.
\end{aligned}
$$

$\Diamond$

21.6 Compound Poisson Random Process

A Poisson counting random process increments its value by one for each new arrival. In some applications we may not know the increment in advance. An example would be to determine the average amount of all transactions within a bank for a given day. In this case the amount obtained is the sum of all deposits and withdrawals. To model these transactions we could assume that customers arrive at the bank according to a Poisson random process. If, for example, each customer deposited one dollar, then at the end of the day, say at time t_0, the total amount of the transactions $X(t_0)$ could be written as

$$X(t_0) = \sum_{i=1}^{N(t_0)} 1 = N(t_0).$$

This is the standard Poisson counting random process. If, however, there are withdrawals, then this would no longer hold. Furthermore, if the deposits and withdrawals are unknown to us before they are made, then we would need to model each one by a random variable, say U_i. The random variable would take on positive values for deposits and negative values for withdrawals and probabilities could be assigned to the possible values of U_i. The total dollar amount of the transactions at the end of the day would be

$$\sum_{i=1}^{N(t_0)} U_i.$$

With this motivation we will consider the more general case in which the U_i's are either discrete or continuous random variables, and denote the total at time t by the random process $X(t)$. This random process is therefore given by

$$X(t) = \sum_{i=1}^{N(t)} U_i \qquad t \geq 0. \tag{21.11}$$

It is a continuous-time random process but can be either continuous-valued or discrete-valued depending upon whether the U_i's are continuous or discrete random variables. We furthermore assume that the U_i's are IID random variables. Hence, $X(t)$ is similar to the usual sum of IID random variables except that the *number of terms in the sum is random* and the number of terms is distributed according to a Poisson random process. This random process is called a *compound Poisson random process*.

In summary, we let $X(t) = \sum_{i=1}^{N(t)} U_i$ for $t \geq 0$, where the U_i's are IID random variables and $N(t)$ is a Poisson counting random process with arrival rate λ. Also, we define $X(0) = 0$, and furthermore assume that the U_i's and $N(t)$ are independent of each other for all t.

We next determine the marginal PMF or PDF of $X(t)$. To do so we will use characteristic functions in conjunction with conditioning arguments. The key to success here is to turn the sum with a *random* number of terms into one with a *fixed* number by conditioning. Then, the usual characteristic function approach described in Section 14.6 will be applicable. Hence, consider for a fixed $t = t_0$ the random variable $X(t_0)$ and write its characteristic function as

$$
\begin{aligned}
\phi_{X(t_0)}(\omega) &= E[\exp(j\omega X(t_0))] \qquad\qquad\qquad\text{(definition)} \\[2mm]
&= E\left[\exp\left(j\omega \sum_{i=1}^{N(t_0)} U_i\right)\right] \\[2mm]
&= E_{N(t_0)}\left[E_{U_1,\dots,U_k|N(t_0)}\left[\exp\left(j\omega \sum_{i=1}^{k} U_i\right)\bigg| N(t_0) = k\right]\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(see Problem 21.18)} \\[2mm]
&= E_{N(t_0)}\left[E_{U_1,\dots,U_k}\left[\exp\left(j\omega \sum_{i=1}^{k} U_i\right)\right]\right] \quad (U_i\text{'s independent of } N(t_0)) \\[2mm]
&= E_{N(t_0)}\left[E_{U_1,\dots,U_k}\left[\prod_{i=1}^{k} \exp(j\omega U_i)\right]\right] \\[2mm]
&= E_{N(t_0)}\left[\prod_{i=1}^{k} E_{U_i}[\exp(j\omega U_i)]\right] \qquad (U_i\text{'s are independent}) \\[2mm]
&= E_{N(t_0)}\left[\prod_{i=1}^{k} \phi_{U_i}(\omega)\right] \qquad\qquad\text{(definition of char. function)} \\[2mm]
&= E_{N(t_0)}\left[\phi_{U_1}^k(\omega)\right] \qquad\qquad\qquad (U_i\text{'s identically dist.}) \\[2mm]
&= \sum_{k=0}^{\infty} \phi_{U_1}^k(\omega) p_{N(t_0)}[k] \\[2mm]
&= \sum_{k=0}^{\infty} \phi_{U_1}^k(\omega) \exp(-\lambda t_0)\frac{(\lambda t_0)^k}{k!} \\[2mm]
&= \exp(-\lambda t_0) \sum_{k=0}^{\infty} \frac{(\lambda t_0 \phi_{U_1}(\omega))^k}{k!} \\[2mm]
&= \exp(-\lambda t_0) \exp(\lambda t_0 \phi_{U_1}(\omega))
\end{aligned}
$$

so that finally we have the characteristic function

$$
\phi_{X(t_0)}(\omega) = \exp[\lambda t_0(\phi_{U_1}(\omega) - 1)]. \qquad\qquad (21.12)
$$

To determine the PMF or PDF of $X(t_0)$ we would need to take the inverse Fourier transform of the characteristic function. As a check, if we let $U_i = 1$ for all i so that

from (21.11) $X(t_0) = N(t_0)$, then since

$$\phi_{U_1}(\omega) = E[\exp(j\omega U_1)] = \exp(j\omega)$$

we have the usual characteristic function of a Poisson random variable (see Table 6.1)

$$\phi_{X(t_0)}(\omega) = \exp[\lambda t_0(\exp(j\omega) - 1)].$$

(The derivation of (21.12) can be shown to hold for this choice of the U_i's, which are degenerate random variables.) An example follows.

Example 21.6 – Poisson random process with dropped arrivals

Consider a Poisson random process in which some of the arrivals are dropped. This means for example that a Geiger counter may not record radioactive particles if their intensity is too low. Assume that the probability of dropping an arrival is $1 - p$, and that this event is independent of the Poisson arrival process. Then, we wish to determine the PMF of the number of arrivals within the time interval $[0, t_0]$. Thus, the number of arrivals can be represented as

$$X(t_0) = \sum_{i=1}^{N(t_0)} U_i$$

where $U_i = 1$ if the ith arrival is counted and $U_i = 0$ if it is dropped. Assuming that the U_i's are IID, we have a compound Poisson random process. The characteristic function of $X(t_0)$ is found using (21.12) where we note that

$$\begin{aligned}\phi_{U_1}(\omega) &= E[\exp(j\omega U_1)] \\ &= p\exp(j\omega) + (1 - p)\end{aligned}$$

so that from (21.12)

$$\begin{aligned}\phi_{X(t_0)}(\omega) &= \exp[\lambda t_0(p\exp(j\omega) + (1 - p) - 1)] \\ &= \exp[p\lambda t_0(\exp(j\omega) - 1)].\end{aligned}$$

But this is just the characteristic of a Poisson counting random process with arrival rate of $p\lambda$. Hence, by dropping arrivals the arrival rate is reduced but $X(t)$ is still a Poisson counting process, a very reasonable result.

$$\diamond$$

Since the characteristic function of a compound Poisson random process is available, we can use it to easily find the moments of $X(t_0)$. In particular, we now determine the mean, leaving the variance as a problem (see Problem 21.22). Using (21.12) we

have

$$E[X(t_0)] = \frac{1}{j} \frac{d\phi_{X(t_0)}(\omega)}{d\omega}\Big|_{\omega=0} \qquad \text{(using (6.13))}$$

$$= \frac{1}{j}\lambda t_0 \frac{d\phi_{U_1}(\omega)}{d\omega} \exp[\lambda t_0(\phi_{U_1}(\omega) - 1)]\Big|_{\omega=0}$$

$$= \lambda t_0 \frac{1}{j} \frac{d\phi_{U_1}(\omega)}{d\omega}\Big|_{\omega=0}$$

since $\phi_{U_1}(0) = 1$. But

$$E[U_1] = \frac{1}{j} \frac{d\phi_{U_1}(\omega)}{d\omega}\Big|_{\omega=0}$$

so that the average value is

$$E[X(t_0)] = \lambda t_0 E[U_1] = E[N(t_0)]E[U_1]. \qquad (21.13)$$

It is seen that the average value of $X(t_0)$ is just the average value of U_1 times the expected number of arrivals. This result also holds even if the U_i's only have the same mean, without the IID assumption (see Problem 21.25 and the real-world problem). An example follows.

Example 21.7 – Expected number of points scored in basketball game

A basketball player, dubbed the "Poisson pistol Pete" of college basketball, shoots the ball at an average rate of 1 shot per minute according to a Poisson random process. He shoots a 2 point shot with a probability of 0.6 and a 3 point shot with a probability of 0.4. If his 2 point field goal percentage is 50% and his 3 point field goal percentage is 30%, what is his expected total number of points scored in a 40 minute game? (We assume that the referees "let them play" so that no fouls are called and hence no free throw points.) The average number of points is $E[N(t_0)]E[U_1]$, where $t_0 = 2400$ seconds and U_1 is a random variable that denotes his points made for the first shot (the distribution for each shot is identical). We first determine the PMF for U_1, where we have implicitly assumed that the U_i's are IID random variables. From the problem description we have that

$$U_1 = \begin{cases} 2 & \text{if 2 point shot attempted and made} \\ 3 & \text{if 3 point shot attempted and made} \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} p_{U_1}[2] &= P[\text{2 point shot attempted and made}] \\ &= P[\text{2 point shot made} \mid \text{2 point shot attempted}]P[\text{2 point shot attempted}] \\ &= 0.5(0.6) = 0.3 \end{aligned}$$

and similarly $p_{U_1}[3] = 0.3(0.4) = 0.12$ and therefore, $p_{U_1}[0] = 0.58$. The expected value becomes $E[U_1] = 2(0.3) + 3(0.12) = 0.96$ and therefore the expected number of points scored is

$$
\begin{aligned}
E[N(t_0)]E[U_1] &= \lambda t_0 E[U_1] \\
&= \frac{1}{60}(2400)(0.96) \\
&= 38.4 \text{ points per game.}
\end{aligned}
$$

$\diamond$

21.7 Computer Simulation

To generate a realization of a Poisson random process on a computer is relatively simple. It relies on the property that the interarrival times are IID $\exp(\lambda)$ random variables. We observe from Figure 21.4 that the ith interarrival time is $Z_i = T_i - T_{i-1}$, where T_i is the ith arrival time. Hence,

$$
T_i = T_{i-1} + Z_i \qquad i = 1, 2, \dots
$$

where we define $T_0 = 0$. Each Z_i has the PDF $\exp(\lambda)$ and the Z_i's are IID. Hence, to generate a realization of each Z_i we use the inverse probability integral transformation technique (see Section 10.9) to yield

$$
Z_i = \frac{1}{\lambda} \ln \frac{1}{1 - U_i}
$$

where $U_i \sim \mathcal{U}(0,1)$ and the U_i's are IID. A typical realization using the following MATLAB code is shown in Figure 21.5a for $\lambda = 2$. The arrivals are indicated now by $+$'s for easier viewing. If we were to increase the arrival rate to $\lambda = 5$, then a typical realization is shown in Figure 21.5b.

```
clear all
rand('state',0)
lambda=2; % set arrival rate
T=5; % set time interval in seconds
for i=1:1000
    z(i,1)=(1/lambda)*log(1/(1-rand(1,1))); % generate interarrival times
    if i==1 % generate arrival time
        t(i,1)=z(i);
        else
            t(i,1)=t(i-1)+z(i,1);
        end
        if t(i)>T % test to see if desired time interval has elapsed
```
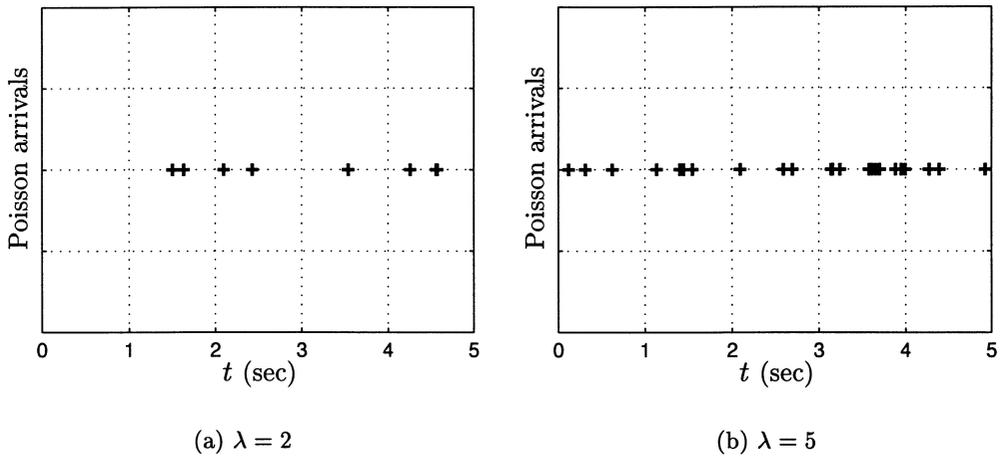
Figure 21.5: Realizations of Poisson random process.

```
        break
    end
end
M=length(t)-1; % number of arrivals in interval [0,T]
arrivals=t(1:M); % arrival times in interval [0,T]
```

21.8 Real-World Example – Automobile Traffic Signal Planning

An important responsibility of traffic engineers is to decide which intersections require traffic lights. Although general guidelines are available [Federal Highway Ad. 1988], new situations constantly arise that warrant a reassessment of the situation—principally an unusually high accident rate [Imada 2001]. In this example, we suppose that a particular intersection, which has two stop signs, is prone to accidents. The situation is depicted in Figure 21.6, where it is seen that the two intersecting streets are one-way streets with a stop sign at the corner of each one. A traffic engineer believes that the high accident rate is due to motorists who ignore the stop signs and proceed at full speed through the intersection. If this is indeed the case, then the installation of a traffic light is warranted. To determine if the accident rate is consistent with his belief that motorists are "running" the stop signs, he wishes to determine the average number of accidents that would occur if this is true. As shown in Figure 21.6, if 2 vehicles arrive at the intersection within a given time interval, an accident will occur. It is assumed the two cars are identical and move with the same speed. The traffic engineer then models the arrivals as two indepen-
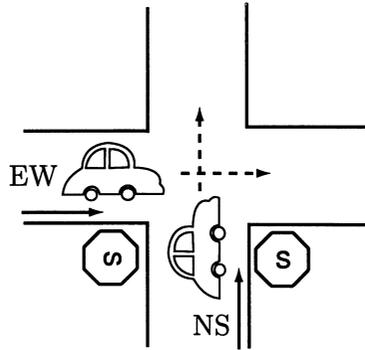
Figure 21.6: Intersection with two automobiles approaching at constant speed.

dent Poisson random processes, one for each direction of travel. A typical set of car arrivals based on this assumption is shown in Figure 21.7. Specifically, an accident
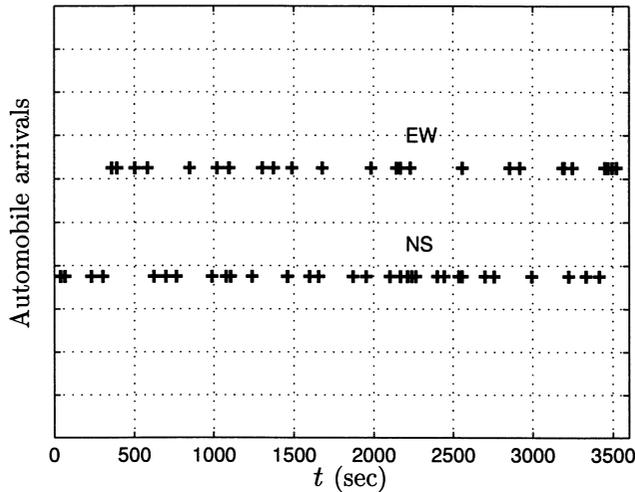

Figure 21.7: Automobile arrivals.

will occur if any two arrivals satisfy $|T^{\mathrm{EW}} - T^{\mathrm{NS}}| \leq \tau$, where T^{EW} and T^{NS} refer to the arrival time at the center of the intersection from the east-west direction and the north-south direction, respectively, and τ is some minimum time for which the cars can pass each other without colliding. The actual value of τ can be estimated using $\tau = d/c$, where d is the length of a car and c is its speed. As an example, if we assume that $d = 22$ ft and $c = 44$ ft/sec (about 30 mph), then $\tau = 0.5$ sec. An accident will occur if two arrivals are within one-half second of each other. In Figure 21.7 this does not occur, but there is a near miss as can be seen in Figure 21.8, which is an expanded version. The east-west car arrives at $t = 2167.5$ seconds while the north-south car arrives at $t = 2168.4$ seconds.
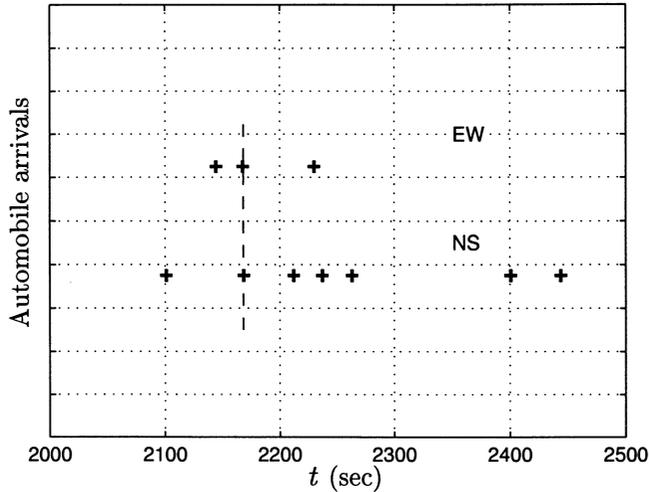
Figure 21.8: Automobile arrivals—expanded version of Figure 21.7. There is a near miss at $t = 2168$ seconds, shown by the dashed vertical line.

We now describe how to determine the average number of accidents per day. This can be obtained by defining a set of indicator random variables (see Example 11.4) as

$$I_i = \begin{cases} 1 & \text{if there is at least one NS arrival with } |T_i^{\mathrm{EW}} - T^{\mathrm{NS}}| \leq \tau \\ 0 & \text{otherwise.} \end{cases}$$

Here T^{NS} can be *any* NS arrival time and T_i^{EW} is the ith arrival time for the EW traffic. (More explicitly, the event for which the indicator random variable is 1 occurs when $\min_{j=1,2,\ldots} |T_i^{\mathrm{EW}} - T_j^{\mathrm{NS}}| \leq \tau$, where T_j^{NS} is the jth arrival for the NS traffic.) Now the number of accidents in the time interval $[0, t]$ is

$$X(t) = \sum_{i=1}^{N(t)} I_i \qquad (21.14)$$

where $N(t)$ is the Poisson counting random process for the EW traffic. To find the expected value of $X(t)$ we note that the equation (21.13), although originally derived under the assumption that the U_i's are IID, is also valid under the weaker assumption that the means of the U_i's are the same as shown in Problem 21.25. Since the I_i's will be seen shortly to have the same mean, the expected value of (21.14) is from (21.13) with $U_1 = I_1$

$$E[X(t)] = \lambda t E[I_1]. \qquad (21.15)$$

Now to evaluate $E[I_i]$, we note that

$$E[I_i] = P[|T_i^{\mathrm{EW}} - T^{\mathrm{NS}}| \leq \tau]$$

and the probability can be found using a conditioning approach (see (13.12)). This produces

$$P[|T_i^{\text{EW}} - T^{\text{NS}}| \leq \tau] = \int_0^\infty P[|T_i^{\text{EW}} - T^{\text{NS}}| \leq \tau | T_i^{\text{EW}} = t] p_{T_i}(t) dt.$$

Proceeding we have that

$$
\begin{aligned}
P[|T_i^{\text{EW}} - T^{\text{NS}}| \leq \tau] &= \int_0^\infty P[|t - T^{\text{NS}}| \leq \tau | T_i^{\text{EW}} = t] p_{T_i}(t) dt \\
&= \int_0^\infty P[|t - T^{\text{NS}}| \leq \tau] p_{T_i}(t) dt \quad (T_i^{\text{EW}}, T^{NS} \text{ are independent}) \\
&= \int_0^\infty P[t - \tau \leq T^{\text{NS}} \leq t + \tau] p_{T_i}(t) dt. \quad\quad (21.16)
\end{aligned}
$$

Note that $t - \tau \leq T^{\text{NS}} \leq t + \tau$ is the event that the NS traffic will have at least one arrival (and hence an accident) in the interval $[t - \tau, t + \tau]$. Its probability is just

$$
\begin{aligned}
P[t - \tau \leq T^{\text{NS}} \leq t + \tau] &= P[\text{one or more arrivals in } [t - \tau, t + \tau]] \\
&= 1 - P[\text{no arrival in } [t - \tau, t + \tau]] \\
&= 1 - P[\text{no arrivals in } [0, 2\tau]] \quad\quad \text{(increment stationarity)} \\
&= 1 - P[N(2\tau) = 0] \\
&= 1 - \exp(-2\lambda\tau) \quad\quad \text{(from (21.2))}
\end{aligned}
$$

and is not dependent on t. Thus,

$$
\begin{aligned}
E[I_i] &= P[|T_i^{\text{EW}} - T^{\text{NS}}| \leq \tau] \\
&= \int_0^\infty (1 - \exp(-2\lambda\tau)) p_{T_i}(t) dt \quad\quad \text{(from (21.16))} \\
&= 1 - \exp(-2\lambda\tau)
\end{aligned}
$$

for all i, and therefore all the I_i's have the same mean. From (21.15)

$$E[X(t)] = \lambda t (1 - \exp(-2\lambda\tau)).$$

For the same example as before with $\tau = 0.5$, the average number of accidents per second is

$$\frac{E[X(t)]}{t} = \lambda(1 - \exp(-\lambda)).$$

For a more meaningful measure we convert this to the average number of accidents per hour, which is $(E[X(t)]/t)3600$. This is plotted versus λ', where λ' is in arrivals per hour, in Figure 21.9. Specifically, it is given by

$$
\begin{aligned}
3600 \frac{E[X(t)]}{t} &= 3600\lambda(1 - \exp(-\lambda)) \\
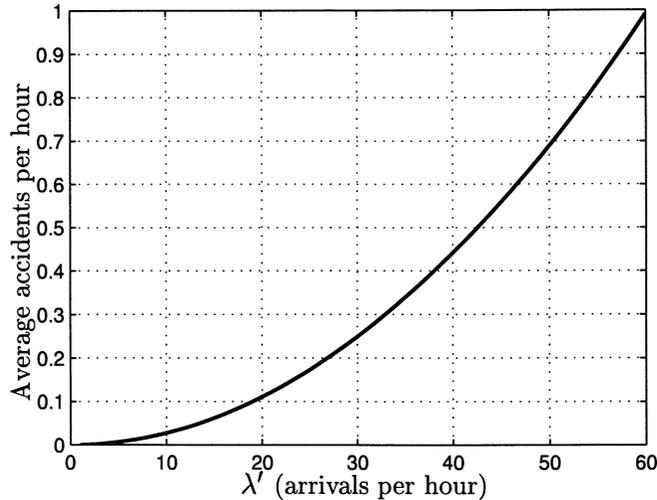&= \lambda' \left[1 - \exp\left(-\frac{\lambda'}{3600}\right)\right]
\end{aligned}
$$

Figure 21.9: Average number of accidents per hour versus arrival rate (in per hour units).

where $\lambda' =$ arrivals per hour $= 3600\lambda$. As seen in Figure 21.9 for about 1 arrival every 3 minutes or 20 arrivals per hour, we will have an average of 0.1 accidents per hour or about an average of one accident every two days. This assumes a busy intersection for about 5 hours per day. Thus, if the traffic engineer notices an accident nearly every other day, he will request that a traffic light be put in.

References

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Federal Highway Administration, *Manual on Uniform Traffic Control Devices*, U.S. Govt. Printing Office, 1988.

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Problems

21.1 (t) Prove that the differential equation describing $P_k(t) = P[N(t) = k]$ for a Poisson counting random process is given by (21.3). To do so use Figure 21.3 with *either* k arrivals in $[0, t - \Delta t]$ and no arrivals in $(t - \Delta t, t]$ *or* $k - 1$ arrivals in $[0, t - \Delta t]$ and one arrival in $(t - \Delta t, t]$. Since there can be at most one arrival in a time interval of length Δt, these are the only possibilities.

21.2 (t) Solve the differential equation of (21.3) by taking the (one-sided) Laplace transform of both sides, noting that $P_k(0^+) = 0$. Explain why the latter condition is consistent with the assumptions of a Poisson random process. You should be able to show that the Laplace transform of $P_k(t)$ is

$$\mathcal{P}_k(s) = \frac{\lambda^k}{(s+\lambda)^{k+1}}$$

by finding $\mathcal{P}_1(s)$ from $\mathcal{P}_0(s)$, and then $\mathcal{P}_2(s)$ from $\mathcal{P}_1(s)$, etc. The desired inverse Laplace transform is found by referring to a table of Laplace transforms.

21.3 (⌣) (f) Find the probability of 6 arrivals of a Poisson random process in the time interval $[7, 12]$ if $\lambda = 1$. Next determine the average number of arrivals for the same time interval.

21.4 (w) For a Poisson random process with an arrival rate of 2 arrivals per second, find the probability of exactly 2 arrivals in 5 successive time intervals of length 1 second each.

21.5 (f) What is the probability of a single arrival for a Poisson random process with arrival rate λ in the time interval $[t, t + \Delta t]$ if $\Delta t \to 0$?

21.6 (w) Telephone calls come into a service center at an average rate of one per 5 seconds. What is the probability that there will be more than 12 calls in the first one minute?

21.7 (⌣) (f,c) For a Poisson random process with an arrival rate of λ use a computer simulation to estimate the arrival rate if $\lambda = 2$ and also if $\lambda = 5$. To do so relate λ to the average number of arrivals in $[0, t]$. Hint: Use the MATLAB code in Section 21.7.

21.8 (w) Two independent Poisson random processes both have an arrival rate of λ. What is the expected time of the first arrival observed from either of the two random processes? Explain your results. Hint: Let this time be denoted by T and note that $T = \min(T_1^{(1)}, T_1^{(2)})$, where $T_1^{(i)}$ is the first arrival time of the ith random process. Then, note that $P[T > t] = P[T_1^{(1)} > t, T_1^{(2)} > t]$.

21.9 (t) In this problem we prove that the sum of two independent Poisson counting random processes is another Poisson counting random process whose arrival rate is the sum of the arrival rates of the two random processes. Let the Poisson counting random processes be $N_1(t)$ and $N_2(t)$ and consider the increments $N(t_2) - N(t_1)$ and $N(t_4) - N(t_3)$ for nonoverlapping time intervals. Argue that the corresponding increments for the sum random process are independent and stationary, knowing that this is true for each individual random process. Then, use characteristic functions to prove that if $N_1(t) \sim \text{Pois}(\lambda_1 t)$ and

$N_2(t) \sim \text{Pois}(\lambda_2 t)$ and $N_1(t)$ and $N_2(t)$ are independent, then $N_1(t) + N_2(t) \sim$ Pois$((\lambda_1 + \lambda_2)t)$.

21.10 (☺) (w) If $N(t)$ is a Poisson counting random process, determine $E[N(t_2) - N(t_1)]$ and var$(N(t_2) - N(t_1))$.

21.11 (w) Commuters arrive at a subway station that has 3 turnstyles with the arrivals at each turnstyle characterized by an independent Poisson random process with arrival rate of λ commuters per second. Determine the probability of a total of k arrivals in the time interval $[0, t]$. Hint: See Problem 21.9.

21.12 (t) In this problem we present an alternate proof that the Poisson random process has no memory as described by (21.7). It is based on the observation that a Poisson random process is the limiting form of a Bernoulli random process as explained in Section 21.1. Consider first the geometric PMF of the first success or arrival which is $P[X = k] = (1-p)^{k-1}p$ for $k = 1, 2, \ldots$. Then show that

$$P[X > k_1 + k_2 | X > k_1] = (1-p)^{k_2}.$$

Next let $p = \lambda \Delta t$ and $k_1 = \xi_1 / \Delta t$ and $k_2 = \xi_2 / \Delta t$ and prove that as $\Delta t \to 0$

$$P[X > k_1 + k_2 | X > k_1] = P[X\Delta t > k_1 \Delta t + k_2 \Delta t | X\Delta t > k_1 \Delta t] \to \exp(-\lambda \xi_2).$$

Hint: As $x \to 0$, $(1 - ax)^{1/x} \to \exp(-a)$.

21.13 (☺) (w) Taxi cabs arrive at the rate of 1 per minute at a taxi stand. If a person has already waited 10 minutes for a cab, what is the probability that he will have to wait less than 1 additional minute?

21.14 (w) A computer memory has the capacity to store 10^6 words. If requests for word storage follow a Poisson random process with a request rate of 1 per millisecond, how long on average will it be before the memory capacity is exceeded?

21.15 (t) If $X_1 \sim \exp(\lambda)$, $X_2 \sim \exp(\lambda)$, and X_1 and X_2 are independent random variables, derive the PDF of the sum by using a convolution integral.

21.16 (t) We give an alternate derivation of the PDF for the kth arrival time of a Poisson random process. This PDF can be expressed as

$$\lim_{\Delta t \to 0} \frac{P[t - \Delta t \le T_k \le t]}{\Delta t}.$$

Use the fact that the event $\{t - \Delta t \le T_k \le t\}$ can only occur as $\Delta t \to 0$ if there are $k - 1$ arrivals in $[0, t - \Delta t]$ and 1 arrival in $(t - \Delta t, t]$.

21.17 (☺) (**w**) People arrive at a football game at a rate of 100 per minute. If the 1000th person is to receive a seat at the 50th yard line (which is highly desirable), how long should you wait before entering the stadium?

21.18 (**t**) Prove that if X, Y, Z are jointly distributed continuous random variables, then $E_{X,Y,Z}[g(X,Y,Z)] = E_Z[E_{X,Y|Z}[g(X,Y,Z)|z]]$ by expressing the expectations using integrals. You may wish to refer back to Section 13.6.

21.19 (**t**) The Poisson random process exhibits the *Markov property*. This says that the conditional probability of $N(t)$ based on past samples of the random process only depends upon the most recent sample. Mathematically, if $t_3 > t_2 > t_1$, then

$$P[N(t_3) = k_3|N(t_2) = k_2, N(t_1) = k_1] = P[N(t_3) = k_3|N(t_2) = k_2].$$

Prove that this is true by making use of the property that the increments are independent. Specifically, consider the equivalent probability

$$P[N(t_3) - N(t_2) = k_3 - k_2|N(t_2) = k_2, N(t_1) - N(0) = k_1]$$

and also explain why this probability is equivalent.

21.20 (☺) (**c**) Use a computer simulation to generate multiple realizations of a Poisson random process with $\lambda = 1$. Then, use the simulation to estimate $P[T_2 \leq 1]$. Compare your result to the true value. Hint: Use the MATLAB code in Section 21.7.

21.21 (**w**) An airport has two security screening lines. An employee directs the incoming travelers to one of the two lines at random. If the incoming travelers arrive at the airport with a rate of λ travelers per second, what is the arrival rate at each of the two security screening lines? What assumptions are implicit in arriving at your answer?

21.22 (**t**) Prove that the variance of a compound Poisson random process is $\text{var}(X(t_0)) = \lambda t_0 E[U_1^2]$. If you guessed that the result would be $\lambda t_0 \text{var}(U_1)$, then evaluate your guess for a Poisson random process (let $U_i = 1$).

21.23 (☺) (**f**) A compound Poisson random process $X(t)$ is composed of random variables U_i that can take on the values ± 1 with $P[U_i = 1] = p$. What is the expected value of $X(t)$?

21.24 (**c**) Perform a computer simulation to lend credibility to the expected number of points scored in the basketball game described in Example 21.7.

21.25 (t) Derive (21.13) for the case where the U_i's have the same mean and are independent of $N(t_0)$. Start your derivation with the expression

$$E[X(t_0)] = E_{N(t_0)}\left[E_{U_1,\ldots,U_k|N(t_0)}\left[\sum_{i=1}^{k} U_i \bigg| N(t_0) = k\right]\right]$$

and then follow the same approach as given in Section 21.6. You do not need the characteristic function to do this.

Appendix 21A

Joint PDF for Interarrival Times

We prove in this appendix that the first two interarrival times Z_1, Z_2 are IID with $Z_i \sim \exp(\lambda)$. The general case of any number of interarrival times can similarly be proven to be IID with an $\exp(\lambda)$ PDF. We now refer to Figure 21.4 and prove that the joint CDF factors and each marginal CDF is that corresponding to the $\exp(\lambda)$ PDF. The joint CDF is given as

$$P[Z_1 \leq \xi_1, Z_2 \leq \xi_2] = \int_0^{\xi_1} P[Z_2 \leq \xi_2 | Z_1 = z_1] p_{Z_1}(z_1) dz_1 \qquad (21A.1)$$

which follows from (13.12) where $A = \{z_2 : z_2 \leq \xi_2\}$. But if $Z_1 = z_1$, then $Z_2 \leq \xi_2$ if and only if $N(z_1 + \xi_2) - N(z_1) \geq 1$ since an arrival must have occurred in $[z_1, z_1 + \xi_2]$. Hence,

$$P[Z_2 \leq \xi_2 | Z_1 = z_1] = P[N(z_1 + \xi_2) - N(z_1) \geq 1 | Z_1 = z_1]$$

and because the event $Z_1 = z_1$ is equivalent to the increment $N(z_1) - N(0) = 1$, and the increments are independent and stationary, we have

$$
\begin{aligned}
P[Z_2 \leq \xi_2 | Z_1 = z_1] &= P[N(z_1 + \xi_2) - N(z_1) \geq 1 | Z_1 = z_1] \\
&= P[N(z_1 + \xi_2) - N(z_1) \geq 1] \qquad \text{(independence)} \\
&= P[N(\xi_2) \geq 1] \qquad \text{(stationarity)}.
\end{aligned}
$$

Using this in (21A.1) produces

$$
\begin{aligned}
P[Z_1 \le \xi_1, Z_2 \le \xi_2] &= \int_0^{\xi_1} P[N(\xi_2) \ge 1] p_{Z_1}(z_1) dz_1 \\
&= \int_0^{\xi_1} [1 - P[N(\xi_2) < 1]] p_{Z_1}(z_1) dz_1 \\
&= \int_0^{\xi_1} (1 - P[N(\xi_2) = 0]) p_{Z_1}(z_1) dz_1 \\
&= \int_0^{\xi_1} (1 - \exp(-\lambda \xi_2)) p_{Z_1}(z_1) dz_1 \\
&= [1 - \exp(-\lambda \xi_2)] \int_0^{\xi_1} p_{Z_1}(z_1) dz_1 \\
&= [1 - \exp(-\lambda \xi_2)] P[Z_1 \le \xi_1] \\
&= [1 - \exp(-\lambda \xi_2)] P[N(\xi_1) \ge 1] \\
&= [1 - \exp(-\lambda \xi_2)][1 - P[N(\xi_1) < 1]] \\
&= [1 - \exp(-\lambda \xi_2)][1 - \exp(-\lambda \xi_1)] \\
&= P[Z_1 \le z_1] P[Z_2 \le z_2].
\end{aligned}
$$

It is seen that the joint CDF factors into the product of the marginal CDFs, where each marginal is the CDF of an $\exp(\lambda)$ random variable. Thus, the first two inter-arrival times are IID with PDF $\exp(\lambda)$.