Chapter 6
Gravitational Three-Body Problem

After solving the one- and two-body problems, generalizing to the three-body problem should be easy, right? No! In fact, it was the gravitational three-body problem that led Henri Poincaré to discover dynamical "chaos" [1]. Some systems are so sensitive to initial conditions that a tiny shift today can dramatically change the long-term behavior. The Solar System is actually an example: despite being well-approximated by the two-body problem, planetary motion is formally chaotic because of gravitational interactions among planets [2]. We cannot solve the three-body problem in general, but we can gain valuable insights from two cases that are simplified but still relevant for systems ranging from satellites near Earth to planets around distant stars.

6.1 Two "Stars" and One "Planet"

First consider a three-body problem in which two of the objects are much more massive than the third. Let's use the language of a "planet" (mass m) moving around two "stars" (masses M_1 and M_2), although we will examine a variety of systems. We assume $m \ll M_1, M_2$ so the planet does not affect the stars' motion. Let's further assume the stars have circular orbits, and the planet moves in their orbital plane. This is clearly a restricted version of the three-body problem, but it is one that has some interesting applications.

6.1.1 Theory: Lagrange Points

To analyze this problem, it is convenient to work in a reference frame that rotates at the same angular frequency as the stars so we can keep the stars fixed and focus on the planet. We have to be careful, though, because Newton's laws in their usual

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form hold only in an inertial (i.e., non-rotating) reference frame. In Sect. A.3 we find that acceleration measured in a rotating reference frame ($\mathbf{a}_{\text{rot}}$) is related to acceleration measured in a fixed reference frame ($\mathbf{a}_{\text{fixed}}$) via

$$\mathbf{a}_{\text{fixed}} = \mathbf{a}_{\text{rot}} + \Omega \times (\Omega \times \mathbf{r}) + 2\Omega \times \mathbf{v}_{\text{rot}} + \frac{d\Omega}{dt} \times \mathbf{r}$$

where Ω is a vector that points along the rotation axis and has an amplitude equal to the rotational frequency $\omega = 2\pi/P$. Newton's second law relates the true force to the acceleration in the fixed frame: $\mathbf{F}_{\text{true}} = m\,\mathbf{a}_{\text{fixed}}$. We can retain the form of this law in the rotating frame if we define an effective force such that $\mathbf{F}_{\text{eff}} = m\,\mathbf{a}_{\text{rot}}$. Clearly we need

$$\mathbf{F}_{\text{eff}} = \mathbf{F}_{\text{true}} - m\,\Omega \times (\Omega \times \mathbf{r}) - 2m\,\Omega \times \mathbf{v}_{\text{rot}} - m\,\frac{d\Omega}{dt} \times \mathbf{r} \tag{6.1}$$

The second term is known as the **centrifugal force**, and it is what you feel "pulling" you outward on a merry-go-round. The third term is known as the **Coriolis force**, and it is important for systems like airplanes and weather moving around the rotating Earth. The fourth term is known as the **Euler force**, and it applies only if the rotation rate is not uniform; it vanishes for problems like ours in which Ω is constant.

It is important to remember that these are not real forces; they are just consequences of working in a rotating reference frame, and are sometimes called "fictitious forces." Nevertheless, they do need to be taken into account when working in a rotating frame.[1]

With the planet's motion restricted to the orbital plane of the stars, $\mathbf{r}$ is perpendicular to Ω and the centrifugal force simplifies: $\Omega \times (\Omega \times \mathbf{r}) = -\omega^2\,\mathbf{r}$. If we neglect the Coriolis force (because it depends on the speed of the planet and is generally small for the systems we consider), then the effective force is

$$\mathbf{F}_{\text{eff}} = \mathbf{F}_{\text{true}} + m\,\omega^2\,\mathbf{r}$$

The associated potential energy is

$$U_{\text{eff}} = -\int \mathbf{F}_{\text{eff}} \cdot d\mathbf{r} = -\int \mathbf{F}_{\text{true}} \cdot d\mathbf{r} - \int m\,\omega^2\,\mathbf{r} \cdot d\mathbf{r} = U_{\text{true}} - \frac{1}{2}\,m\,\omega^2 r^2$$

(In principle there is a constant of integration, but it only affects the unobservable zeropoint of the potential.) We know U_{true} for the two stars (see Eq. 2.12), so we can write down the effective potential,

[1] And they certainly don't feel fictitious when you make a sharp, fast turn in a car!

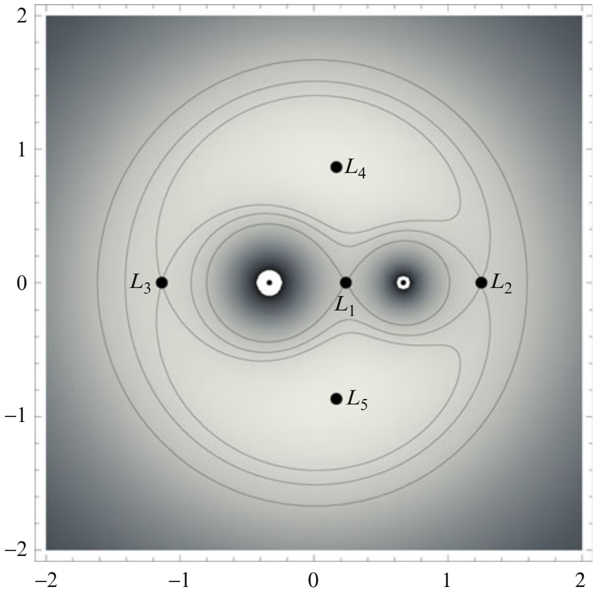


Fig. 6.1 Contour plot of the effective potential for a restricted three-body problem in which the primary objects have a 2:1 mass ratio. The Lagrange points are labeled; L_1, L_2, and L_3 are saddle points, while L_4 and L_5 are local maxima. Contours are chosen to pass through L_1–L_3. The *two small dots* mark the primary masses; they are surrounded by white regions only because the *grayscale* does not capture the divergence $\Phi_{\text{eff}} \to -\infty$ near M_1 and M_2

$$\Phi_{\text{eff}} = \frac{U_{\text{eff}}}{m} = -\frac{GM_1}{|\mathbf{r} - \mathbf{r}_1|} - \frac{GM_2}{|\mathbf{r} - \mathbf{r}_2|} - \frac{1}{2}\omega^2 r^2 \tag{6.2}$$

This function is plotted in Fig. 6.1 for an illustrative example. Since Φ_{eff} is a function in two dimensions, it can have three types of critical points where the derivatives vanish: local minima, local maxima, and saddle points. In the restricted three-body problem, the effective potential has three saddle points, which all lie on the line joining the two stars, and two local maxima, which make equilateral triangles with the two stars (regardless of the stars' masses; see Problem 6.3). These are collectively known as **Lagrange points** after Joseph-Louis Lagrange, and they are labeled as follows:

- L_1: between the two stars
- L_2: "behind" the less massive star
- L_3: "behind" the more massive star
- L_4/L_5: leading/trailing by 60°

Formally, the saddle points L_1–L_3 are unstable equilibria: a particle at rest can stay put, but any little nudge will cause it to roll away. It is possible, though, to find small orbits around L_1, L_2, or L_3 [3]. Despite being local maxima, L_4 and L_5 turn out to

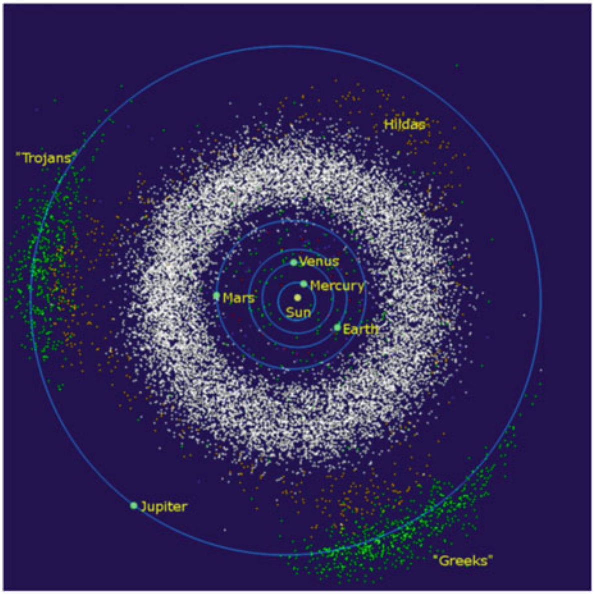


Fig. 6.2 Locations of known asteroids in the inner Solar System. Main belt asteroids are shown in *white*, while Trojan asteroids associated with Jupiter are colored *green*. The Trojans are divided into the "Trojan" and "Greek" camps (Credit: Wikimedia Commons)

be stable equilibria if the mass ratio is $M_1/M_2 > 24.96$. The analysis of stability involves the Coriolis force, which goes beyond the level of detail we are considering here [4].

6.1.2 Applications

Lagrange points are important for natural and artificial objects in our own Solar System, and for certain types of binary star systems as well.

Sometimes we want to place a satellite away from Earth but in a location where it will not drift off. The Lagrange points for the Sun/Earth system are natural choices. Several satellites observing the Sun, including the Solar and Heliospheric Observatory, are at L_1. Several telescopes, including the Wilkinson Microwave Anisotropy Probe and the Planck spacecraft (both observing the Cosmic Microwave Background radiation; see Sect. 20.1), are at L_2.

The L_4 and L_5 Lagrange points of the Sun/Jupiter system host a few thousand asteroids collectively known as "Trojan" asteroids (see Fig. 6.2). These objects are not actually fixed at L_4 or L_5; they move in sizable regions but are trapped in stable orbits around the Lagrange points (again, see [4] for more about stability). There are also some Trojan asteroids associated with Neptune, Mars, and even Earth [5].

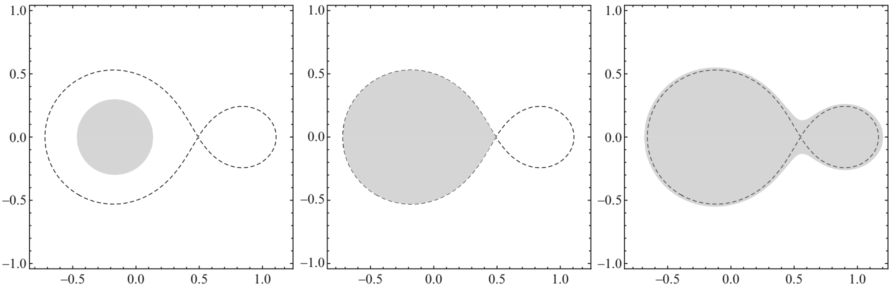


Fig. 6.3 If one component of a binary (*gray*) expands and fills its Roche lobe (*middle*), mass can flow out and envelop the companion (*right*). Here the *dashed line* shows the effective potential contour that passes through the Lagrange point L_1

The Lagrange point L_1 plays a prominent role in binary systems with two stars close together. If one star puffs up (for example, when it becomes a red giant; see Sect. 16.3), then matter near the surface might actually feel stronger gravity from the companion than from its own star. In that case mass can begin to flow from one star to the other. The equipotential contour running through L_1 marks the transition zone, which we call the **Roche lobe**. This scenario, which is pictured in Fig. 6.3, can have several consequences:

- *Accretion.* To conserve angular momentum, the transferred matter often settles into a disk around the second star and then slowly spirals in.
- *Energy release.* When matter drops from L_1 onto the accretion disk or star, potential energy is converted into kinetic energy, which is further converted into heat and light; we can observe X-rays from accretion onto neutron stars and black holes.
- *Nova.* If the second star is a white dwarf, hydrogen can accumulate and heat up to the point that nuclear fusion occurs on the surface; this can make the system much brighter for a few weeks or months, in a phenomenon we call a nova.
- *Supernova.* If enough mass accumulates on a white dwarf, it can carry the white dwarf over the “Chandrasekhar limit” of about $1.4\,M_\odot$ (see Sect. 17.2) and cause the white dwarf to explode as a Type Ia supernova; these objects have become important probes of the expanding universe (see Sect. 18.2).

6.2 One “Planet” and Two “Moons”

Now consider a different limit in which one object far outweighs the other two ($M_1 \gg m_2, m_3$). This limit can describe a variety of systems, but we will initially use the language of a “planet” with two “moons.”

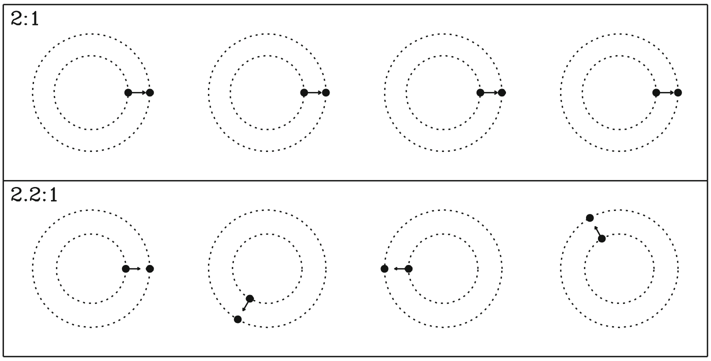


Fig. 6.4 The *top row* shows four successive points of closest approach for two bodies in a 2:1 orbital resonance. At each snapshot in time, the *dots* show the positions of the two bodies, and the *arrow* indicates the direction of the force exerted by the outer body on the inner body. The *bottom row* show a case that is not in resonance, with a frequency ratio of 2.2:1

6.2.1 Theory: Resonances

The planet dominates the gravitational field and keeps the moons moving in Keplerian orbits, but whenever the moons approach one another they exchange a small gravitational "kick." In general, the kicks occur at different locations in the orbits, so they have different directions and tend to average out over time (see the bottom row of Fig. 6.4). Suppose, however, that the inner moon completes exactly two orbits while the outer moon completes one:

$$P_2 = \frac{1}{2} P_3 \quad \Leftrightarrow \quad \omega_2 = 2\omega_3$$

In this case the kicks happen at the same place in the orbit and in the same direction (see the top row of Fig. 6.4) so they tend to add up over time. Any *integer* combination of orbits can likewise let the gravitational kicks combine coherently. If the inner moon goes around m times[2] while the outer moon goes around n times, then

$$mP_2 = nP_3 \quad \Leftrightarrow \quad \frac{\omega_2}{\omega_3} = \frac{m}{n}$$

and we call this an **m:n resonance** (for example, a 2:1 resonance, 3:2 resonance, etc.). In any single orbit the gravity between the moons is weak compared with the gravity from the planet, but the accumulated perturbations can have some interesting consequences.

[2]Here we briefly use m as an integer, not a mass.

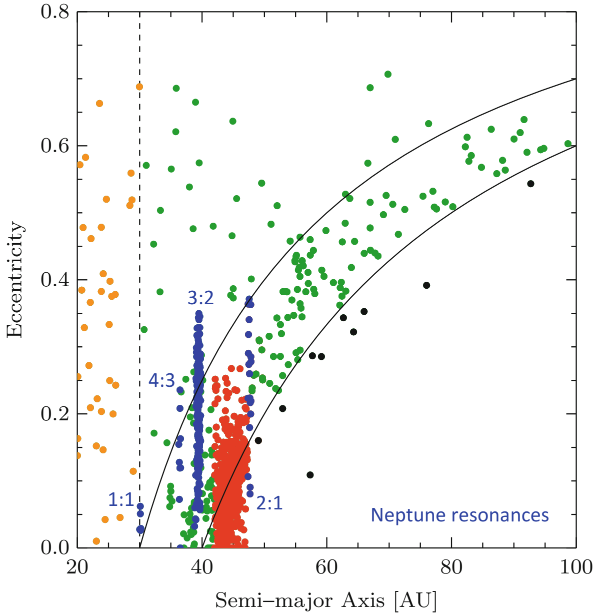


Fig. 6.5 Orbital properties of objects in the outer Solar System. Different colors indicate different classes of objects; we focus on the *blue points*, which are in orbital resonances with Neptune. The *dashed line* marks the semimajor axis of Neptune; the *solid curves* indicate orbits whose perihelion is interior to the semimajor axis of Neptune (*upper curve*) or Pluto (*lower curve*) (Credit: David Jewitt, UCLA)

6.2.2 Applications

One effect of orbital resonances is to lock groups of objects into related orbits. We see this in the Jupiter system; here are the orbital periods and frequencies for three of the moons that Galileo discovered:

	P (day)	ω (day^{-1})
Io	1.769	3.552
Europa	3.551	1.769
Ganymede	7.155	0.878

These moons are in a joint 4:2:1 resonance. The mutual gravitational interaction causes Io's orbit to be more elongated than it would have been otherwise, which couples with the tidal force from Jupiter (Sect. 5.2.2) to make Io the most geologically active body in the Solar System. We also see resonances in the outer Solar System. Figure 6.5 shows the distribution of semimajor axes and eccentricities for known "trans-Neptunian objects." There are notable groupings of objects in 1:1, 4:3, 3:2, and 2:1 resonances with Neptune. (Pluto is part of the group in the 3:2 resonance.)

We might ask how objects come to be in resonance. With the trans-Neptunian objects, an intriguing possibility is that Neptune migrated outward during the planet formation process, causing the location of the resonance to travel outward as well. If the moving resonance captured an object like Pluto, the gravitational interaction would have caused Pluto to migrate such that it remained trapped in resonance [6]. In this way Neptune's migration may have swept a number of objects into resonance. Ongoing research is examining whether a similar process happened among Jupiter's moons to create the resonance between Io, Europa, and Ganymede [7].

In a complementary action, resonances can also clear gaps in extended structures. One example is the dark band called the **Cassini division** in Saturn's rings. Objects cannot stay in this region because they would be in resonance with one of Saturn's moons (see Problem 6.5); the gravitational kicks would elongate the orbit, move the apocenter into the outer ring, and cause these objects to collide with other ring constituents [8]. Another example involves asteroids that lie in the "main belt" between the orbits of Mars and Jupiter. While the distribution of positions in space looks fairly continuous (Fig. 6.2), the distribution of semimajor axes has conspicuous dips at certain values (notably 2.5 and 3.3 AU; see Fig. 6.6).[3] These **Kirkwood gaps** are associated with Jupiter resonances (especially 3:1, 3:2, and 7:3). Even the outer edge of the asteroid belt seems to have been sculpted by a 2:1 resonance with Jupiter.

Problems

6.1. A staple of science fiction is the idea that you could spin a spaceship or space station so that the centrifugal force simulates gravity. How fast would a spaceship with a radius of 10 m have to spin to mimic the gravity on the surface of Earth? How about a space station with a radius of 100 m?

6.2. There are a few known three-body solutions beyond the restricted three-body problem and resonances. Lagrange found a solution with the three bodies forming an equilateral triangle. For this problem, assume the masses are the same.

(a) If the initial velocities are zero, what will happen to the system? Estimate how long it takes the system to reach its final state.
(b) Find the rotational velocity required to balance the gravitational attraction and keep the masses moving along a circular obit.

[3] You might ask why the Kirkwood gaps are not apparent in a snapshot of positions in space. Since asteroid orbits can be moderately elliptical, an asteroid with a given semimajor axis can be found at a range of radii. The gaps in a plot of semimajor axis get smeared out in a plot of position.

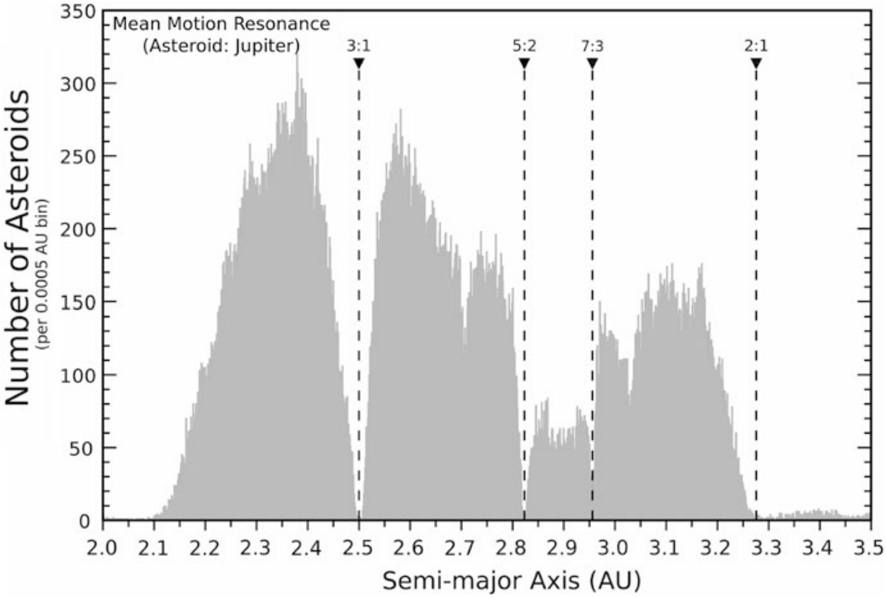


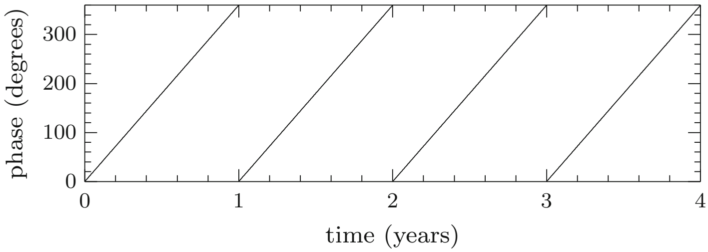
Fig. 6.6 Distribution of semimajor axes for main-belt asteroids. The "Kirkwood gaps" in the distribution coincide with orbital resonances with Jupiter (*dashed lines*) (Credit: NASA/JPL-Caltech)

(c) Is the circular rotating configuration stable? Give a qualitative description of what happens if the velocity is slightly larger or smaller than the "critical" velocity from part (b), or if one of the masses moves slightly inward or outward.

6.3. Consider the restricted three-body problem from Sect. 6.1. Let's show that $\mathbf{F}_{\text{eff}} = 0$ at the Lagrange point L_4. Recall that L_4 makes an equilateral triangle with the two masses.

(a) What is the net gravitational force on a particle of mass m at L_4? Work in Cartesian coordinates, and express your answer in terms of M_1, M_2, and a.
(b) Convert the result from part (a) into polar coordinates centered on the M_1/M_2 center of mass. You should find that the force is radial.
(c) Compute the centrifugal force at L_4 in terms of M_1, M_2, and a. Hint: use Kepler's laws to find ω.
(d) Show that the effective force vanishes at L_4.

6.4. Here is a way to find the locations of closest approach for two orbiting bodies. Consider a planet going around a star in a circular orbit. Its phase angle increases steadily with time, going from 0° to 360° in one period, then jumping back down to 0° and repeating. If the orbital period is 1 yr, the plot looks like this:



(a) Overplot the phase angle for a planet in a 2:1 resonance with the first. Show that the closest approaches always occur at the same phase (top row of Fig. 6.4).
(b) Now consider a frequency ratio of 2.2:1. Show that the closest approaches do *not* occur at the same phase (bottom row of Fig. 6.4).
(c) What does a 3:1 resonance look like? Show both the phase plot and the closest approach configurations.

6.5. The Cassini division is approximately 118,000 km from the center of Saturn. Below are the orbital radii of some of Saturn's major moons. Which one is responsible for the Cassini division? How do you know? (Hint: you do not explicitly need the orbital periods.)

Mimas	185,000 km
Enceladus	238,000 km
Tethys	295,000 km
Dione	377,000 km
Rhea	527,000 km
Titan	1,222,000 km
Iapetus	3,560,000 km

6.6. How common are resonances between planets in extrasolar planetary systems? Use exoplanet data available online to look for resonances.

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