Chapter 1
Introduction: Tools of the Trade

A concise way to state the scientific method in astrophysics is this: We use theory to make quantitative predictions that can be compared with observations. Sometimes we can solve the relevant equations with pencil and paper in a modest number of steps, but other times we cannot. How do we proceed? Often we can use physical insight and approximate calculations to understand the salient features of a system without sweating the details. Before diving into technical material, it is good to see how physical reasoning and estimation techniques (such as toy models, scaling relations, Taylor series approximations, and dimensional analysis) offer a potent approach to astrophysics.

1.1 What Is Gravity?

Understanding gravity opens the door to studying many fascinating systems, so it is a natural place to begin. Plus, it provides a nice way to illustrate the analytic tools that infuse our inquiry. You can probably recite Newton's law of gravity,

$$F = \frac{GMm}{r^2} \tag{1.1}$$

but where does it come from? Put yourself in Isaac Newton's shoes and imagine you are trying to understand the motion of planets. Johannes Kepler has combed through reams of observational data and distilled three laws of planetary motion:

I. Planets move in elliptical orbits, with the Sun at one focus.
II. A line that connects a planet to the Sun sweeps out equal areas in equal times.
III. A planet's orbital period P and average distance from the Sun a are related by

$$P^2 \propto a^3$$

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These are examples of **empirical laws**; they are extracted from, and provide a powerful summary of, observational data, but they do not explain in any physical way *why* planets move as they do. Empirical laws can, however, offer clues that help us find physical explanations, if we know how to reason with them.

The first step is to recognize that Kepler's third law is an example of a **scaling relation**. It answers the question: If you move a planet farther from the Sun, will its orbital period increase or decrease, and by how much? The second step is to see if we can relate the scaling relation we know to something we want to learn. While I cannot say for certain, I imagine Newton's reasoning was something like this: Galileo famously demonstrated that objects of different mass fall at the same rate under the influence of gravity. Since a more massive object has more inertia, it must feel more gravity; the gravitational force should therefore be proportional to m. Then by Newton's third law of motion (equal and opposite reactions),[1] the force must be proportional to the product Mm. Surely gravity depends on the distance between two objects; intuitively it should decrease with distance, so let's postulate

$$F \propto \frac{Mm}{r^n}$$

where n is unknown. Let's call the constant of proportionality K and write

$$F = \frac{KMm}{r^n} \tag{1.2}$$

The third step is to connect the two scaling relations. Here we might introduce a **toy model** that is deliberately simple but (we hope) rich enough to capture the essential physics. To build a toy model for motion under the influence of gravity, we ignore Kepler's lesson about ellipses and just consider circles. From Newton's laws of motion, we know the force required to keep an object of mass m in a circular orbit of radius r and speed v is

$$F = \frac{mv^2}{r} = \frac{4\pi^2 mr}{P^2} \tag{1.3}$$

where we replace the orbital speed v with the period $P = 2\pi r/v$ in order to connect with Kepler III. We then equate the force we have available (1.2) with the force we need to explain the motion (1.3):

$$\frac{KMm}{r^n} = \frac{4\pi^2 mr}{P^2}$$

[1] Newton's laws of motion are independent of his law of gravity. We will discuss them later; for now we take them as given.

Rearranging yields

$$P^2 = \frac{4\pi^2}{KM} r^{n+1}$$

If we want to explain Kepler's third law ($P^2 \propto r^3$), we apparently need the gravitational force to follow an inverse square law ($n = 2$). This argument is only heuristic; it cannot be taken as *proof* of Eq. (1.1). But imagine you were Newton and had no one to tell you the law of gravity. An analysis like this would strongly suggest the hypothesis that gravity is described by an inverse square law.[2]

Once we know the gravitational force law, we might wonder how it affects our everyday experience on Earth. Strictly speaking, we already have everything we need to determine how gravity weakens with height (h) above the surface of Earth (indicated by the radius $R_\oplus$):

$$F = \frac{GM_\oplus m}{(R_\oplus + h)^2}$$

This formula can be a little unwieldy, though, if we just want to know what happens when we climb a mountain or fly in an airplane. Is there any way to simplify the analysis when h is much smaller than $R_\oplus$? Yes! Rewriting F slightly lets us make the following approximation:

$$F = \frac{GM_\oplus m}{R_\oplus^2}\left(1 + \frac{h}{R_\oplus}\right)^{-2} \approx \frac{GM_\oplus m}{R_\oplus^2}\left[1 - 2\frac{h}{R_\oplus} + \mathcal{O}\left(\left(\frac{h}{R_\oplus}\right)^2\right)\right] \quad (1.4)$$

If $h \ll R_\oplus$ then the second term in square brackets is much smaller than the first, and the third term is smaller still so we can neglect it without making a significant error. What we have done here is make a **Taylor series expansion** of F. This is a form of estimation that we will use from time to time when we encounter functions that are cumbersome, or we want to examine a function's behavior over some fairly narrow range. In Eq. (1.4), the Taylor series shows that at "lowest order" (i.e., in the first term) the force of gravity is independent of height above the surface of Earth. In elementary mechanics classes we often write this as $F = mg$ where[3]

$$g = \frac{GM_\oplus}{R_\oplus^2} = \frac{(6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}) \times (5.97 \times 10^{24}\,\mathrm{kg})}{(6.38 \times 10^6\,\mathrm{m})^2} = 9.80\,\mathrm{m\,s^{-2}}$$

[2] See p. 57 of *Isaac Newton* by James Gleick [1] for more discussion.

[3] Notice how I write and keep track of all units when doing the calculation. I *strongly* encourage you to get in the habit of doing this; it will help you catch errors and remember to convert units when necessary.

The minus sign in the second term of Eq. (1.4) then says gravity weakens with height. While we knew that already, the approximation offers a simple way to quantify this effect. Suppose we ask how much gravity varies when you go up in a building or an airplane, or into the upper parts of Earth's atmosphere:

Example	h	$2h/R_\oplus$	$1-2h/R_\oplus$
Building	$\sim$6 m	2×10^{-6}	0.999998
Airplane	$\sim$6 km	2×10^{-3}	0.998
Upper atmosphere	$\sim$60 km	0.02	0.98

These numbers help us understand that you have to go pretty high (relative to the atmosphere) for any change to be significant.

To recap: we have combined an empirical scaling relation with a toy model to deduce the form of the gravitational force law. We did not do any complicated math; rather, we used careful physical reasoning. We also used a Taylor series expansion to examine how gravity varies with height. I hope this book will help you cultivate these types of analysis skills, which can be quite valuable throughout astrophysics and beyond.

1.2 Dimensions and Units

Most of the quantities we discuss in physics and astrophysics come as numbers with some **units** attached (such as meters or light-years). The units are crucial; the numbers are meaningless without them. That said, units themselves are merely conventions for how we express measurements. The more fundamental quantities are **dimensions** (such as length). The distinction may seem subtle, but it is important because units are fungible while dimensions are not. Analyzing the dimensions that matter for a particular problem can be a good first step, as we are about to see.

In this book we use a combination of SI and astrophysical units. While it may seem unnecessarily complicated to mix different sets of units, there can be some advantages. Using certain units can help build your intuition about the relevant scales for different problems (e.g., it is more enlightening to specify star masses in units of the mass of the Sun than in kilograms). Also, *knowing* that you may encounter different sets of units can make you more vigilant about checking them. As a general rule:

In calculations, always check dimensions and units!

1.2.1 Fundamental Dimensions

The three key dimensions we use in physics are length, mass, and time. Here are their units in the SI system:

	Dimension	Unit
Length	$[L]$	m
Mass	$[M]$	kg
Time	$[T]$	s

Other familiar quantities involve combinations of the fundamental dimensions:

velocity	$\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$	$[LT^{-1}]$
acceleration	$\mathbf{a} = \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}$	$[LT^{-2}]$
force	$\mathbf{F} = m\mathbf{a}$	$[MLT^{-2}]$
kinetic energy	$K = \frac{1}{2}mv^2$	$[ML^2T^{-2}]$
momentum	$\mathbf{p} = m\mathbf{v}$	$[MLT^{-1}]$
angular momentum	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$	$[ML^2T^{-1}]$
pressure	$P = \frac{\text{force}}{\text{area}}$	$[ML^{-1}T^{-2}]$
number density	$n = \frac{\text{number}}{\text{volume}}$	$[L^{-3}]$
mass density	$\rho = \frac{\text{mass}}{\text{volume}}$	$[ML^{-3}]$

We sometimes invent special units to measure certain quantities. Some of the special units are clearly combinations of fundamental dimensions (and their associated units):

Force	Newton	$\mathrm{N} = \mathrm{kg\,m\,s^{-2}}$
Energy	Joule	$\mathrm{J} = \mathrm{N\,m}$
Energy	Electron volt	$\mathrm{eV} = 1.60 \times 10^{-19}\,\mathrm{J}$

Other special units might seem to be unique but turn out to be composites as well:

- **Temperature** is often measured on the Fahrenheit, Celsius, or Kelvin scale, but it is actually a measure of energy. We can always convert a temperature in Kelvins to an equivalent energy using $E = k_B\,T$ where

$$k_B = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}} = 8.62 \times 10^{-5}\,\mathrm{eV\,K^{-1}}$$

is Boltzmann's constant.[4] Astronomers sometimes invoke the equivalence between temperature and energy by reporting the "temperature" of hot gas in keV.

- **Charge** has a special unit—the Coulomb—in the SI system of units, but it can actually be expressed in terms of the three fundamental dimensions. In the Gaussian system of units, the force between charges q_1 and q_2 separated by a distance r is written with no proportionality constant[5]:

$$F = \frac{q_1 q_2}{r^2}$$

With this convention, we can identify the dimensions of charge as follows:

$$\begin{aligned} q_1 q_2 &= r^2 F \\ [Q^2] &= [L^2 \times MLT^{-2}] \\ \Rightarrow \quad [Q] &= [M^{1/2} L^{3/2} T^{-1}] \end{aligned}$$

This is one case in which I favor the Gaussian system, because thinking of charge in terms of the three fundamental dimensions turns out to be very helpful for dimensional analysis (as we will see below). In centimeter-gram-second units the value of the electron charge is $e = 4.8032 \times 10^{-10}\,\mathrm{g}^{1/2}\,\mathrm{cm}^{3/2}\,\mathrm{s}^{-1}$. Converting to meter-kilogram-second units yields $e = 1.5189 \times 10^{-14}\,\mathrm{kg}^{1/2}\,\mathrm{m}^{3/2}\,\mathrm{s}^{-1}$.

1.2.2 Constants of Nature

There are some special, fundamental numbers in physics:

Speed of light (in vacuum)	c	$= 2.9979 \times 10^{8}\,\mathrm{m\,s^{-1}}$	$[LT^{-1}]$
Newton's grav. constant	G	$= 6.6738 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$	$[M^{-1}L^3T^{-2}]$
Planck's constant	$\hbar$	$= 1.0546 \times 10^{-34}\,\mathrm{kg\,m^2\,s^{-1}}$	$[ML^2T^{-1}]$
Electron charge	e	$= 1.5189 \times 10^{-14}\,\mathrm{kg^{1/2}\,m^{3/2}\,s^{-1}}$	$[M^{1/2}L^{3/2}T^{-1}]$
Electron mass	m_e	$= 9.1094 \times 10^{-31}\,\mathrm{kg}$	$[M]$
Proton mass	m_p	$= 1.6726 \times 10^{-27}\,\mathrm{kg}$	$[M]$
Neutron mass	m_n	$= 1.6749 \times 10^{-27}\,\mathrm{kg}$	$[M]$

[4] We often drop the subscript B to simplify the notation. Any k that appears in conjunction with T is probably Boltzmann's constant.

[5] You might ask whether we could do something similar to redefine the dimensions of mass. The answer is no, because mass appears not only in $F = GMm/r^2$ but also in $F = ma$. We cannot eliminate proportionality constants from both relations at the same time.

These allow conversions between the fundamental dimensions:

- Time $\leftrightarrow$ length: $\ell = ct$ or $t = \ell/c$ (think of a "light-year")
- Mass $\leftrightarrow$ length: $\ell = GM/c^2$
- Energy $\leftrightarrow$ mass: $E = mc^2$
- Energy $\leftrightarrow$ time: $E = h\nu$ where ν is frequency (or inverse time)

Using these conversions, you could argue in principle that there is really one fundamental dimension: length. Theoretical studies of general relativity or quantum mechanics often do such conversions. We will stick with length, mass, and time, though, because they are familiar and keeping track of all three dimensions can help us check and interpret calculations.

1.2.3 Astrophysical Units

There are some numbers that are used so frequently in astrophysics that they act as a de facto set of units. Using astrophysical units can help us interpret quantities quickly; for example, it is easier to get an impression of an exoplanet's properties if we quote its mass and radius as $0.7\ M_{\rm Jupiter}$ and $1.6\ R_{\rm Jupiter}$ than if we specify them as 1.3×10^{27} kg and 1.1×10^8 m. We need to remember, though, that the quantities we take as reference values are not *fundamental*; they just happen to be quantities that are familiar in our corner of the universe. (Part of our goal as astrophysicists is to see if we can explain why these quantities have the values they do.) Here are some of the quantities we will use as astrophysical units:

Mass	Earth mass	$M_\oplus$	=	5.974×10^{24} kg
	Jupiter mass	M_J	=	1.899×10^{27} kg
	Solar mass	$M_\odot$	=	1.989×10^{30} kg
length	Earth radius	$R_\oplus$	=	6.378×10^6 m
	Jupiter radius	R_J	=	7.149×10^7 m
	Solar radius	$R_\odot$	=	6.955×10^8 m
	Astronomical unit	AU	=	1.496×10^{11} m
	Light-year	ly	=	9.461×10^{15} m
	Parsec	pc	=	3.086×10^{16} m

Our earlier discussion of Kepler's third law illustrates the value of picking good units. The proportionality means there is some constant K such that $P^2 = Ka^3$. When we study planets orbiting the Sun, we can eliminate K by taking a ratio with respect to Earth:

$$P^2 = Ka^3 \quad \text{and} \quad P_\oplus^2 = Ka_\oplus^3 \quad \Rightarrow \quad \left(\frac{P}{P_\oplus}\right)^2 = \left(\frac{a}{a_\oplus}\right)^3$$

Since $P_\oplus = 1\,\text{yr}$ and $a_\oplus = 1\,\text{AU}$ (by definition), we can write

$$\left(\frac{P}{1\,\text{yr}}\right)^2 = \left(\frac{a}{1\,\text{AU}}\right)^3 \tag{1.5}$$

If we measure a planet's distance from the Sun in AU and orbital period in years, we can write $P^2 = a^3$ without any additional constants.[6] Using appropriate units for a problem can simplify things quite a bit.

1.2.4 Dimensional Analysis

Thinking about dimensions can be a good way to begin analyzing a particular system. Before doing any detailed calculations, we might be able to make an "educated guess" about the properties of a system just by finding combinations of constants and scales that have the right dimensions. This approach cannot pin down numerical factors of order unity (e.g., 2, π, etc.), but those are rarely essential for conceptual understanding. Nor can it tell us what to do if we find several combinations of constants and scales that have the right dimensions. If that happens, we can use physical reasoning to choose among the possibilities. Let's see how this works in a few examples.

Planetary Motion

Consider a planet orbiting at distance r from a star of mass M, and suppose we want to determine the period of the orbit. To make a dimensional analysis estimate, we start by listing the scales or constants that are involved in the problem. We are given r and M, and we know gravity plays a role, so we write this list:

Distance	r	$[L]$
Mass	M	$[M]$
Gravity	G	$[M^{-1}L^3T^{-2}]$

If we want to form a combination that has dimensions of time, we clearly need to start with $G^{-1/2}$. Then we include $M^{-1/2}$ to eliminate mass, and $r^{3/2}$ to eliminate length. Thus, we guess that the expression for orbital period should look like

[6] This works only for objects orbiting the Sun, because K depends on the mass of the central object. Equation (1.5) implies that we can write $K = 1\,\text{yr}^2\,\text{AU}^{-3}$ for motion around the Sun.

$$P \sim \frac{r^{3/2}}{(GM)^{1/2}}$$

Does our guess make sense? Consider the scalings: as M increases, the gravitational force gets stronger, so things move faster and P decreases. Also, as r increases, P increases with the specific relation

$$P^2 \propto r^3$$

This is the scaling in Kepler's third law! In other words, we can recover Kepler III from dimensional analysis alone. The exact calculation for a circular orbit (which we did in Sect. 1.1) gives

$$P = 2\pi \frac{r^{3/2}}{(GM)^{1/2}}$$

Our dimensional analysis estimate was right up to a factor of $2\pi \approx 6$, which is not bad for such a simple analysis. Even more important is the fact that we got the scalings correct.

Black Hole

In Einstein's general theory of relativity, a point mass has an "event horizon" out of which no physical object can escape (see Sect. 10.6). What is the radius of the event horizon of a black hole with mass M? Again, we begin by listing the scales and constants we think are relevant:

Mass	M	$[M]$
Gravity	G	$[M^{-1}L^3T^{-2}]$
Relativity	c	$[LT^{-1}]$

The combination that has dimensions of length is

$$R \sim \frac{GM}{c^2}$$

The exact answer is the Schwarzschild radius of a black hole,

$$R_S = \frac{2GM}{c^2}$$

Here dimensional analysis comes within a factor of 2.

Suppose we had incorrectly invoked quantum mechanics rather than gravity. Then we would have used $\hbar = [ML^2T^{-1}]$ and constructed

$$R \sim \frac{\hbar}{Mc}$$

Think about this for a moment: it would imply that more massive objects have smaller event horizons. That would not make sense! There may be different combinations of scales and constants with the dimensions we are looking for, but considering the physical scalings can help us identify the best choice.

Atom

How big is an atom? The size is determined by electrons orbiting under the influence of the electric force from the nucleus. The force must involve the electric charge, while an electron's response to the force is affected by its mass. And the whole problem is quantum mechanical in nature. Thus, we have:

Quantum mechanics	$\hbar$	$[ML^2T^{-1}]$
Electric force	e	$[M^{1/2}L^{3/2}T^{-1}]$
Electron mass	m_e	$[M]$

The combination with dimensions of length is

$$\frac{\hbar^2}{m_e\, e^2}$$

The scalings with e and m_e make sense: increasing the charge would strengthen the electric force and pull the electrons closer, while increasing that mass would mean the electrons do not move as much (less acceleration for a given force). In fact, the combination we have found is the **Bohr radius** a_0, which is the radius of the lowest electron energy level in the Bohr model of the hydrogen atom (see Sect. 13.4.1). We take it as characteristic of the sizes of atoms.

1.3 Using the Tools

In Part II of this book we will encounter gases in various astrophysical contexts. Even before we study the details, we can use dimensional analysis to understand the key properties of the gases, and then deduce a few features—some straightforward, some unexpected—of different types of stars.

1.3.1 Phases of an Electron Gas

Our first goal is to uncover the **equation of state** relating the pressure of a gas to its other physical properties. In many settings we will study, the gas is ionized and most of the pressure comes from free electrons; hence we consider an electron gas. There are different scenarios depending on whether the behavior of the gas depends on quantum physics, relativity, both, or neither.

Ideal Gas

If quantum physics is not important, we can think of the gas as being made of point particles that hardly interact with one another; this is a classic "ideal" gas. Pressure is caused by particles bouncing off the walls of any container holding the gas. Dimensionally, pressure is force per unit area so

$$[P] = [ML^{-1}T^{-2}]$$

What quantities might influence the pressure of a classical ideal electron gas? The speed with which particles hit the wall depends on the temperature, and the rate at which that happens depends on how many particles there are. Are temperature (or equivalent energy) and number density enough?

Temperature	kT	$[ML^2T^{-2}]$
Number density	n	$[L^{-3}]$

In fact, simply multiplying these quantities gives dimensions of pressure, so we put

$$P \sim n\,k\,T$$

A detailed analysis reveals that there are no dimensionless factors, and we have actually recovered the famous **ideal gas law** (see Sect. 12.1.3).

We might wonder whether relativity is important for an ideal gas. That is the subject of Problem 1.5.

Classical Degenerate Gas

What happens when the density increases significantly? As the electrons squeeze closer together, the main contribution to pressure comes from the fact that different particles cannot occupy the same quantum state; in effect, the Pauli exclusion principle kicks in to create what is known as **electron degeneracy pressure**. This pressure would exist even if the temperature were zero, so the equation of state must not involve T. What does it depend on?

Number density	n	$[L^{-3}]$
Particle mass	m_e	$[M]$
Quantum mechanics	$\hbar$	$[ML^2T^{-1}]$

To this point we have built dimensional analysis estimates basically by trial and error. We can be more systematic, though. Let's postulate that the equation of state has the form

$$P \sim \hbar^\alpha \, m_e^\beta \, n^\gamma$$

where the exponents α, β, and γ are to be determined. Plugging in the dimensions, we obtain

$$\begin{aligned}[M\, L^{-1}\, T^{-2}] &\sim [M^\alpha \, L^{2\alpha} \, T^{-\alpha} \times M^\beta \times L^{-3\gamma}] \\ &\sim [M^{\alpha+\beta} \, L^{2\alpha-3\gamma} \, T^{-\alpha}]\end{aligned}$$

To match the dimensions on the left- and right-hand sides, we need

$$\begin{aligned}1 &= \alpha + \beta \\ -1 &= 2\alpha - 3\gamma \\ -2 &= -\alpha\end{aligned}$$

This is a system of three equations in three unknowns, whose solution is $\alpha = 2$, $\beta = -1$, and $\gamma = 5/3$. Thus, our equation of state for a degenerate electron gas is

$$P \sim \frac{\hbar^2}{m_e} \, n^{5/3}$$

A complete analysis gives a dimensionless factor of $(3\pi^2)^{2/3}/5 = 1.91$ (see Sect. 17.1).

To find the transition between an ideal gas and a degenerate gas, we want to find the point at which the two systems have comparable pressures. This is equivalent to requiring

$$P_{\text{ideal}} \sim P_{\text{deg}} \quad \Rightarrow \quad kT \sim \frac{\hbar^2}{m_e} \, n^{2/3}$$

To find the transition between a classical and relativistic system, we can estimate a typical speed

$$v \sim \frac{\hbar}{m_e} \, n^{1/3}$$

and find that v becomes comparable to c when the density reaches

$$n \sim \left(\frac{m_e c}{\hbar}\right)^3 \sim 2 \times 10^{37}\,\mathrm{m}^{-3}$$

Relativistic Degenerate Gas

Finally we come to the case of a degenerate gas in which the particles are moving near the speed of light. The energy of relativistic particles is dominated by motion rather than mass, so m_e presumably drops out of the equation of state and c enters. Thus, our list of ingredients becomes:

Number density	n	$[L^{-3}]$
Relativity	c	$[LT^{-1}]$
Quantum mechanics	$\hbar$	$[ML^2T^{-1}]$

As before, we put

$$P \sim \hbar^\alpha\, c^\beta\, n^\gamma$$
$$[ML^{-1}T^{-2}] = [M^\alpha\, L^{2\alpha+\beta-3\gamma}\, T^{-\alpha-\beta}]$$

and solve to find $\alpha = 1$, $\beta = 1$, $\gamma = 4/3$. This yields the equation of state

$$P \sim \hbar\, c\, n^{4/3}$$

Where is the transition between a relativistic ideal gas and a relativistic degenerate gas? That would correspond to

$$P_{\mathrm{ideal}} \sim P_{\mathrm{deg}} \quad \Rightarrow \quad kT \sim \hbar\, c\, n^{1/3} \quad \Rightarrow \quad n \sim \left(\frac{kT}{\hbar c}\right)^3$$

Phase Diagram

To recap, here are the equations of state we have estimated for the various scenarios we have considered:

Ideal gas	$P \sim n\,k\,T$
Classical degenerate gas	$P \sim \hbar^2\, m_e^{-1}\, n^{5/3}$
Relativistic degenerate gas	$P \sim \hbar\, c\, n^{4/3}$

All of these expressions have dimensions of pressure, but our physical reasoning has let us understand which expression corresponds to which physical context.

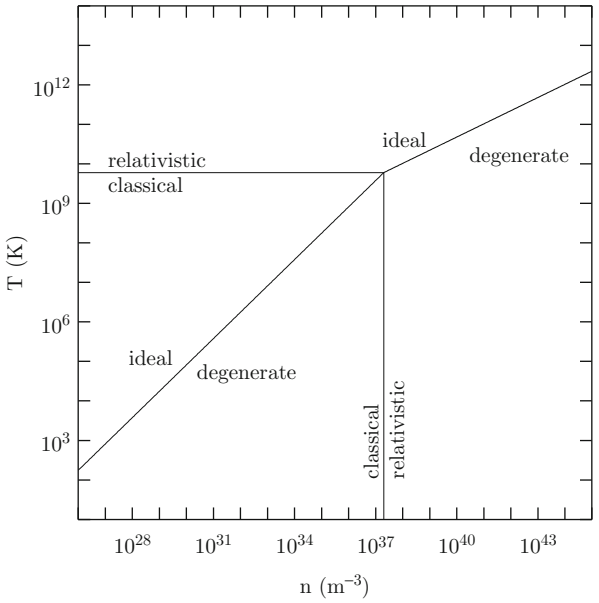

Fig. 1.1 Phase diagram for an electron gas, identifying the regimes discussed in the text: classical ideal gas, relativistic ideal gas, classical degenerate gas, and relativistic degenerate gas

We also found the transitions between different regimes, so we can sketch a phase diagram as shown in Fig. 1.1. The boundaries between the different regions are not sharp (because we have only done dimensional analysis, which is not exact). But this analysis does give a general picture of the type of gas we will encounter in different settings.

1.3.2 Stars, Familiar and Exotic

The preceding analysis may have seemed esoteric, but it proves to be very useful for understanding different kinds of stars. To make the connection, let's shift from microscopic quantities like density and pressure to the macroscopic quantities that we typically use to characterize an astrophysical object: mass M and radius R. We can relate them as follows:

$$\text{mass density} \quad \rho \sim \frac{M}{R^3}$$

$$\text{number density} \quad n \sim \frac{M}{m_p\, R^3}$$

$$\text{pressure} \quad P \sim \frac{G\, M^2/R^2}{R^2} \sim \frac{G\, M^2}{R^4}$$

(Note that m_p appears in the number density because protons dominate the mass even if electrons dominate the pressure.) Now we can answer some interesting questions about stars.

Ideal Gas

What is the temperature of a normal star composed of ideal gas?

$$\begin{aligned} & P \sim nkT \\ \Rightarrow \quad & T \sim \frac{P}{nk} \sim \frac{GM^2R^{-4}}{Mm_p^{-1}R^{-3}k} \sim \frac{GMm_p}{kR} \end{aligned}$$

For the Sun, plugging in numbers gives

$$\begin{aligned} T &\sim \frac{(6.67\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})\times(1.99\times10^{30}\,\mathrm{kg})\times(1.67\times10^{-27}\,\mathrm{kg})}{(1.38\times10^{-23}\,\mathrm{kg\,m^2\,s^{-2}\,K^{-1}})\times(6.96\times10^{8}\,\mathrm{m})} \\ &\sim 2\times10^{7}\,\mathrm{K} \end{aligned}$$

This estimate agrees surprisingly well with detailed stellar models (see Sect. 16.2.2).

Classical Degenerate Gas

What would a star composed of a degenerate electron gas be like?

$$\begin{aligned} P &\sim \hbar^2 m_e^{-1} n^{5/3} \\ GM^2R^{-4} &\sim \hbar^2 m_e^{-1}(Mm_p^{-1}R^{-3})^{5/3} \\ \Rightarrow \quad R &\sim \frac{\hbar^2}{Gm_e m_p^{5/3} M^{1/3}} \end{aligned}$$

The scaling $R \propto M^{-1/3}$ implies that more massive stars are *smaller*. While this may seem counterintuitive, it is confirmed by more detailed calculations (see Sect. 17.2). Consider a **white dwarf** with $M \sim M_\odot$:

$$\begin{aligned} R &\sim \frac{(1.05\times10^{-34}\,\mathrm{kg\,m^2\,s^{-1}})^2}{(9.11\times10^{-31}\,\mathrm{kg})\times(1.67\times10^{-27}\,\mathrm{kg})^{5/3}} \\ &\quad\times\frac{1}{(6.67\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})\times(1.99\times10^{30}\,\mathrm{kg})^{1/3}} \\ &\sim 6\times10^{6}\,\mathrm{m} \end{aligned}$$

A white dwarf is roughly the size of Earth.

Relativistic Degenerate Gas

Now let's actually consider a *neutron* gas. What would a star composed of a relativistic degenerate neutron gas be like?

$$P \sim \hbar c n^{4/3}$$
$$\frac{GM^2}{R^4} \sim \hbar c \left(\frac{M}{m_n R^3}\right)^{4/3}$$
$$\Rightarrow \quad M \sim \frac{1}{m_n^2}\left(\frac{\hbar c}{G}\right)^{3/2} \sim 4 \times 10^{30}\,\mathrm{kg} \sim 2\,M_\odot$$

All stars composed of a (highly) relativistic degenerate neutron gas have roughly the same mass. In order for them to be relativistic, we need:

$$n \gtrsim \left(\frac{m_n c}{\hbar}\right)^3$$
$$\Rightarrow \quad R \lesssim \frac{1}{m_n^2}\left(\frac{\hbar^3}{G c}\right)^{1/2} \sim 3\,\mathrm{km}$$

Such a star would be a little more massive that the Sun, but only as big as a city. In fact, we observe this kind of object as a **neutron star**. (Real neutron stars are probably not ultra-relativistic, but this analysis still gives a useful sense of the physics. See Chap. 17 for more discussion.)

Problems

1.1. Use dimensional analysis to derive a relationship between the total mass M of a gravitationally bound system, its size R, and the typical speed v of its components. Then use it to answer the following questions.

(a) At what speed does the Earth orbit the Sun?
(b) Globular clusters typically contain $\sim 10^6$ stars moving at speeds of $\sim 10\,\mathrm{km\,s^{-1}}$. How big are they?
(c) Spiral galaxies are typically about 10 kpc in size and rotate such that the stars move at $\sim 200\,\mathrm{km\,s^{-1}}$. Estimate the mass of a spiral galaxy (in $M_\odot$).

1.2. Type Ia supernovae are exploding stars that have played an important role in observational cosmology (see Chap. 18).

(a) The exploding stars are white dwarfs that have a mass of about $1.4\,M_\odot$ and a radius of about 5,000 km. Use dimensional analysis to estimate the gravitational binding energy of such a star.

(b) The explosion is powered by nuclear fusion. How much mass must be converted to energy ($E = mc^2$) in order to overcome the binding energy and explode the star?

1.3. Explosions such as Type Ia supernovae produce blast waves.

(a) Use dimensional analysis to estimate the size R of a blast wave at time t after an explosion with energy E, propagating into a medium of ambient density ρ. (Hint: these are all the quantities you need; gravity is not directly relevant here.)
(b) Information about the first atomic bomb tests was kept secret, but the physicist Geoffrey Taylor estimated the energy of one test from published photographs showing a fireball expanding through the air [2]. If the blast wave reached 100 m just 0.02 s after the explosion, what was the energy? What mass was converted into energy? (Hint: you will need to look up the density of air.)
(c) How large would the remnant of a supernova ($E \sim 10^{44}$ J) be 1,000 years after the explosion, as it expands into the interstellar medium with a typical density of 10^6 hydrogen atom per cubic meter?

1.4. The universe is believed to be about 14 billion years old. Use dimensional analysis to estimate the average density of the universe. About how many hydrogen atoms are there in 1 m^3 of "empty" space?

1.5. Can we treat the center of the Sun as a classical ideal gas? Let's find out.

(a) Consider a gas at temperature T composed of particles of mass m. Use dimensional analysis to estimate the typical speed of the particles.
(b) Recall our estimate of the Sun's central temperature, $T \sim 2 \times 10^7$ K. This is hot enough to ionize atoms, so electrons and nuclei move independently. What is the typical speed of electrons? Of hydrogen nuclei? Are they relativistic?
(c) At roughly what temperature does an electron gas become relativistic ($v \sim c$)?

1.6. Light carries momentum, so it creates pressure when it shines on something. This has led people to propose using "solar sails" on interplanetary or interstellar spacecraft.

(a) Use dimensional analysis to estimate the light pressure at a distance d from a star with luminosity L (energy per unit time).
(b) Estimate the force on a solar sail with an area of 1 km^2 that is 1 AU from the Sun.
(c) Suppose that sail is pulling a 10 ton spacecraft. How long would it take to reach Jupiter's orbit (5.2 AU from the Sun)? For simplicity, assume the acceleration remains constant even though it actually varies with distance from the star.

References

1. J. Gleick, *Isaac Newton* (Vintage Books, New York, 2004)
2. G. Taylor, Proc. R. Soc. Lond. Ser. A. Math. Phys. Sci. **201**(1065), 175 (1950)