Chapter 3
Gravitational One-Body Problem

Newton's laws of motion and gravity come together to explain the motion of planets around the Sun, plus a wide range of other astrophysical systems. In this chapter we study systems in which the source of gravity (e.g., the Sun) is stationary and a single object (e.g., a planet) is in motion. While Newton's third law tells us that a planet's gravitational pull must also cause the Sun to move, the Sun is so much more massive than any of its planets that its motion can be neglected as a first approximation. In Chap. 4 we will generalize to the case in which both objects move.

3.1 Deriving Kepler's Laws

Kepler's laws provide a great way to analyze orbital motion, since they are already focused on relevant properties of orbits, but in their initial form they were purely empirical and limited to motion around the Sun. If we can use Newton's laws to justify and generalize Kepler's, then we can use the latter to analyze orbital motion in a wide range of settings.

Since Kepler taught us to work with ellipses, we begin by reviewing their geometry. An ellipse is specified mathematically as the solution of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{3.1}$$

We can assume $a > b$ without loss of generality, so Eq. (3.1) is written in a coordinate system where the long or "major" axis of the ellipse is along the x-axis, while the short or "minor" axis is along the y-axis. There are two special points inside the ellipse called **foci** (plural of **focus**) that are a distance $c = \sqrt{a^2 - b^2}$ from the center along the major axis. They are special because the combined distance to the two foci is constant along the ellipse. We define the **eccentricity** of an ellipse to be the dimensionless ratio $e = c/a$, such that a circle has $e = 0$ and more elongated

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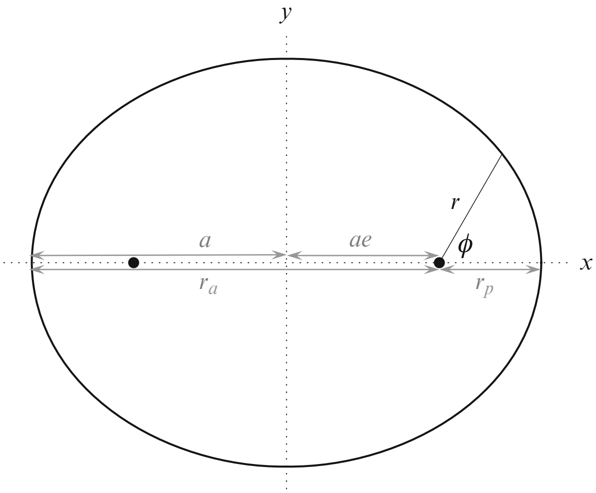

Fig. 3.1 An ellipse with eccentricity $e = 0.6$. The distance from the center to the curve is a along the major axis, and the foci (points) are located a distance ae from the center. The area of the ellipse is $A = \pi a^2(1 - e^2)^{1/2}$. In the text we use polar coordinates (r, ϕ) centered on one focus. The pericenter and apocenter distances are indicated: $r_p = a(1 - e)$ and $r_a = a(1 + e)$

ellipses have higher eccentricities up to the limit $e = 1$. Using the eccentricity we can rewrite $b = a\sqrt{1 - e^2}$ and then specify the size and shape of an ellipse using (a, e) instead of (a, b).

Kepler also taught us that the Sun is at one focus of an elliptical orbit, so if we introduce polar coordinates (r, ϕ) centered on the Sun then we have (see Fig. 3.1)

$$x = ae + r\cos\phi \qquad y = r\sin\phi$$

Plugging into Eq. (3.1) yields

$$\frac{(ae + r\cos\phi)^2}{a^2} + \frac{(r\sin\phi)^2}{a^2(1 - e^2)} = 1$$

Rearranging, we can write this as

$$\frac{1 - e^2\cos^2\phi}{1 - e^2}\,\frac{r^2}{a^2} + 2e\cos\phi\,\frac{r}{a} - 1 + e^2 = 0$$

This is a quadratic equation for r, so it has two solutions. Taking the positive solution (since radius must be positive), we obtain

$$r = \frac{a(1 - e^2)}{1 + e\cos\phi} \tag{3.2}$$

This is the equation for an ellipse in polar coordinates centered on a focus. The points on the ellipse that are closest and farthest from the star have $\phi = 0$ and $\phi = \pi$, respectively; these are known as **pericenter** and **apocenter**.[1] Their radii are

$$\text{pericenter, } r_p = a(1-e) \qquad \text{apocenter, } r_a = a(1+e) \tag{3.3}$$

Our goal now is to connect the geometry to the physical principles represented by Newton's laws. Since the gravitational force is radial it makes sense to use spherical polar coordinates in which the acceleration vector has the form (see Sect. A.2)

$$\begin{aligned}\mathbf{a} = &\left[\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - r\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\right]\hat{\mathbf{r}} \\ &+ \left[r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t} - r\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\right]\hat{\boldsymbol{\theta}} \\ &+ \left[r\sin\theta\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + 2\sin\theta\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\phi}{\mathrm{d}t} + 2r\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\phi}{\mathrm{d}t}\right]\hat{\boldsymbol{\phi}}\end{aligned}$$

Newton's second law gives $\mathbf{a} = \mathbf{F}/m$, which yields the three component equations

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - r\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = -\frac{GM}{r^2} \tag{3.4a}$$

$$r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t} - r\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = 0 \tag{3.4b}$$

$$r\sin\theta\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + 2\sin\theta\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\phi}{\mathrm{d}t} + 2r\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\phi}{\mathrm{d}t} = 0 \tag{3.4c}$$

We can solve Eq. (3.4b) if θ is fixed to $\pi/2$, so the motion is confined to a plane (which we are taking to be the equatorial plane). Then Eq. (3.4c) simplifies to

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) = 0 \tag{3.5}$$

If we recall the specific angular momentum from Eq. (2.3),

$$\ell = |\boldsymbol{\ell}| = |\mathbf{r}\times\mathbf{v}| = \left|(r\,\hat{\mathbf{r}})\times\left(\frac{\mathrm{d}r}{\mathrm{d}t}\,\hat{\mathbf{r}} + r\,\frac{\mathrm{d}\phi}{\mathrm{d}t}\,\hat{\boldsymbol{\phi}}\right)\right| = r^2\,\frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{3.6}$$

[1]Special versions of these terms are used for certain situations: *perigee/apogee* for an orbit around Earth, and *perihelion/aphelion* for an orbit around the Sun.

then we see that Eq. (3.5) says angular momentum is conserved (also see Sect. 2.2). This, finally, lets us rewrite the radial equation (3.4a) as

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{\ell^2}{r^3} = -\frac{GM}{r^2} \tag{3.7}$$

To solve this equation, let's shift from t to ϕ as the independent variable and also make the substitution $r = 1/u$. The derivative becomes

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}(1/u)}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\phi}\ell u^2 = -\ell\frac{\mathrm{d}u}{\mathrm{d}\phi}$$

In the second step we use the chain rule of derivatives, and in the third step we use $\mathrm{d}\phi/\mathrm{d}t = \ell/r^2$. By a similar analysis, the second derivative is

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}\phi}\left(-\ell\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)\ell u^2 = -\ell^2 u^2\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2}$$

Plugging this into Eq. (3.7) and simplifying yields

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{GM}{\ell^2} \tag{3.8}$$

If the right-hand side were zero, this would be the equation for a simple harmonic oscillator and the solution would have the form $u_0 = B\cos\phi$ where B is some constant. To deal with the constant on the right-hand side, we just need to add GM/ℓ^2 to u_0 (which works because the constant does not affect the derivative term). In other words, the solution has the form $u = B\cos\phi + GM/\ell^2$. Without loss of generality, we can define a new constant e such that $B = GMe/\ell^2$ and our final solution is

$$\frac{1}{r(\phi)} = u(\phi) = \frac{GM}{\ell^2}(1 + e\cos\phi) \tag{3.9}$$

Comparing with Eq. (3.2), we see that this curve describes an ellipse, and the constant e we have defined here is nothing more than the eccentricity of the ellipse.

To examine Kepler's second law, we need to consider the area dA swept out by a planet's motion in some small time interval dt. From the geometry shown in Fig. 3.2, the area is

$$\mathrm{d}A = \frac{1}{2}\left|\mathbf{r}\times\mathbf{v}\,\mathrm{d}t\right| = \frac{1}{2}\left|\boldsymbol{\ell}\right|\mathrm{d}t \quad\Rightarrow\quad \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\ell}{2} \tag{3.10}$$

This is constant because angular momentum is conserved. Thus, Kepler's second law is a direct consequence of the fact that gravity is a central force.

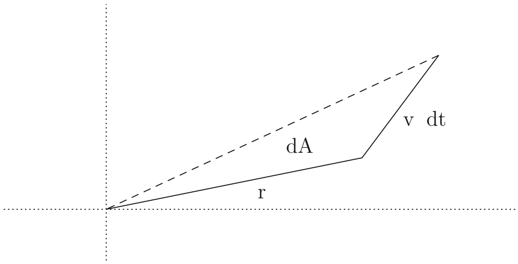

Fig. 3.2 A particle at position **r** moving with velocity **v** for an infinitesimal time interval dt sweeps out a small triangle. By the properties of the cross product, the area of the triangle is d$A = (1/2)|\mathbf{r} \times \mathbf{v}\, \mathrm{d}t|$

Now we come to Kepler's third law. By comparing Eqs. (3.2) and (3.9), we can express the specific angular momentum in terms of the orbital elements as

$$\ell = \left[GMa(1-e^2)\right]^{1/2} \tag{3.11}$$

Then from Eq. (3.10) the rate at which area is swept out is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}\left[GMa(1-e^2)\right]^{1/2}$$

Since this is constant, the area swept out in one period is

$$A = \frac{\mathrm{d}A}{\mathrm{d}t} \times P = \frac{P}{2}\left[GMa(1-e^2)\right]^{1/2} \tag{3.12}$$

But this has to equal the area of the ellipse, which is

$$A = \pi\, a\, b = \pi\, a^2 \left(1-e^2\right)^{1/2} \tag{3.13}$$

Equating (3.12) and (3.13) and solving for P yields

$$P^2 = \frac{4\pi^2}{GM}\, a^3 \tag{3.14}$$

This is Kepler's third law, but now in a general form that explicitly shows the proportionality factor between P^2 and a^3, which depends on the mass of the central object.

To recap, here again are Kepler's empirical laws of planetary motion, along with Newton's physical explanation of them:

I. The orbit is an ellipse because that shape is the solution of Newton's laws of motion under the influence of an inverse square gravitational force.
II. The rate at which area is swept out is constant because of conservation of angular momentum, which holds because gravity is a central force.
III. The relation $P^2 \propto a^3$ holds because gravity has an inverse square force law.

3.2 Using Kepler III: Motion → Mass

With Newton's generalization, Kepler's third law becomes a powerful principle for astrophysics. Rearranging Eq. (3.14), we can write

$$M = \frac{4\pi^2 a^3}{GP^2} \tag{3.15}$$

This form is notable because the right-hand side involves quantities we can measure—the size and period of an orbit—while the left-hand side is something we may want to know—the mass of an astrophysical object. As we explore applications, we will encounter a number of practical challenges (mostly related to measuring a accurately), but the fundamental principle remains valid: if we can observe motion and interpret it using Newton's laws, we can infer mass. Mass is a key property of astronomical systems that is difficult to measure directly, so the motion→mass principle is valuable in a wide range of contexts.

3.2.1 The Black Hole at the Center of the Milky Way

At the center of the Milky Way galaxy is a compact source known as **Sagittarius A*** (often abbreviated as Sgr A*) that emits light across the electromagnetic spectrum. At radio wavelengths, high-resolution observations have constrained the size to be $\lesssim$0.3 AU [1]. At X-ray wavelengths corresponding to photon energies[2] between 2 and 10 keV, the luminosity is greater than 10^{26} J s^{-1} [2]. What could be so energetic yet compact?

Beginning in the 1990s, powerful telescopes and clever observational techniques made it possible to resolve individual stars in the Galactic Center, as shown in Fig. 3.3. Dedicated observers discovered that the stars are moving, mapped the motions, and ultimately found that the orbits appear to be ellipses with Sgr A* as a common focus. In other words, the stars orbiting Sgr A* form a Keplerian system that is directly analogous to the planets orbiting the Sun.

We can therefore use the motion → mass principle to measure the mass of Sgr A*. Stars #2, 16, and 19 (labeled in Fig. 3.3) are particularly important because they have been tracked long enough to pass pericenter, so their orbits are well constrained. Fitting ellipses to the motion yields the following orbital parameters (taken from Ghez et al. [3]; see Gillessen et al. [4] for updated data):

Star	P (yr)	a (AU)	r_p (AU)
2	14.53	919	122
16	36	1,680	45
19	37.3	1,720	287

[2]X-ray astronomers often quote energy rather than frequency or wavelength using the quantum relation $E = h\nu = hc/\lambda$.

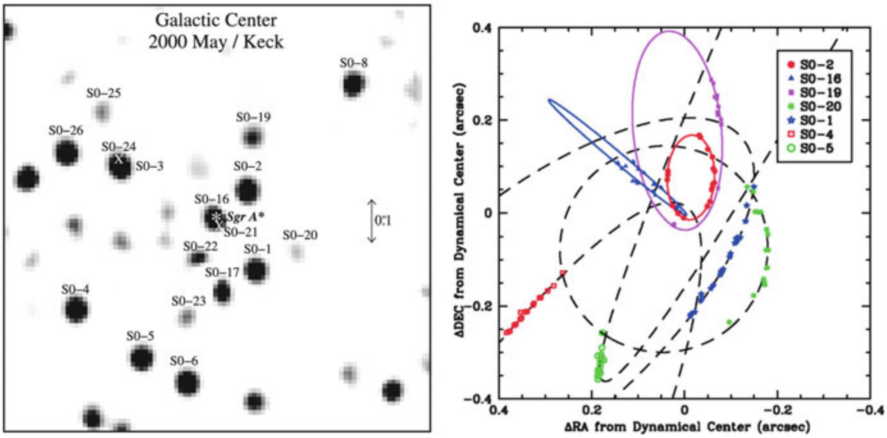

Fig. 3.3 Stars near the Galactic Center. The *left panel* shows a snapshot from May 2000, while the *right panel* shows some of the orbits traced over time (plotted on a different scale) (Credit: Ghez et al. [3]. Reproduced by permission of the AAS)

Applying Eq. (3.15) to star #2 yields

$$M = \frac{4\pi^2 \times (919 \times 1.50 \times 10^{11}\ \text{m})^3}{(6.67 \times 10^{-11}\ \text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}) \times (14.53 \times 3.16 \times 10^7\ \text{s})^2}$$
$$= 7.3 \times 10^{36}\ \text{kg}$$
$$= 3.7 \times 10^6\ M_\odot$$

Repeating the analysis for other stars gives consistent results. In other words, from the motions of stars we conclude that there is an object with nearly four million times the mass of the Sun lying at the center of the Milky Way. From the radio and X-ray observations, and the pericenter distances, we know this object is luminous and compact. What could it be? The only plausible answer is a black hole—indeed, a **supermassive black hole (SMBH)**.

At this point you may have some questions:

- **Why did we treat this as a one-body problem?**
  The black hole is even more massive relative to the stars than the Sun is compared to the planets, so its reflex motion is negligible.
- **Could Sgr A* be anything other than a black hole?**
  Could it be a single star? No: in Chap. 16 we will see that there is no way for a single star to be anywhere near this massive. Could it be a cluster of stars? Again, no: in Sect. 3.3.2 below we will see that such a massive and compact star cluster would "evaporate" due to stellar dynamical effects.

- **If it is a black hole, why haven't we used relativity?**
  As we will see in Chap. 10, relativistic effects become important on scales comparable to the Schwarzschild radius of a black hole. For Sgr A* this is

$$R_S = \frac{2GM}{c^2} = 1.1 \times 10^{10}\,\text{m} = 0.07\,\text{AU}$$

  Even star #16 stays far enough from the black hole that Newtonian gravity gives a reasonable approximation to the motion.
- **Can we see the event horizon?**
  The Galactic Center is about $R_{\text{GC}} \approx 8\,\text{kpc}$ away, so the angle subtended by the black hole's event horizon is (using the small-angle approximation)

$$\begin{aligned}\phi &\approx \frac{R_S}{R_{\text{GC}}} \\ &= \frac{1.1 \times 10^{10}\,\text{m}}{8 \times 3.09 \times 10^{19}\,\text{m}} \\ &= 4.5 \times 10^{-11}\,\text{rad} \times \frac{180\,\text{deg}}{\pi\,\text{rad}} \times \frac{3{,}600\,\text{arcsec}}{1\,\text{deg}} \\ &= 9.3 \times 10^{-6}\ \text{arcsec}\end{aligned}$$

  At optical wavelengths, the best resolution that can be achieved today is 0.05–0.1 arcsec (with the Hubble Space Telescope, or adaptive optics from the ground). At radio wavelengths, it is possible to use an array of telescopes with a technique called interferometry to achieve a resolution of 10^{-4} arcsec or better. While observations have not directly revealed the event horizon, they do seem to be on the verge of resolving some of the interesting structure in Sgr A* [1].

3.2.2 Supermassive Black Holes in Other Galaxies

Our galaxy is not the only one with a supermassive black hole at the center; evidence is growing that every massive galaxy hosts such an object. In most cases we cannot study the black holes in as much detail as Sgr A*, but we can still use the motion→mass principle to infer their masses.

NGC 4258

After the Milky Way, the galaxy with the best constraints on a supermassive black hole is NGC 4258. (The name refers to the galaxy's entry in the *New General Catalogue of Nebulae and Clusters of Stars* [5].) Radio observations reveal water

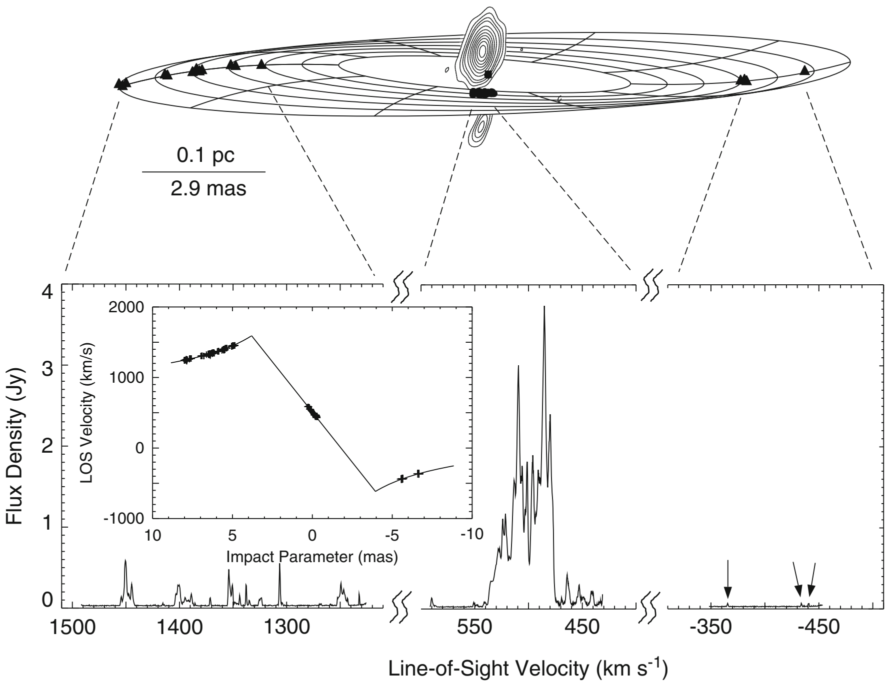

Fig. 3.4 The *top panel* shows a sketch of the disk of gas orbiting the black hole at the center of NGC 4258, with some maser positions indicated. The *bottom panel* shows the radio spectrum. The inset shows the line-of-sight velocity as a function of position, along with a Keplerian rotation curve. (The middle part of the Keplerian curve corresponds to "sideways" motion in the front part of the rotating disk) (Reprinted by permission from Macmillan Publishers Ltd: Herrnstein et al. [6], © 1999)

masers[3] orbiting the center of the galaxy. While the orbital period is too long for us to see the masers shift position, we can still measure motion. Masers emit light at very specific wavelengths, but if they are moving toward or away from us the emission is shifted to shorter or longer wavelengths by the **Doppler effect**. For non-relativistic motion, the shift in wavelength is $\Delta\lambda/\lambda_e = v/c$ where λ_e is the emitted wavelength, and v is the component of velocity along the line of sight with the convention that $v > 0$ if the object is moving away from us and $v < 0$ if it is moving toward us. (See Sect. 10.2.4 for a full discussion of the relativistic Doppler effect.) Figure 3.4 shows that masers closer to the center of NGC 4258 move faster, and the motion is consistent with orbits around an object with mass $(3.9\pm0.1)\times10^7\ M_\odot$ [6].

[3]Maser originally stood for "microwave amplification by stimulated emission of radiation," although "microwave" is now sometimes replaced by "molecular." A laser is similar to a maser except that it operates in the visible portion of the electromagnetic spectrum (the "l" stands for "light," specifically meaning visible light).

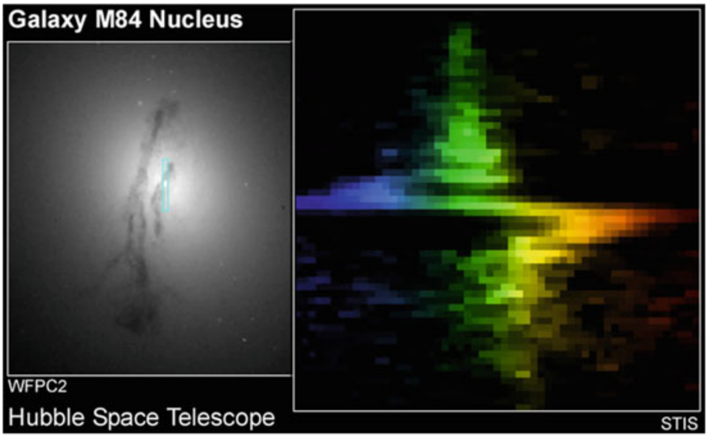

Fig. 3.5 On the *left* is an image of the galaxy NGC 4374 (also known as M84), taken with the Wide Field and Planetary Camera 2 on the Hubble Space Telescope. The *small box* shows the region whose spectrum was recorded with the Space Telescope Imaging Spectrograph, as shown on the right. The *zigzag pattern* is created by the Doppler shift of light from stars and gas orbiting a supermassive black hole at the center of the galaxy (Credit: Gary Bower, Richard Green (NOAO), the STIS Instrument Definition Team, and NASA)

The current upper limit on the size of the object is 0.16 pc, so the size constraint is not nearly as strong as for Sgr A*. Nevertheless, astronomers believe the central object is a black hole.

NGC 4374

At present there are no other galaxies where we can observe *individual* objects moving around the center of the galaxy. Still, we can measure *collections* of stars or gas moving around in the centers of many galaxies. As an example, Fig. 3.5 shows an optical spectrum of the galaxy NGC 4374. The light from stars and gas on one side of the galaxy center is shifted toward bluer wavelengths by the Doppler effect, while the light from stars and gas on the other side of the center is shifted toward redder wavelengths. Also, objects closer to the center move faster. The motion again reveals a central massive object, this time with a mass of nearly $9 \times 10^8\, M_\odot$ [7].

A Supermassive Black Hole in Every Galaxy?

Similar observations in other galaxies have shown that whenever we can make good measurements we find evidence for supermassive black holes. Astronomers now suspect that every massive galaxy harbors a central black hole, and the black hole masses range from a few million to more than a billion times the mass of

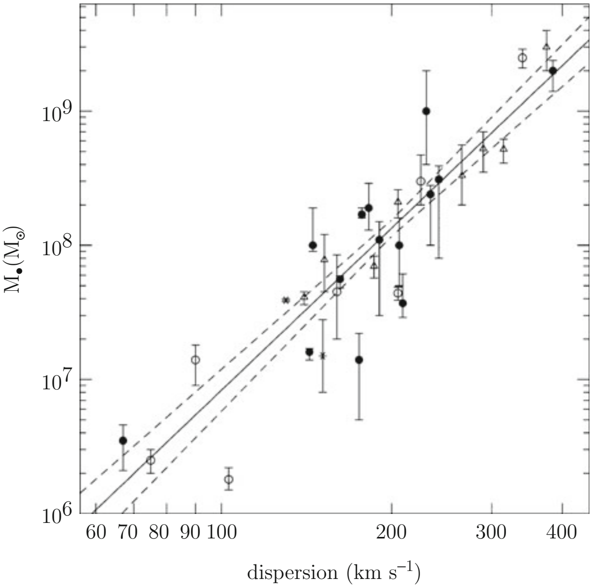

Fig. 3.6 Relation between *black hole mass* (indicated here by $M_\bullet$) and galaxy velocity dispersion. The *solid line* shows the best fit to the data, which scales as $M \propto \sigma^{4.02}$. The *dashed lines* show uncertainties in the fit (Credit: Tremaine et al. [8]. Reproduced by permission of the AAS. (See [9] for an updated version of this relation))

the Sun. What's more, the mass of the black hole appears to be closely related to the properties of the galaxy in which it resides.

We will study galaxies later (in Chaps. 7 and 8), but for now we note that most galaxies can be described in terms of two types of structures: a flat *disk* in which the star orbits lie mostly in a plane; and a rounder *spheroid* in which the star orbits have random orientations. Spiral galaxies usually have large disks surrounding smaller spheroids known as bulges, while elliptical galaxies are pure spheroids. Since the motion in spheroids is random, we characterize it by examining the distribution of star velocities (strictly speaking, the component along the line of sight) and computing the statistical standard deviation, which we call the **velocity dispersion**.

A striking discovery about supermassive black holes is that the black hole mass is correlated with the velocity dispersion of the spheroidal component of its host galaxy, as shown in Fig. 3.6. You may wonder: why should it be remarkable that motion (σ) is closely related to mass (M)? Most of the stars used to measure the velocity dispersion lie far enough from the black hole that they should hardly notice its gravity.[4] Yet the stars seem to know how much the black hole weighs—or, conversely, the black hole knows how fast the stars move. Astronomers are still

[4]We quantify this idea in terms of a gravitational "sphere of influence" in Sect. 3.3.1.

trying to understand how this came to be: observers are trying to see whether the M-σ relation was the same in the past, while theorists are trying to understand whether the processes by which black holes and spheroids grow might be related to one another. The final answers are not known, but the discovery of the M-σ relation has sparked a lot of new research.

3.2.3 Active Galactic Nuclei

Direct motion-based measurements of black hole masses can be made only in relatively nearby galaxies, where we can resolve the motion on small scales. Nevertheless, strong indirect evidence suggests that supermassive black holes are common in galaxies throughout the universe.

The evidence comes from Active Galactic Nuclei (AGN)—an umbrella term for galaxies that emit huge amounts of energy from their centers. There are many types of AGN but for our purposes there are two key features. First, these objects can be very luminous, reaching $L \sim 10^{12} L_{\odot}$. Second, AGN can vary on time scales as short as $\Delta t \sim 1$ h. The variability lets us place an upper limit on the size, because a source can change coherently only if information about the physical conditions can travel across the source. If we imagine that something changes in the middle of the source, the time it would take for that information to reach the edge is $\Delta t \gtrsim R/c$ (and perhaps much longer if the information propagates at less than the speed of light). If $\Delta t \sim 1$ h then we can infer

$$R \lesssim c\,\Delta t \sim 3.0 \times 10^8\,\mathrm{m\,s^{-1}} \times 3{,}600\,\mathrm{s} \sim 10^{12}\,\mathrm{m} \sim 7\,\mathrm{AU}$$

In other words, an AGN can be *as bright as a galaxy, but smaller than the Solar System!* What might be so energetic? A supermassive black hole.

You may ask: Aren't black holes supposed to be black? How can they emit so much energy? While nothing can escape a black hole once it has fallen in, a lot of energy can be emitted *as matter approaches a black hole*. Imagine mass falling in at a rate $\dot{M} = \mathrm{d}M/\mathrm{d}t$. In time $\mathrm{d}t$, an amount $\dot{M}\,\mathrm{d}t$ falls into the black hole, and as it falls from infinity to the event horizon it releases potential energy

$$\mathrm{d}U \sim -\frac{GM}{R_s}\,\dot{M}\,\mathrm{d}t$$

(We will use Newtonian gravity for this simple estimate.) As atoms fall in, their kinetic energy must increase to conserve energy. As they speed up, they bump into one another more and more often, causing the gas to heat up and radiate. If all the potential energy that was liberated gets converted to light, the total luminosity (light energy per unit time) could be as large as

$$L \sim \left|\frac{\mathrm{d}U}{\mathrm{d}t}\right| \sim \frac{GM}{R_s}\,\dot{M} \sim \frac{GM}{2GM/c^2}\,\dot{M} \sim \frac{1}{2}\,\dot{M}\,c^2$$

As mass falls into a black hole, a significant fraction of its rest mass energy could be converted into light.

There are some caveats. The energy release is probably gradual; it does not all happen at the event horizon. Some of the energy might even vanish into the black hole. Also, a proper analysis should account for relativity. Detailed analyses indicate that the energy release has the form (e.g., [10])

$$L \approx \varepsilon \dot{M} c^2$$

where the "efficiency" is $\varepsilon \sim 0.06 - 0.42$ and $\varepsilon \approx 0.1$ is a typical value. Even so, it is fair to say that black holes are the most efficient machines in the universe for converting mass into energy.[5]

3.3 Related Concepts

Let us briefly step away from the main story to address two topics that arose in Sect. 3.2. The notion of a gravitational sphere of influence is important for interpreting the M-σ relation, and it is an interesting variant of the one-body problem. The concept of stellar dynamical evaporation is important for interpreting constraints on supermassive black holes (particularly Sgr A*), and it provides a nice application of dimensional analysis.

3.3.1 Sphere of Influence

In Sect. 3.2.2 we mentioned that astronomers were surprised to find a tight relation between the masses of supermassive black holes and the velocity dispersions of the spheroids in which they are embedded. Why was that a surprise? To find out, let's estimate the size of the region in which a black hole has a significant influence on the motions of stars. To be more specific, let's define a black hole's "sphere of influence" to be the region where the gravity from the black hole is stronger than the gravity from the rest of the matter in the galaxy. At radius r, the strength of the gravitational force from the black hole is

$$F_{\rm bh}(r) = \frac{G M_{\rm bh} m}{r^2}$$

What about the force from the galaxy? For simplicity, let's assume the galaxy is spherically symmetric. From Eq. (2.11), the force is then

[5]For comparison, the energy released by fusion in stars corresponds to an efficiency $\varepsilon = 0.007$ (see Sect. 15.2).

$$F_{\text{gal}}(r) = \frac{GM_{\text{gal}}(r)\,m}{r^2}$$

where $M_{\text{gal}}(r)$ is the mass enclosed by a sphere of radius r. In Chaps. 7 and 8 we will see that a simple model for a galaxy with velocity dispersion σ is the isothermal sphere, which has density

$$\rho_{\text{gal}}(r) = \frac{\sigma^2}{2\pi G r^2}$$

The mass enclosed by radius r is

$$M_{\text{gal}}(r) \;=\; \int_0^r \frac{\sigma^2}{2\pi G (r')^2}\, 4\pi (r')^2\, \mathrm{d}r' \;=\; \frac{2\sigma^2}{G}\, r$$

so the gravitational force from the galaxy is

$$F_{\text{gal}}(r) \;=\; \frac{Gm}{r^2}\,\frac{2\sigma^2 r}{G} \;=\; \frac{2\sigma^2 m}{r}$$

In order to have the force from the black hole exceed the force from the rest of the mass in the galaxy, we need

$$\frac{GM_{\text{bh}} m}{r^2} > \frac{2\sigma^2 m}{r}$$

Thus, the black hole's sphere of influence is the region with $r < R_0$ where

$$R_0 \equiv \frac{GM_{\text{bh}}}{2\sigma^2}$$

From the observed M-σ relation, a galaxy with $\sigma \approx 200\,\text{km s}^{-1}$ hosts a black hole of about $M_{\text{bh}} \approx 10^8\, M_\odot$. By our estimate, the black hole's sphere of influence is then

$$R_0 = \frac{(6.67 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}) \times (10^8 \times 1.99 \times 10^{30}\,\text{kg})}{2 \times (2 \times 10^5\,\text{m s}^{-1})^2} = 1.7 \times 10^{17}\,\text{m} = 5.4\,\text{pc}$$

For a massive galaxy with $\sigma \approx 330\,\text{km s}^{-1}$ that hosts a huge black hole with $M_{\text{bh}} \approx 10^9\, M_\odot$, we get

$$R_0 \;=\; 6.1 \times 10^{17}\,\text{m} \;=\; 20\,\text{pc}$$

These distances are very small compared with the size of a galaxy (which is typically measured in kpc). In other words, even a supermassive black hole does not have enough mass compared with its galaxy to have a strong effect on the entire galaxy.

The M-σ relation must arise from some indirect connection between the way galaxies form and the way supermassive black holes grow inside galaxies.

3.3.2 Stellar Dynamical Evaporation

In Sect. 3.2.1 we learned that stellar motions reveal Sgr A* to be massive and compact, but they do not definitively prove it to be a black hole, so we should consider alternatives. We said it cannot be a single star, but could it be a cluster of stars?

If millions of stars are confined to a small space, they will occasionally pass very close to each other. Since gravity gets strong when separations get small, close interactions can impart enough force to eject one of the stars from the cluster. Let's use dimensional analysis to estimate the time it would take for a star cluster to "evaporate" in this way.[6] Suppose there are N stars of mass m (so the total mass is $M = Nm$), in a region of size R. For dimensional analysis, what do we have to work with?

Cluster mass	M	$[M]$
Star mass	m	$[M]$
Number of stars	N	—
Cluster size	R	$[L]$
Gravity	G	$[M^{-1}L^3T^{-2}]$

We need $G^{-1/2}$ to get a time, and then we need $R^{3/2}$ to eliminate length. We have a choice of mass: M or m. Since the evaporation interactions involve individual stars, I think the key mass is m. There may also be some factor that depends on the number of stars N; we will come back to that in a moment. To this point, our analysis of dimensions gives a guess of the form

$$t_{\text{evap}} \sim \frac{R^{3/2}}{(Gm)^{1/2}}$$

Now let's consider the number of stars. I imagine that there are two places where N enters. First, since stars are ejected one by one the time it takes to evaporate the cluster should have a factor of N. Second, if we pack more stars into a fixed space, gravity will be stronger, and the stars will move faster. That will cause interactions to happen more quickly, decreasing the evaporation time. In Problem 1.1 you used dimensional analysis to estimate the typical velocity of stars in a gravitationally bound system; the upshot is that speed scales as

[6] See Sect. 3.2 of *Galaxies in the Universe* by Sparke and Gallagher [11] for a complementary analysis of evaporation.

$v \propto N^{1/2}$, which suggests that the evaporation time should have a factor of $N^{-1/2}$. Incorporating both of these factors yields

$$t_{\text{evap}} \sim \frac{R^{3/2}}{(Gm)^{1/2}} \frac{N}{N^{1/2}} \sim \left(\frac{NR^3}{Gm}\right)^{1/2} \sim \left(\frac{MR^3}{Gm^2}\right)^{1/2}$$

Let's plug in numbers. Our mass estimate for Sgr A* is $M = 3.7 \times 10^6\, M_\odot$. Strictly speaking, all we know from the motion is that the mass is confined within a region $R < 45$ AU. If we assume all stars are like the Sun, we have $m \sim M_\odot$. Then:

$$\begin{aligned} t_{\text{evap}} &\sim \left[\frac{(3.7 \times 10^6 \times 1.99 \times 10^{30}\,\text{kg}) \times (45 \times 1.50 \times 10^{11}\,\text{m})^3}{(6.67 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}) \times (1.99 \times 10^{30}\,\text{kg})^2}\right]^{1/2} \\ &\sim 2.9 \times 10^{12}\,\text{s} \\ &\sim 90{,}000\,\text{yr} \end{aligned}$$

While this estimate from dimensional analysis may be fairly crude, it certainly indicates that if Sgr A* were a cluster of normal stars it would have evaporated long ago.

We are left with the conclusion that Sgr A* is probably a black hole. Even though we have not yet detected the event horizon—that is the holy grail of black hole studies—we have assembled a strong case in which the Kepler's laws and the motion → mass principle have played a key role.

Problems

3.1. Sketch the orbital speed v as a function of orbital size r for a planet in a circular orbit about the Sun. This is known as a *Keplerian rotation curve*.

3.2. Consider a rocket orbiting Earth in an orbit that is initially circular.

(a) If the rocket fires a short burst from its engine to apply a force in the *same* direction as its motion, what happens to the shape of the orbit? Sketch the before and after orbits. Hint: think about the kinetic and potential energies just before and just after the burst, and refer back to Problem 2.4.
(b) Repeat part (a) with the engine firing in the *opposite* direction.
(c) How would a rocket have to fire its engine if it wanted to move to an orbit that is larger but still circular?

3.3. Suppose a comet orbits the Sun with a period of 27 years, and the closest it gets to the Sun is 3 AU. At the point in its orbit when it is moving slowest, how far is the comet from the Sun?

3.4. If the Moon orbited above Earth's equator at a distance of 42,200 km from Earth's center how would it appear to an observer on Earth? Describe the cycle of phases the observer would see.

3.5. Use the orbital data for Jupiter's Galilean moons to compute Jupiter's mass. Verify that all four moons give consistent results.

	P (days)	a (10^3 km)
Io	1.769	421.7
Europa	3.551	670.9
Ganymede	7.155	1,070.4
Callisto	16.689	1,882.7

3.6. Derive expressions for the orbital speeds at pericenter and apocenter of an elliptical orbit. Then consider the stars observed to orbit the black hole at the center of the Milky Way. Which star moves fastest at pericenter? (Be quantitative.)

3.7. Suppose you discover an extrasolar planet orbiting a star of mass $2M_\odot$ with an orbital period of 3 months. What is the semimajor axis of the planet's orbit?

3.8. The black hole in NGC 4374 has been studied using the Doppler shift of light with a wavelength of about 6,600 Å. What is the wavelength shift of light emitted from gas that orbits the black hole at a distance of 30 pc?

3.9. Revisit the analysis of a black hole's sphere of influence (Sect. 3.3.1) assuming a uniform density of stars. Express your answer in terms of $\rho_{\rm stars}$ and equivalently in terms of the mass and radius of a spherical galaxy with uniform density.

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