# Chapter 15
# Nuclear Fusion

To this point we have viewed star temperatures and luminosities as empirical quantities, but now we seek to understand them in terms of physical processes that occur deep within stars. Nuclear fusion provides a potent power source, and studying fusion reveals a link between properties of astrophysical objects and reactions that occurs on scales more than 20 orders of magnitude smaller.

## 15.1   What Powers the Sun?

Before delving into the physics of fusion, it is worthwhile to consider why that is the only source of energy that could power the Sun. At stake is the total energy emitted by the Sun during its lifetime. The Sun's present luminosity is $3.84 \times 10^{26} \,\mathrm{J\,s^{-1}}$, and the age of the Solar System inferred from radiometric dating of rocks and meteorites is about $4.5\,\mathrm{Gyr} \approx 1.4 \times 10^{17}\,\mathrm{s}$. Even if the Sun was a little dimmer in the past (see Sect. 16.3.1), we can safely estimate that the Sun has already released more than $10^{43}$ J in light. What is the source of that energy?

**Gravitational energy?** The gravitational potential energy of an object of mass $M$ and radius $R$ is $U \sim -GM^2/R$ (see Sect. 8.2.1). If the object shrinks without losing mass then $U$ decreases (becomes more negative). The process of releasing energy by gravitational contraction is known as the **Kelvin-Helmholtz mechanism**. If the Sun was initially large enough that its gravitational potential energy was roughly zero, the amount of gravitational energy it could have released by now is

$$\Delta U \;\sim\; \frac{GM_\odot^2}{R_\odot} \;\sim\; 4 \times 10^{41}\,\mathrm{J}$$

There is a dimensionless factor of order unity that depends on how the density changes with radius, and an additional factor of $1/2$ because, according to the virial theorem (see Sect. 8.1.3) only half of the energy can be radiated (the other half

goes into kinetic energy). Thus, as an order-of-magnitude estimate we can say that something like $E_{tot} \sim 10^{41}$ J could have been released by gravitational collapse. At its current luminosity the Sun would radiate this amount of energy in a time

$$t \sim \frac{E_{tot}}{L_\odot} \sim \frac{10^{41} \text{ J}}{3.84 \times 10^{26} \text{ J s}^{-1}} \sim 3 \times 10^{14} \text{ s} \sim 8 \times 10^6 \text{ yr}$$

This is far too short compared with the age of the Solar System. The Sun cannot be powered by gravitational energy alone.

**Chemical energy?** Chemical reactions can release energy by rearranging electrons in atoms and molecules. The energy scale is set by the difference between electron energy levels. If we optimistically assume that each atom can release $E_1 = 10$ eV, the energy available from all atoms is $E_{tot} = NE_1$. If we assume the Sun is pure hydrogen, the total number of atoms is $N \sim M_\odot/m_p \sim 1.2 \times 10^{57}$. Then

$$E_{tot} \sim 1.2 \times 10^{58} \text{ eV} \times \frac{1.60 \times 10^{-19} \text{ J}}{1 \text{ eV}} \sim 1.9 \times 10^{39} \text{ J}$$

With this total energy, the Sun could shine at its current rate for

$$t \sim \frac{E_{tot}}{L_\odot} \sim \frac{1.9 \times 10^{39} \text{ J}}{3.84 \times 10^{26} \text{ J s}^{-1}} \sim 5.0 \times 10^{12} \text{ s} \sim 160{,}000 \text{ yr}$$

The Sun cannot be powered by chemical energy, either.

**Nuclear energy?** Nuclear reactions involve energies in the range of MeV, and thus provide something like a million times more energy than chemical reactions. Specifically, nuclear fusion involves the conversion of mass into energy via $E = mc^2$. If the entire mass of the Sun were converted to energy in this way, the total energy released would be

$$E_{tot} \sim M_\odot c^2 \sim (1.99 \times 10^{30} \text{ kg}) \times (3.0 \times 10^8 \text{ m s}^{-1})^2 \sim 2 \times 10^{47} \text{ J}$$

which would correspond to a lifetime of

$$t \sim \frac{2 \times 10^{47} \text{ J}}{3.84 \times 10^{26} \text{ J s}^{-1}} \sim 5 \times 10^{20} \text{ s} \sim 10^{13} \text{ yr}$$

In practice, the total energy available from fusion is less than this, because only a small fraction of mass is converted into energy in each reaction, and only a portion of matter in the Sun can undergo fusion. (We quantify these fractions below.) Even so, nuclear fusion provides ample energy to power the Sun.

## 15.2 Physics of Fusion

Fusion occurs when two lighter nuclei combine to create a heavier nucleus. The process can release energy because the mass of the final nucleus may be less than the combined mass of the starting nuclei; the "missing" mass gets converted to energy (such that the total mass/energy is conserved). In this section we study the physics of fusion to understand the conditions under which fusion can occur and the rate at which energy is released.[1]

### 15.2.1 Mass and Energy Scales

To discuss fusion we need to be more precise than we have been about the masses of particles and nuclei. Given the focus on energy, it is common to quote masses in terms of equivalent energies using $m = E/c^2$; the corresponding unit is $\mathrm{MeV}/c^2$. The masses of the three familiar fundamental particles are:

| | |
|---|---|
| Proton | $m_p = 938.272\,\mathrm{MeV}/c^2$ |
| Neutron | $m_n = 939.565\,\mathrm{MeV}/c^2$ |
| Electron | $m_e = 0.511\,\mathrm{MeV}/c^2$ |

We will see in Sect. 15.3 that the reaction powering the Sun is the fusion of 4 hydrogen nuclei into a helium nucleus. The mass involved are[2]

$$4\,\text{hydrogen} \quad 4m_\mathrm{H} = 3{,}753.09\,\mathrm{MeV}/c^2$$

$$\text{helium} \quad m_\mathrm{He} = 3{,}727.38\,\mathrm{MeV}/c^2$$

The amount of mass that gets converted to energy is[3]

$$\Delta m = 25.71\,\mathrm{MeV}/c^2$$

The fraction of the original mass that goes into energy is sometimes called the **efficiency** of a nuclear reaction,

$$\epsilon = \frac{m_\text{start} - m_\text{end}}{m_\text{start}} = \frac{\Delta m}{m_\text{start}}$$

---

[1]Parts of this presentation draw on the books by Carroll and Ostlie [1] and Maoz [2].

[2]These are masses of nuclei; they do not include electrons. In the core of the Sun where fusion occurs, atoms are ionized (see Sect. 16.2.2).

[3]There may be a little additional energy released when electrons annihilate with anti-electrons (see Sect. 15.3.3), but we are focusing on nuclear masses.

With this definition we can write the energy released when some mass $M$ undergoes fusion as

$$E = \epsilon M c^2$$

For hydrogen fusing into helium, the efficiency is $\epsilon = 0.007$.

How much fusion energy is available in the Sun? A first estimate is:

$$E \sim 0.007 \times M_\odot c^2 \sim 1.3 \times 10^{45} \, \text{J}$$

This could power the Sun for a lifetime of

$$t \sim \frac{1.3 \times 10^{45} \, \text{J}}{3.84 \times 10^{26} \, \text{J s}^{-1}} \sim 3.3 \times 10^{18} \, \text{s} \sim 10^{11} \, \text{yr}$$

In practice, only about 10 % of the Sun's mass can undergo fusion, so the actual energy and lifetime are about a factor of 10 smaller.

### 15.2.2   Requirements for Fusion

Why does only a portion of the Sun's mass ever participate in fusion? Fusion requires high temperatures and densities that occur only in the core of the Sun. To understand why, note that nuclei have positive charges and thus repel each other through the Coulomb force. Fusion can occur only if the nuclei get close enough for the **strong nuclear force** to take over and create an attractive force that binds nuclear particles together (see Fig. 15.1). The strong force operates over scales of femtometers (1 fm $= 10^{-15}$ m), so we need to consider the conditions under which nuclei are able to overcome Coulomb repulsion and get close enough for the strong force to come into play.

**Classical Analysis**

The center of the Sun is hot, so nuclei are zipping around and bumping into one another. Is this enough to overcome the **Coulomb barrier** between nuclei? Thinking in terms of classical physics, we would say that fusion can occur only if the kinetic energy is above the height of the Coulomb barrier. For a Maxwell-Boltzmann distribution of velocities (see Sect. 12.1.2), the typical kinetic energy is

$$\text{kinetic energy} \sim \frac{3}{2} k T$$

If $r_s$ is the scale on which the strong nuclear force acts, the height of the Coulomb barrier is
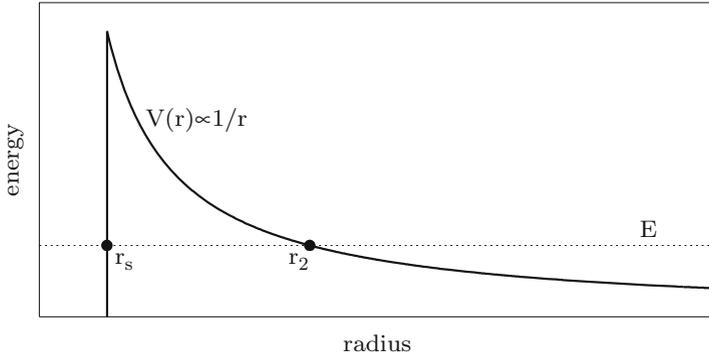
**Fig. 15.1** A simple Coulomb barrier between two nuclei. At large separations, Coulomb repulsion leads to a $V(r) \propto 1/r$ potential. At small separations, the strong nuclear force creates a deep potential well. If a particle approaches from the right with an energy $E$ that is lower than the peak, a classical analysis would say the particle can never reach the inner potential well. According to quantum mechanics, however, the particle can tunnel through the barrier. The tunneling probability is analyzed in Sect. 15.2.3

$$\text{Coulomb barrier} \sim \frac{Z_1 Z_2 e^2}{r_s}$$

where $Z_1$ and $Z_2$ are the atomic numbers of the two nuclei. In order for the kinetic energy to exceed the Coulomb barrier, the temperature must be higher than

$$T_{\text{fusion}} \sim \frac{2Z_1 Z_2 e^2}{3kr_s} \qquad \text{(classical)}$$

For hydrogen fusion, taking $r_s \sim 1\,\text{fm}$ gives

$$T_{\text{fusion}} \sim \frac{2 \times (1.52 \times 10^{-14}\,\text{kg}^{1/2}\,\text{m}^{3/2}\,\text{s}^{-1})^2}{3 \times (1.38 \times 10^{-23}\,\text{kg}\,\text{m}^2\,\text{s}^{-2}\,\text{K}^{-1}) \times (10^{-15}\,\text{m})} \sim 10^{10}\,\text{K}$$

The center of the Sun is hot ($1.6 \times 10^7$ K), but not this hot. According to classical physics, fusion should not occur in the Sun.

**Quantum Analysis**

The picture changes when we consider quantum physics, thanks to **tunneling**. In the language of quantum mechanics, we discuss not particles themselves but rather their wavefunctions, which characterize the probability of finding particles in particular positions (or other quantum states). Approaching a barrier in the potential, a wavefunction need not vanish; it can penetrate the barrier and, if the barrier is not

too thick, come out the other side with an amplitude that is smaller but still finite. This corresponds to a finite probability for the particle to "jump through" the barrier.

Tunneling means that particles do not necessarily have to have enough energy to go over the Coulomb barrier; they just need enough energy to tunnel through it. To make a toy model of this effect, recall wave/particle duality. We have already seen that wavelength and momentum are related for light through the relation $\lambda = h/p$ (see Sect. 13.1.4). Louis de Broglie suggested that the same idea applies to massive particles like electrons and nuclei. We imagine, then, that bringing two nuclei within a de Broglie wavelength of one another will allow them to tunnel through the Coulomb barrier and fuse (we will be more careful about the criterion below). Using $p = h/\lambda$, we can write the kinetic energy as

$$\text{kinetic energy} \ = \ \frac{1}{2}\mu v^2 \ = \ \frac{p^2}{2\mu} \ = \ \frac{h^2}{2\mu\lambda^2}$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the system.[4] The Coulomb barrier has the same form as before, but now we take the separation to be $\lambda$:

$$\text{Coulomb barrier} \sim \frac{Z_1 Z_2 e^2}{\lambda}$$

The kinetic energy matches the Coulomb barrier for

$$\lambda \sim \frac{h^2}{2 Z_1 Z_2 e^2 \mu}$$

Again equating the kinetic energy to $(3/2)kT$ gives

$$T_{\text{fusion}} \ \sim \ \frac{h^2}{3\mu k \lambda^2} \ \sim \ \frac{4 Z_1^2 Z_2^2 e^4 \mu}{3 k h^2} \qquad \text{(quantum)}$$

Plugging in numbers for hydrogen (with reduced mass $\mu = m_p/2$) yields

$$T_{\text{fusion}} \sim \frac{4 \times (1.52 \times 10^{-14}\,\text{kg}^{1/2}\,\text{m}^{3/2}\,\text{s}^{-1})^4 \times (8.36 \times 10^{-28}\,\text{kg})}{3 \times (1.38 \times 10^{-23}\,\text{kg m}^2\,\text{s}^{-2}\,\text{K}^{-1}) \times (6.626 \times 10^{-34}\,\text{kg m}^2\,\text{s}^{-1})^2} \sim 10^7\,\text{K}$$

This is quite close to the central temperature of the Sun. While our toy model should not be taken too literally, it does suggest that tunneling enables fusion in the Sun and therefore deserves a more careful treatment.

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[4]We can work in the center of mass frame and convert the two-body Coulomb problem into an equivalent one-body problem, as we did with gravity in Sect. 4.1. The relevant mass is then the reduced mass.

### 15.2.3 Cross Section

The preceding analysis gives a general sense of the conditions required for fusion to be possible. Now let's get more specific. In Chap. 12 we saw that it is useful to discuss interactions between particles in terms of the **cross section**. Previously we pictured billiard ball collisions and took the cross section to be the physical size of the objects, but now we introduce a more general definition:

$$\sigma(E) = \frac{\text{number of reactions / nucleus / time}}{\text{number of incident particles / area / time}} = [L^2] \qquad (15.1)$$

We write $\sigma(E)$ for nuclear reactions because the ability to cross the Coulomb barrier depends on energy. While the full energy dependence may be quite complicated, there are some key factors that we can identify.

First, the "size" of a nucleus is either its physical size or its de Broglie wavelength, whichever is larger. To estimate the de Broglie wavelength, we take the momentum to be $p \sim \mu v_{\text{rms}}$ where $\mu$ is the reduced mass and $v_{\text{rms}} = \sqrt{3kT/\mu}$ is the typical particle speed (see Eq. 12.6). At the center of the Sun, $T = 1.6 \times 10^7$ K (see Sect. 16.2.2) and hence $kT = 2.2 \times 10^{-16}$ J. Using $\mu = m_p/2$ for hydrogen then yields

$$\lambda \sim \frac{h}{(3\mu kT)^{1/2}}$$

$$\sim \frac{(6.626 \times 10^{-34} \text{ kg m}^2 \text{ s}^{-1})}{[3 \times (8.36 \times 10^{-28} \text{ kg}) \times (2.2 \times 10^{-16} \text{ kg m}^2 \text{ s}^{-2})]^{1/2}}$$

$$\sim 10^{-12} \text{ m}$$

This is much larger than the physical size of a nucleus ($\sim 10^{-15}$ m), so we can approximate the effective size of a nucleus as $\lambda$. Therefore we expect the cross section to scale as

$$\sigma(E) \propto \lambda^2 \propto \left(\frac{h}{\mu v}\right)^2 \propto \frac{1}{E} \qquad (15.2)$$

Second, we need to account for the tunneling probability. Below we derive the probability for tunneling through a simple, fixed Coulomb barrier and show that it has the form

$$P = e^{-(E_c/E)^{1/2}} \qquad (15.3)$$

where the energy scale $E_c$ is defined by

$$E_c \equiv \frac{2\pi^2 Z_1^2 Z_2^2 e^4 \mu}{\hbar^2} \qquad (15.4)$$

Even if the real interaction is more complicated than we have considered here, the two factors we have identified in Eqs. (15.2) and (15.3) should capture the strongest energy dependence. We therefore write

$$\sigma(E) = \frac{S(E)}{E} \, e^{-(E_c/E)^{1/2}} \tag{15.5}$$

and bundle any remaining effects into $S(E)$, which is called the **nuclear S-factor**. In principle, $S(E)$ needs to be determined for each reaction (mainly from experimental data; e.g., [3,4]). However, in the next section we will see that knowing the details of $S(E)$ is not essential for a general understanding of fusion (although it is vital for detailed investigations, of course).

**Tunneling Probability**

Quantum tunneling through a barrier that is wide compared with the de Broglie wavelength of a particle can be analyzed in the standard WKB (Wentzel-Kramers-Brillouin) approximation of quantum mechanics. (For more details, see Chap. 8 of *Introduction to Quantum Mechanics* by Griffiths [5] or a similar textbook.) Consider a barrier with potential $V(r)$. Conservation of energy gives the momentum:

$$E = \frac{p^2}{2\mu} + V(r) \quad \Rightarrow \quad p(r) = \{2\mu[E - V(r)]\}^{1/2}$$

where $\mu$ is the reduced mass of the system. If the energy $E$ is below the peak of $V$, then $p(r)$ is imaginary within the barrier, but that is not a problem in a quantum analysis. The WKB approximation gives the probability that the particle can tunnel through the barrier as $P$ such that

$$\ln P \approx -\frac{2}{\hbar} \int_{r_s}^{r_2} |p(r)| \, dr$$

where $r_s$ and $r_2$ are indicated in Fig. 15.1. With a simple Coulomb barrier

$$V(r) = \frac{Z_1 Z_2 e^2}{r} \qquad (r > r_s)$$

we can write the outer radius as

$$r_2 = \frac{Z_1 Z_2 e^2}{E} \tag{15.6}$$

The integral can then be evaluated as follows:

$$\ln P \approx -\frac{2}{\hbar}(2\mu)^{1/2} \int_{r_s}^{r_2} \left( \frac{Z_1 Z_2 e^2}{r} - E \right)^{1/2} dr$$

$$\approx -\frac{2}{\hbar} \left(2Z_1 Z_2 e^2 \mu\right)^{1/2} \int_{r_s}^{r_2} \left(\frac{1}{r} - \frac{1}{r_2}\right)^{1/2} \mathrm{d}r$$

where we use Eq. (15.6) to replace $E$ with $r_2$. In the limit $r_2 \gg r_s$, the integral evaluates to $(\pi/2)r_2^{1/2}$, yielding

$$\ln P \approx -\frac{\pi}{\hbar} \left(2Z_1 Z_2 e^2 \mu r_2\right)^{1/2} \approx -\frac{\pi Z_1 Z_2 e^2}{\hbar} \left(\frac{2\mu}{E}\right)^{1/2}$$

where we now use Eq. (15.6) to convert back to $E$. We can rewrite the result as

$$\ln P \approx -\left(\frac{E_c}{E}\right)^{1/2} \quad \text{where} \quad E_c = \frac{2\pi^2 Z_1^2 Z_2^2 e^4 \mu}{\hbar^2} \tag{15.7}$$

### 15.2.4   Reaction Rate

The cross section is one important factor in determining how many reactions occur. Another is the sheer number of nuclei that have the required energy. To figure out how to compute the reaction rate, let's go back to the definition of cross section (Eq. 15.1). The particles that can interact with a target nucleus in some time $\mathrm{d}t$ are those in a cylinder whose cross sectional area is $\sigma$ and length is $v\,\mathrm{d}t$. The number of reactions per nucleus in time $\mathrm{d}t$ can therefore be written as

$$\text{\# reactions per nucleus} = \int n_2(v)\,\sigma\,v\,\mathrm{d}t\,\mathrm{d}v$$

where $n_2(v)\,\mathrm{d}v$ is the number density of particles with speeds between $v$ and $v + \mathrm{d}v$.

Consider a volume $V$ that contains $N_1$ target nuclei. To get the total number of reactions for all the nuclei, multiply the previous expression by $N_1$. To get the number of reactions per unit volume, divide by $V$. Finally, to get the number of reactions per unit volume per unit time, divide by $\mathrm{d}t$. The result is the **reaction rate**,

$$r_{12} \equiv \text{number of reactions per unit volume, per unit time} = n_1 \int n_2(v)\,\sigma\,v\,\mathrm{d}v$$

where $n_1 = N_1/V$ is the number density of nuclei. Using the Maxwell-Boltzmann distribution for $n_2(v)$ gives[5]

$$r_{12} = n_1\,n_2 \left(\frac{\mu}{2\pi kT}\right)^{3/2} 4\pi \int_0^\infty v^3\,e^{-\mu v^2/2kT}\,\sigma\,\mathrm{d}v$$

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[5]Now $n_2$ indicates the total number density of reactants. We could write it as $n_{2,\text{tot}}$ following Eq. (12.3), but we omit "tot" to simplify the notation.
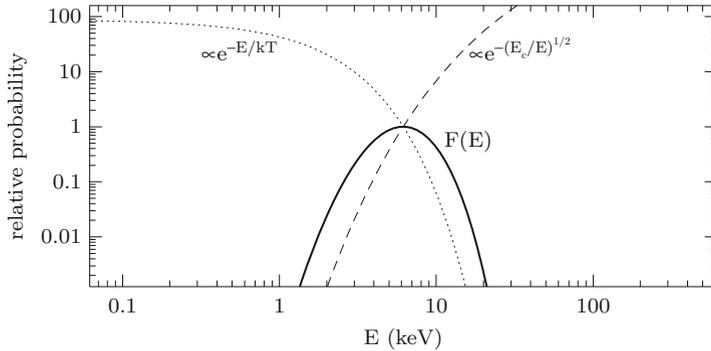
**Fig. 15.2** The *dotted curve* shows the Boltzmann factor as a function of energy, while the *dashed curve* shows the tunneling probability factor for hydrogen in the Sun. The *solid curve* shows their product, which reveals the Gamow peak. For clarity, all curves are scaled to pass through 1 at $E_0$

Changing integration variables to $E = \mu v^2/2$ transforms the integral to

$$
r_{12} = n_1 \, n_2 \left( \frac{\mu}{2\pi kT} \right)^{3/2} 4\pi \int_0^\infty \left( \frac{2E}{\mu} \right)^{3/2} e^{-E/kT} \, \sigma \, \frac{dE}{(2\mu E)^{1/2}}
$$

$$
= \left( \frac{2}{kT} \right)^{3/2} \frac{n_1 \, n_2}{(\pi\mu)^{1/2}} \int_0^\infty E \, \sigma \, e^{-E/kT} \, dE
$$

$$
= \left( \frac{2}{kT} \right)^{3/2} \frac{n_1 \, n_2}{(\pi\mu)^{1/2}} \int_0^\infty S(E) \, e^{-(E_c/E)^{1/2}} \, e^{-E/kT} \, dE \qquad (15.8)
$$

In the last step we use Eq. (15.5) to substitute for $\sigma(E)$.

Notice the two exponential factors in the integrand. The Boltzmann factor $e^{-E/kT}$ decreases as energy increases, but the tunneling probability factor $e^{-(E_c/E)^{1/2}}$ increases with energy. The product of the two is a strongly peaked curve, as shown in Fig. 15.2. In Problem 15.3 you can show that the **Gamow peak** occurs at energy

$$
E_0 = \left[ \frac{1}{4} \, E_c \, (kT)^2 \right]^{1/3} \qquad (15.9)
$$

Most of the contribution to the integral in Eq. (15.8) comes from energies near $E_0$. If the S-factor does not vary too rapidly (which is seen to be true for many reactions),[6] we can approximate $S(E)$ by its value at $E_0$ and pull $S(E_0)$ out of the integral. Then the integral can be evaluated analytically, using the method outlined in Problem 15.4.

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[6]The exception is reactions in which "resonances" enhance the cross section at certain energies. Addressing nuclear resonances involves more detail than we want to get into here (see [3]).

Since fusion typically involves nuclei in a fairly narrow range of energies, the reaction rate can increase dramatically as the temperature increases and more particles are brought into the fusion energy range. It is convenient to characterize such a rapid temperature dependence as a power law relation,

$$r_{12} \propto T^{\alpha}$$

How do we determine the **power law index**, $\alpha$? First, let's write the relation as

$$r_{12} = C\, T^{\alpha}$$

where $C$ is a multiplicative factor that does not depend on temperature. Consider the derivative:

$$\frac{\mathrm{d}r_{12}}{\mathrm{d}T} = \alpha\, C\, T^{\alpha-1} = \frac{\alpha\, r_{12}}{T}$$

Therefore we can compute the power law index as

$$\alpha = \frac{T}{r_{12}} \frac{\mathrm{d}r_{12}}{\mathrm{d}T} \tag{15.10}$$

Alternatively, take the logarithm of the original relation:

$$\ln r_{12} = \ln\left(C\, T^{\alpha}\right) = \ln C + \alpha \ln T$$

Then we can find $\alpha$ using

$$\alpha = \frac{\mathrm{d}(\ln r_{12})}{\mathrm{d}(\ln T)}$$

which is mathematically equivalent to Eq. (15.10). Using either method, you can show in Problem 15.4 that the power law index for the fusion reaction rate has the form

$$\alpha = \left(\frac{E_c}{4kT}\right)^{1/3} - \frac{2}{3} \tag{15.11}$$

In Problem 15.5 you can quantify the temperature dependence of the reaction rates for two channels by which hydrogen fuses into helium.


**Application to Hydrogen in the Sun**

Using the reduced mass $\mu = m_p/2$, the energy factor $E_c$ from Eq. (15.4) is

$$E_c = \frac{2\pi^2 \times (1.52 \times 10^{-14}\,\mathrm{kg^{1/2}\,m^{3/2}\,s^{-1}})^4 \times (8.36 \times 10^{-28}\,\mathrm{kg})}{(1.05 \times 10^{-34}\,\mathrm{kg\,m^2\,s^{-1}})^2}$$

$$= 7.9 \times 10^{-14}\,\mathrm{J}$$

$$= 490\,\mathrm{keV}$$

where we convert to keV because (as we will see) this is a convenient unit given the energy scales in the Sun. For $T = 1.6 \times 10^7$ K, the thermal energy factors are

$$kT = (1.38 \times 10^{-23}\,\mathrm{kg\,m^2\,s^{-2}\,K^{-1}}) \times (1.6 \times 10^7\,\mathrm{K}) = 2.2 \times 10^{-16}\,\mathrm{J} = 1.4\,\mathrm{keV}$$

$$E_{\mathrm{rms}} = \frac{3}{2}\,kT = 2.1\,\mathrm{keV}$$

where $E_{\mathrm{rms}}$ is the typical kinetic energy in the Maxwell-Boltzmann distribution. The Gamow peak is then

$$E_0 = \left[\frac{1}{4} \times (490\,\mathrm{keV}) \times (1.4\,\mathrm{keV})^2\right]^{1/3} = 6.2\,\mathrm{keV}$$

(This corresponds to a temperature of about $7 \times 10^7$ K.) In other words, the typical nuclei involved in fusion have energies about three times the RMS value. While such nuclei are not right at the peak of the Maxwell-Boltzmann distribution, they are still sufficiently abundant to make fusion work as an energy source.

## 15.3   Nuclear Reactions in Stars

Having discussed the general theory of fusion, we can now consider the specific reactions that occur in stars.

### 15.3.1   Cast of Characters

The first step is to list all the particles that might participate in nuclear reactions. The elements in the periodic table are composed of protons, neutrons, and electrons. All three particles have alternate versions called **antimatter** obtained by reversing the electric charge:

| Matter | | Antimatter |
|---|---|---|
| Electron, $e^-$ | $\leftrightarrow$ | Positron, $e^+$ |
| Proton, $p$ | $\leftrightarrow$ | Antiproton, $\bar{p}$ |
| Neutron, $n$ | $\leftrightarrow$ | Antineutron, $\bar{n}$ |

When matter and antimatter come together, they annihilate into energy in the form of photons of light (denoted by $\gamma$):

$$e^- + e^+ \;\rightarrow\; 2\gamma$$
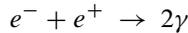
Two photons are needed so the reaction can conserve momentum and energy simultaneously.

In the 1930s physicists noticed that certain nuclear reactions seemed to violate energy and momentum conservation. Wolfgang Pauli proposed that unseen particles carried the "missing" energy and momentum. They had to be neutral (there was no missing charge), so Enrico Fermi christened them **neutrinos** for "little neutral ones." As we will see in Sect. 15.4, we now understand that there are different types of neutrinos, but for the moment we focus on ones associated with electrons:

| Matter | | Antimatter |
|---|---|---|
| Neutrino, $\nu_e$ | $\leftrightarrow$ | Antineutrino, $\bar{\nu}_e$ |

Whenever a nuclear reaction involves an electron or positron, a neutrino or antineutrino is produced as a result of the weak nuclear force.

### 15.3.2  Masses and Binding Energies

When we discuss atomic nuclei, we label them as

$$^A_Z\mathrm{X}$$

where X is the abbreviation for the element in the periodic table, $Z$ is the **atomic number** (the number of protons), and $A$ is the **mass number** (the total number of protons and neutrons). For example, $^4_2\mathrm{He}$ indicates helium with two protons and two neutrons.

To quantify the energetics of fusion, we define the **binding energy** to be the difference between the actual mass of a nucleus ($m_{\mathrm{nuc}}$) and the combined mass that all the constituent protons and neutrons would have if they were isolated:

$$E_b = \left[ Z\,m_p + (A - Z)\,m_n - m_{\mathrm{nuc}} \right] c^2 \tag{15.12}$$

This is the amount of energy that would be released if the nucleus were built from scratch in one step (although nuclei are not actually made that way, as we will see). To compare different nuclei, it can be valuable to consider the binding energy per nucleon, $E_b/A$. Here are the nuclear masses and binding energies of some low-mass isotopes, along with electrons and neutrons for comparison [6][7]:

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[7]These are the masses of bare nuclei, not neutral atoms. Recall that we are considering reactions between nuclei in ionized gas.
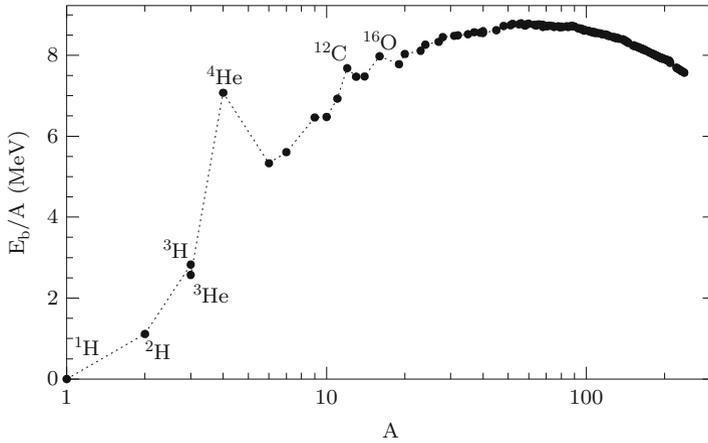
**Fig. 15.3** Binding energy per nucleon, as a function of the mass number, $A$. Some important isotopes are identified (Data from [6])

|          | $m$ ($MeV/c^2$) | $E_b$ (MeV) | $E_b/A$ (MeV) |
|----------|-----------------|-------------|---------------|
| $e^-$    | 0.511           |             |               |
| n        | 939.565         |             |               |
| p/$^1_1$H | 938.272        |             |               |
| $^2_1$H  | 1,875.613       | 2.224       | 1.112         |
| $^3_2$He | 2,808.391       | 7.718       | 2.573         |
| $^4_2$He | 3,727.379       | 28.296      | 7.074         |
| $^7_3$Li | 6,533.834       | 39.244      | 5.606         |
| $^8_4$Be | 7,454.850       | 56.500      | 7.062         |
| $^8_5$B  | 7,472.319       | 37.737      | 4.717         |
| $^{12}_6$C | 11,174.863    | 92.162      | 7.680         |

Notice that helium-4 has a higher binding energy per nucleon than other nearby nuclei. Moving up the periodic table, Fig. 15.3 shows that carbon-12 and oxygen-16 also have higher binding energies per nucleon than adjacent nuclei. That fact makes helium-4, carbon-12, and oxygen-16 unusually stable, which in turn makes them important players in the fusion reactions inside stars, and among the most abundant nuclei in the universe (after hydrogen).

### 15.3.3  Burning Hydrogen Into Helium

The reaction that powers stars for most of their lives[8] is the fusion of four hydrogen nuclei into helium-4. From theory and experiment, the overall reaction is
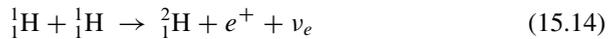
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[8]In Chap. 16 we will examine reactions that occur in the late stages of stellar evolution.

$$4\,{}^{1}_{1}\mathrm{H} \;\rightarrow\; {}^{4}_{2}\mathrm{He} + 2e^{+} + 2\nu_{e} + 2\gamma \tag{15.13}$$
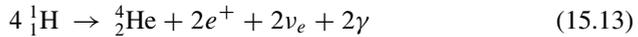
This does not happen in one step, though. In fact, several sequences of reactions have this net effect.

**Proton-Proton (PP) Chain**

One sequence of reactions is called the proton-proton chain. There are actually several variants of this chain. The first, called **PP I**, begins with:

$$ {}^{1}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \;\rightarrow\; {}^{2}_{1}\mathrm{H} + e^{+} + \nu_{e} \tag{15.14}$$

$$ {}^{2}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \;\rightarrow\; {}^{3}_{2}\mathrm{He} + \gamma \tag{15.15}$$

After this pair of reactions occurs twice, there are two helium-3 nuclei that can fuse via

$$ {}^{3}_{2}\mathrm{He} + {}^{3}_{2}\mathrm{He} \;\rightarrow\; {}^{4}_{2}\mathrm{He} + 2\,{}^{1}_{1}\mathrm{H} $$

The first pair of steps uses three hydrogen nuclei; they must occur twice, consuming a total of six hydrogens. The final step produces two new hydrogens, though, so the net effect is that four hydrogens are consumed to create one helium-4 nucleus.

Note that the first reaction requires one proton to be converted into a neutron,[9]
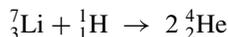
$$ p \;\rightarrow\; n + e^{+} + \nu_{e} $$
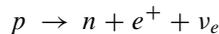
There is something odd here: the neutron weighs *more* than the proton, so the right-hand side has more mass/energy (940.076 MeV, plus the neutrino energy) than the left-hand side (938.272 MeV). In other words, at least 1.8 MeV of energy is absorbed by this reaction. That is okay, though, because the fusion of a proton and neutron into deuterium (${}^{2}_{1}\mathrm{H}$) releases more than enough energy to compensate.

Once some helium-4 exists,[10] there is a chance that the helium-3 produced in the second step above will react with helium-4,
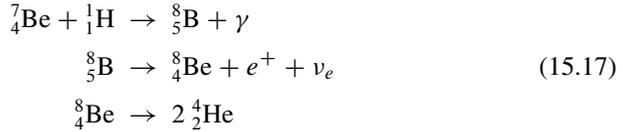
$$ {}^{3}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} \;\rightarrow\; {}^{7}_{4}\mathrm{Be} + \gamma $$
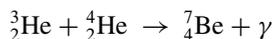
After this, there are two possibilities. One branch called **PP II** involves:

$$ {}^{7}_{4}\mathrm{Be} + e^{-} \;\rightarrow\; {}^{7}_{3}\mathrm{Li} + \nu_{e} \tag{15.16}$$

$$ {}^{7}_{3}\mathrm{Li} + {}^{1}_{1}\mathrm{H} \;\rightarrow\; 2\,{}^{4}_{2}\mathrm{He} $$

---

[9]This is a form of beta decay, which is different from hypothetical spontaneous proton decay.

[10]Helium-4 is produced in the big bang (see Chap. 20), so it can be present in stars even before PP I occurs.

The alternative called **PP III** has:

$$\begin{aligned}
{}^7_4\text{Be} + {}^1_1\text{H} &\rightarrow {}^8_5\text{B} + \gamma \\
{}^8_5\text{B} &\rightarrow {}^8_4\text{Be} + e^+ + \nu_e \\
{}^8_4\text{Be} &\rightarrow 2\,{}^4_2\text{He}
\end{aligned} \tag{15.17}$$

In principle, helium-3 can go straight to helium-4 through the reaction

$$ {}^3_2\text{He} + {}^1_1\text{H} \rightarrow {}^4_2\text{He} + e^+ + \nu_e \tag{15.18} $$
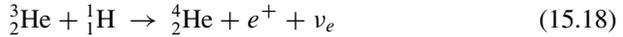
This is known as **PP IV**, or **HeP** because it combines helium (He) with a proton (P). It has not actually been identified in the Sun because the predicted rate is too low.

In addition to Eq. (15.14), there is an alternate way to produce deuterium, which is known as **PEP** because it involves an electron in addition to two protons:

$$ {}^1_1\text{H} + e^- + {}^1_1\text{H} \rightarrow {}^2_1\text{H} + \nu_e \tag{15.19} $$
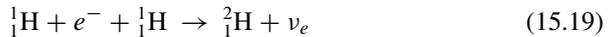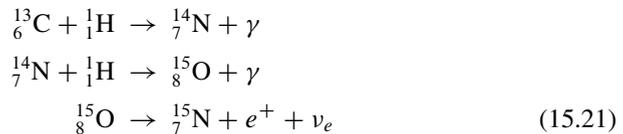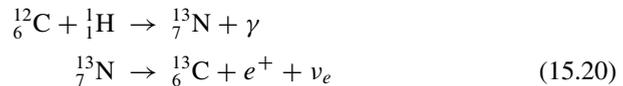
This step replaces Eq. (15.14) in about 1/400 of the reactions in the Sun.

Overall, 87.6 % of the reactions in the Sun follow the PP I branch, 10.7 % follow the PP II branch, and 0.9 % follow the PP III branch. (Only 0.8 % of reactions follow the alternate sequence known as the CNO cycle.) [7]
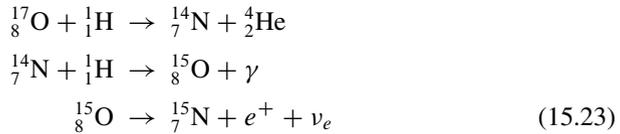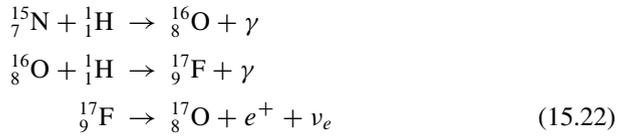
## CNO Cycle

Another sequence of reactions for converting hydrogen into helium uses carbon, nitrogen, and oxygen as catalysts, and hence is called the **CNO cycle**. The main branch looks like this (**CNO I**):

$$\begin{aligned}
{}^{12}_6\text{C} + {}^1_1\text{H} &\rightarrow {}^{13}_7\text{N} + \gamma \\
{}^{13}_7\text{N} &\rightarrow {}^{13}_6\text{C} + e^+ + \nu_e \\
{}^{13}_6\text{C} + {}^1_1\text{H} &\rightarrow {}^{14}_7\text{N} + \gamma \\
{}^{14}_7\text{N} + {}^1_1\text{H} &\rightarrow {}^{15}_8\text{O} + \gamma \\
{}^{15}_8\text{O} &\rightarrow {}^{15}_7\text{N} + e^+ + \nu_e \\
{}^{15}_7\text{N} + {}^1_1\text{H} &\rightarrow {}^{12}_6\text{C} + {}^4_2\text{He}
\end{aligned} \tag{15.20, 15.21}$$

This representation makes it look like carbon-12 is the start- and end-point, but the sequence is in fact a cycle that can be entered at any point. While various forms of carbon, nitrogen, and oxygen are created and consumed in the cycle, the net effect is the conversion of four hydrogens into one helium-4.

An alternative branch (**CNO II**) replaces the $^{15}_{7}\text{N}$ step above with a different series of reactions:

$$^{15}_{7}\text{N} + ^{1}_{1}\text{H} \rightarrow ^{16}_{8}\text{O} + \gamma$$

$$^{16}_{8}\text{O} + ^{1}_{1}\text{H} \rightarrow ^{17}_{9}\text{F} + \gamma$$

$$^{17}_{9}\text{F} \rightarrow ^{17}_{8}\text{O} + e^{+} + \nu_e \tag{15.22}$$

$$^{17}_{8}\text{O} + ^{1}_{1}\text{H} \rightarrow ^{14}_{7}\text{N} + ^{4}_{2}\text{He}$$

$$^{14}_{7}\text{N} + ^{1}_{1}\text{H} \rightarrow ^{15}_{8}\text{O} + \gamma$$

$$^{15}_{8}\text{O} \rightarrow ^{15}_{7}\text{N} + e^{+} + \nu_e \tag{15.23}$$

Notice that this branch does *not* replace the initial carbon. However, this branch is rare (0.04 %), so the overall carbon destruction rate is small. There are yet other branches of the CNO cycle that occur only in massive stars.

**PP or CNO?**

Which chain actually powers stars? As you can compute in Problem 15.5, the CNO cycle has a much stronger temperature dependence than the PP chain. The PP chain therefore dominates at low temperatures while the CNO cycle takes over at high temperatures (see Fig. 15.4). The transition temperature corresponds to the core of a main sequence star with a mass of about $1.2\,M_\odot$. Along the main sequence, in other words, stars with $M \lesssim 1.2\,M_\odot$ are mainly powered by the PP chain, while stars with $M \gtrsim 1.2\,M_\odot$ are mainly powered by the CNO cycle.

## 15.4   Solar Neutrinos

For the most part, we can test the theory of fusion only by building it into stellar models (see Sect. 16.2) and seeing how well the models reproduce properties of stars such as mass, size, and luminosity. With the Sun, however, we have access to the fusion reactions through the neutrinos they produce. These particles interact so weakly that they stream right out of the Sun. Most of them stream through Earth, too, but the few that are caught prove to be very informative.

### *15.4.1   Neutrino Production in the Sun*

John Bahcall and collaborators [8] used a detailed model of the Sun (known as the Standard Solar Model; see Sect. 16.2.2) to predict the rate at which neutrinos
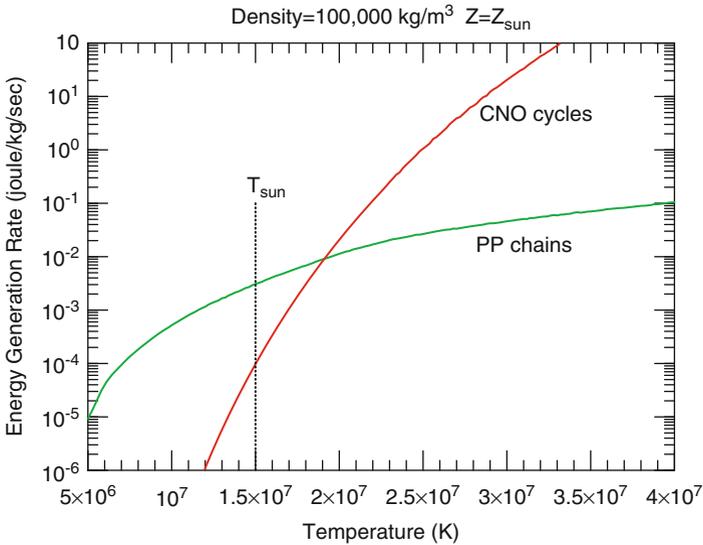
**Fig. 15.4** Energy generation rate as a function of temperature for PP chains and CNO cycles, assuming the same chemical composition as the Sun and a density of $10^5 \, \text{kg} \, \text{m}^{-3}$ (Credit: F. Timmes, ASU)

are produced. Neutrinos hardly notice the matter in the Sun (see Problem 15.7), so the spectrum of neutrino energies at Earth should reflect the intrinsic distribution of energies in the core of the Sun. Figure 15.5 shows that the spectrum for each reaction has a cutoff set by the masses involved in the reaction (see Problem 15.8). This is important because, as we will see, it is easiest to detect neutrinos whose energies are above a few MeV. Most of the solar neutrinos that we detect therefore come from the boron decay reaction in the PP III branch. The boron reaction may be rare in the Sun, but it plays a major role in our understanding of stellar physics.

### 15.4.2  Neutrino Detection (I)

Figure 15.5 suggests that there are enormous numbers of neutrinos passing through Earth. How can we detect them? One reaction that can be used to "capture" a neutrino is

$$^{37}_{17}\text{Cl} + \nu_e \; \rightarrow \; ^{37}_{18}\text{Ar} + e^- \tag{15.24}$$

This reaction involves the isotope chlorine-37, which constitutes about 25 % of natural chlorine. In the 1960s, Ray Davis assembled about 600 tons of tetra-chlorethylene (cleaning fluid) to serve as the world's first neutrino detector. (See [9] for a contemporaneous article.) The idea was to let the tank sit for a few weeks,
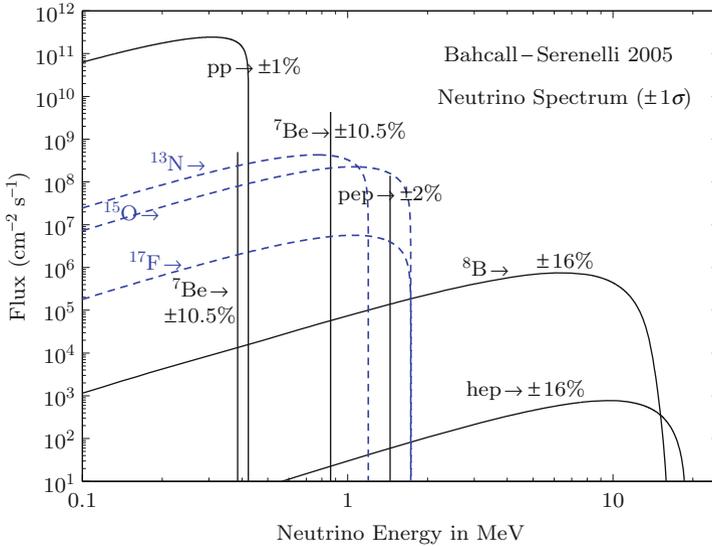
**Fig. 15.5** Predicted fluxes (at Earth) of neutrinos from the Sun, as a function of neutrino energy. *Solid lines* correspond to different reactions in the PP chain, while *dashed lines* correspond to different steps in the CNO cycle. (Reactions that produce neutrinos are numbered in Sect. 15.3.3) (Credit: Bahcall et al. [8]. Reproduced by permission of the AAS)

then collect the argon atoms that were produced when chlorine-37 atoms captured neutrinos. It seems quite remarkable that Davis and his team were able to separate a few dozen argon atoms from $\sim 10^{30}$ chlorine atoms!

The experiment took place nearly 1.5 km underground, in the **Homestake Gold Mine** in South Dakota. It was located in a mine so the overlying rock could act as a natural shield against stray particles that might produce argon. Neutrinos pass through rock with ease (in fact, the Homestake experiment collected neutrinos during both day and night, because neutrinos could travel the long way through Earth and enter the bottom of the tank at night), but most other particles are stopped.

Homestake did detect neutrinos from the Sun, confirming our basic picture of fusion in the Sun's core. For this pioneering work, Davis shared the 2002 Nobel Prize in Physics. However, the measured flux of neutrinos was only about 30 % of the flux predicted by the Standard Solar Model. The discrepancy between theory and observations became known as the **solar neutrino problem**.

### 15.4.3   Neutrino Oscillations

The solar neutrino problem had two possible interpretations: either our model of the Sun was wrong, or neutrino physics was richer than anticipated. The second

possibility was intriguing because the reactions that produce neutrinos in the Sun and the chlorine reactions used to capture neutrinos at Homestake all involve electrons or anti-electrons. According to particle physics, neutrinos associated with electrons are not the only kind that exist. The electron has two sister particles, known as the muon and the tau particle, and they have their own related neutrinos:

| Electron | Muon | Tau |
|----------|------|-----|
| $e^-$ | $\mu$ | $\tau$ |
| $\nu_e$ | $\nu_\mu$ | $\nu_\tau$ |

(All six have associated antiparticles.) These particles are collectively known as **leptons**, and the three different classes are known as **flavors**.

The particle physics became relevant for astrophysics because theorists speculated that neutrinos might be able to **oscillate** between flavor states [10]. If so, then electron neutrinos produced in the Sun might transform to mu or tau neutrinos on the way to Earth, thereby becoming invisible to experiments like Homestake and causing the observed neutrino flux to be lower than expected. Such changes could occur only if the three types of neutrinos have different masses, but at the time it was not known whether they have any mass at all. The possibility that neutrino oscillations might solve the solar neutrino problem—and, conversely, that solar neutrinos might reveal new physics—inspired a new generation of experiments to detect other flavors of neutrinos and constrain their masses.

The oscillation process is modified slightly when neutrinos propagate through matter, because the presence of electrons changes the effective mass states. This process, known as the **MSW effect** after Mikheyev and Smirnov [11] and Wolfenstein [12], can affect neutrinos as they leave the Sun and also as they travel through Earth.

### 15.4.4  Neutrino Detection (II)

In order to test the hypothesis that neutrino oscillations solve the solar neutrino problem, new detectors needed to be sensitive not only to electron neutrinos but also to the mu and tau flavors. The two experiments that played the most significant roles were Super-Kamiokande in Japan and the Sudbury Neutrino Observatory in Canada.

#### Super-Kamiokande

Super-Kamiokande detects neutrinos when they scatter off electrons in 50,000 tons of water. (There is nothing particularly special about water; it is just a convenient

medium that is translucent and feasible to obtain and purify in large quantities.) Formally, the interaction can be written as

$$\nu_x + e^- \; \rightarrow \; \nu_x + e^-$$

where $\nu_x$ refers to any flavor of neutrino. In detail, electron neutrinos have a higher electron scattering cross section than mu or tau neutrinos, but the difference is known and can be factored into data analysis. Super-Kamiokande focuses on neutrinos with energies above 5 MeV in order to discriminate them from background effects. When such an energetic neutrino scatters, it transfers some of its energy to an electron, causing the electron to reach a speed very close to $c$ that can actually be faster than light moves in water.[11] A faster-than-light electron emits a flash of blue light called **Čerenkov radiation** that can be detected with photomultiplier tubes lining the tank.

Electron scattering does not directly distinguish neutrino flavors, but it still offers ways to look for evidence of oscillations. There may be a daily modulation in the solar neutrino flux: during daytime solar neutrinos come down from above, but at night they must travel through Earth before reaching the detector, which subjects them to the MSW effect. There may also be an annual modulation: as Earth's distance from the Sun varies, there is not only a $1/r^2$ change in flux (which can be accounted for) but also a small change in oscillations because of the varying propagation distance. Super-Kamiokande has analyzed these variations, in conjunction with the total flux and energy spectrum, to constrain neutrino oscillations and provide evidence that the neutrino flux at Earth is in fact consistent with predictions from the Standard Solar Model. (See [13] for a discussion of the methodology and initial results, and [14] for more recent results.)

Super-Kamiokande has also been used to study neutrinos that are produced when cosmic rays strike Earth's atmosphere [15], as well as neutrinos in artificial beams produced at accelerators [16]. The various studies are designed to understand different aspects of neutrino oscillations and implications for neutrino masses.

**Sudbury Neutrino Observatory**

Sudbury Neutrino Observatory (SNO) was designed not only to detect but also to distinguish between electron, mu, and tau neutrinos. It used 1,000 tons of **heavy water** ($D_2O$) in order to be sensitive to three different processes:

- **Electron scattering** (ES), as in Super-Kamiokande.
- **Charged current reaction** (CC). An electron neutrino can split a deuterium nucleus into a proton and neutron, and then convert the neutron into a proton and electron:

---

[11]The speed of light in a water is $c/n$ where $n \approx 1.3$ is the index of refraction at visible wavelengths. Electrons can travel faster than light in water without violating the relativistic speed limit $v < c$.

$$\nu_e + {}^2_1\text{H} \ \rightarrow \ {}^1_1\text{H} + {}^1_1\text{H} + e^-$$

(This is the inverse of step (15.14) in the PP chain.) Electrons produced in this reaction can also emit Čerenkov radiation.

• **Neutral current reaction** (NC). Any flavor of neutrino can split a deuterium nucleus without transforming the neutron:

$$\nu_x + {}^2_1\text{H} \ \rightarrow \ {}^1_1\text{H} + n + \nu_x$$
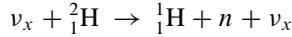
The neutron can then be captured by another nucleus in a reaction that emits an energetic gamma ray. The gamma ray photon then scatters off an electron, accelerating the electron to the point that it can emit Čerenkov radiation.

Like Super-Kamiokande, SNO had a neutrino energy threshold of 5 MeV, so it was mainly sensitive to neutrinos from boron decay.

In the initial experiment, the three channels led to the following neutrino fluxes (in units of $10^6 \, \text{cm}^{-2} \, \text{s}^{-1}$) [17]:

$$\text{CC:} \quad \phi_{\text{CC}} = 1.76^{+0.06}_{-0.05}(\text{stat.})^{+0.09}_{-0.09}(\text{syst.})$$
$$\text{ES:} \quad \phi_{\text{ES}} = 2.39^{+0.24}_{-0.23}(\text{stat.})^{+0.12}_{-0.12}(\text{syst.})$$
$$\text{NC:} \quad \phi_{\text{NC}} = 5.09^{+0.44}_{-0.43}(\text{stat.})^{+0.46}_{-0.43}(\text{syst.})$$

where the two sets of errorbars represent statistical and systematic uncertainties. The fact that the NC flux was higher than the CC and ES fluxes immediately revealed that some of the neutrinos from the Sun are being detected as mu and/or tau neutrinos, i.e., that neutrino oscillation does occur. Decomposing the three channel fluxes into contributions from electron neutrinos or mu/tau neutrinos (combined) yielded

$$\phi_e = 1.76^{+0.05}_{-0.05}(\text{stat.})^{+0.09}_{-0.09}(\text{syst.})$$
$$\phi_{\mu\tau} = 3.41^{+0.45}_{-0.45}(\text{stat.})^{+0.48}_{-0.45}(\text{syst.})$$

There were two key conclusions. First, the total flux of neutrinos agreed with predictions from the Standard Solar Model (5.05 in these units [18]). Second, only about 34 % of the detections involved electron neutrinos. Apparently the original solar neutrino problem arose not because the predictions were incorrect but because the Homestake experiment was unable to detect 2/3 of the solar neutrinos that pass through Earth.

In a second phase of the experiment, SNO added two tons of salt (NaCl) to the heavy water to enhance the sensitivity to NC reactions. (Chlorine is a good target for capturing neutrons released when deuterium is split; there were other technical gains as well.) This experiment yielded neutrino fluxes of [19]

$$\text{CC:} \quad \phi_{\text{CC}} = 1.68^{+0.06}_{-0.06}(\text{stat.})^{+0.08}_{-0.09}(\text{syst.})$$

$$\text{ES:} \quad \phi_{\text{ES}} = 2.35^{+0.22}_{-0.22}(\text{stat.})^{+0.15}_{-0.15}(\text{syst.})$$

$$\text{NC:} \quad \phi_{\text{NC}} = 4.94^{+0.21}_{-0.21}(\text{stat.})^{+0.38}_{-0.34}(\text{syst.})$$

In a third phase, SNO introduced an independent way to detect the NC neutrons by having them captured in helium-3 nuclei. This experiment yielded fluxes of [20]

$$\text{CC:} \quad \phi_{\text{CC}} = 1.67^{+0.05}_{-0.04}(\text{stat.})^{+0.07}_{-0.08}(\text{syst.})$$

$$\text{ES:} \quad \phi_{\text{ES}} = 1.77^{+0.24}_{-0.21}(\text{stat.})^{+0.09}_{-0.10}(\text{syst.})$$

$$\text{NC:} \quad \phi_{\text{NC}} = 5.54^{+0.33}_{-0.31}(\text{stat.})^{+036}_{-0.34}(\text{syst.})$$

Clearly the neutrino flux measurements are reproducible and robust to different methodologies.

The bottom line is that SNO solved the solar neutrino problem by demonstrating that neutrino oscillation is real and measuring a (total) neutrino flux that is consistent with predictions from the Standard Solar Model. The agreement between theory and experiment indicates that we understand quite a lot about what is happening deep inside the Sun. The little neutral ones have proven to be both harbingers of new particle physics and important messengers from the center of our star.

## Problems

**15.1.** Does the fusion of heavy nuclei require higher or lower temperatures than the fusion of light nuclei? Why?

**15.2.** Suppose fusion in the Sun stopped with $^3_2\text{He}$ instead of proceeding to $^4_2\text{He}$. If everything else were the same, would the Sun's lifetime be longer or shorter? By how much?

**15.3.** Show that the function

$$F(E) = e^{-(E_c/E)^{1/2}} e^{-E/kT}$$

has a peak at energy

$$E_0 = \left[ \frac{1}{4} E_c (kT)^2 \right]^{1/3}$$

This is the Gamow peak discussed in Sect. 15.2.4. Make sure to verify that it is a peak (i.e., a local maximum, not a local minimum).

**15.4.** The goal of this problem is to understand how the fusion reaction rate depends on temperature. In Eq. (15.8) we showed that the reaction rate can be written as

$$r_{12} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_1 n_2}{(\pi\mu)^{1/2}} \int_0^\infty S(E)\, e^{-f(E)}\, dE \quad \text{where} \quad f(E) = \frac{E}{kT} + \left(\frac{E_c}{E}\right)^{1/2}$$

The function $e^{-f(E)}$ is sharply peaked near the Gamow peak $E_0$. Assuming that $S(E)$ is reasonably constant near $E_0$, we pull it out of the integral and write

$$r_{12} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_1 n_2}{(\pi\mu)^{1/2}} S(E_0) \int_0^\infty e^{-f(E)}\, dE$$

Our task is to estimate the remaining integral. Here are the steps to do that.

(a) Expand the function $f(E)$ as a Taylor series around the Gamow peak, $E = E_0$. This means you can approximate $f$ with the form

$$f(E) \approx b_0 + b_1(E - E_0) + b_2(E - E_0)^2 + \dots$$

What you need to do is determine the coefficients $b_0$, $b_1$, and $b_2$. Hint: you should find $b_1 = 0$ since $E_0$ is a local maximum of $f(E)$.

(b) Using part (a) and setting $x = E - E_0$ converts the integral into a Gaussian form that can be evaluated using expressions in Sect. A.7.[12] Use this with your values of $b_0$ and $b_2$ to write an approximation for $r_{12}$.

(c) Following Sect. 15.2.4, write the temperature dependence with a power law approximation $r_{12} \propto T^\alpha$ and show that $\alpha$ is given by Eq. (15.11).

**15.5.** Let's compute the temperature dependence of the reaction rate for two specific examples. Even if you have not worked through Problem 15.4, you can still use Eq. (15.11) here. Take the temperature to be $T = 1.6 \times 10^7$ K.

(a) PP chain: Consider the $^1_1$H + $^1_1$H step and compute $kT$, $E_c$, $E_0$, and $\alpha$.
(b) CNO cycle: Consider the $^{14}_7$N + $^1_1$H step and compute $kT$, $E_c$, $E_0$, and $\alpha$.

**15.6.** A brown dwarf is a star that is not massive enough to have normal hydrogen fusion (see Problem 16.5).

(a) A brown dwarf can release energy by the Kelvin-Helmholtz mechanism. Suppose a brown dwarf of mass $M$ and radius $R$ is shrinking at a rate $dR/dt$. What is its luminosity? (As always, state any assumptions you make.)
(b) Consider a brown dwarf with $M = 0.05\, M_\odot$ and $R = 0.1\, R_\odot$, whose radius shrinks by 1 % over $10^9$ yr. What is its luminosity (in units of $L_\odot$)? If it radiates like a blackbody, what is its effective temperature? In what portion of the electromagnetic spectrum does it emit most of its light?

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[12]Section A.7 gives integrals of Gaussian functions over the range $-\infty < x < \infty$. The limits of integration in this problem are far enough from the peak that we can extend them to $\pm\infty$.

(c) Brown dwarfs can fuse deuterium[13] via the reaction $^2_1\text{H} + ^1_1\text{H} \rightarrow ^3_2\text{He}$. If a fraction $f = 4 \times 10^{-5}$ of a brown dwarf's mass is deuterium, how long could deuterium burning power the star at the same luminosity you estimated in part (b)?

**15.7.** The cross section for a typical solar neutrino to interact with an atomic nucleus is $\sigma \sim 10^{-47}\,\text{m}^2$ (it varies with the neutrino energy, but we will focus on order-of-magnitude estimates). Estimate the mean free path of neutrinos in the Sun. You may make reasonable assumptions and approximations to obtain an order-of-magnitude estimate.

**15.8.** In Fig. 15.5, there is an upper limit to the neutrino energy in each reaction. Consider the total mass at the start and end to find the maximum neutrino energy for each of the following reactions (as labeled in the figure):

(a) pp, Eq. (15.14)
(b) $^8$B, Eq. (15.17)
(c) hep, Eq. (15.18)

**15.9.** Estimate the number of solar neutrinos passing through your body each second. Estimate the number of times a neutrino will hit a nucleus in your body during your lifetime (assuming a cross section of $\sigma \sim 10^{-47}\,\text{m}^2$). You will need to make a variety of assumptions and approximations; explain your reasoning.

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[13]They may contain deuterium left over from the big bang (see Sect. 20.2).

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