

# Chapter 13

## Planetary Temperatures

In Chap. 12 we saw how a planet's temperature and gravity combine to determine whether the planet has an atmosphere. In this chapter we study the physical processes that determine the temperature in the first place. Intuitively, we expect a planet close to the Sun to be warmer than a planet farther away, but we seek to quantify that effect. We also consider ways in which a planet's atmosphere can act as a blanket to trap heat. The physical phenomena that play a role here are blackbody radiation and the interaction of light with matter.

### 13.1 Blackbody Radiation

A **blackbody** is a hypothetical object that absorbs all light incident upon it. Since light carries energy, the object must either heat up or get rid of the excess energy. Heating a solid object causes it to glow, and a blackbody emits a characteristic spectrum that depends only on the temperature (not on the size or composition). While real astrophysical objects are not perfect blackbodies, the blackbody spectrum nevertheless provides a good starting point for describing the light emitted by stars and planets.

#### 13.1.1 Luminosity

The relationship between temperature and the total amount of energy emitted by a blackbody was determined empirically by Josef Stefan, and then explained theoretically by Ludwig Boltzmann. (We will consider the theory in Sect. 13.1.2.) The **Stefan-Boltzmann law** for luminosity is

$$L = A \sigma T^4 \tag{13.1}$$

where  $A$  is the surface area of the blackbody,  $T$  is the temperature, and  $\sigma$  is a constant now known as the **Stefan-Boltzmann constant**:

$$\sigma = 5.67 \times 10^{-8} \text{ kg s}^{-3} \text{ K}^{-4}$$

### Example: Sun

The Sun's luminosity is  $L = 3.84 \times 10^{26} \text{ J s}^{-1}$  and radius is  $R = 6.96 \times 10^8 \text{ m}$ . What is its surface temperature? Since the Sun is not a perfect blackbody, we need to be more precise about the question: What is the temperature of a blackbody that has the same size and luminosity as the Sun? This is what we define to be the **effective temperature** of the Sun, and while it may not be precisely the same as the physical temperature in the outer layers of the star it is close enough to be very useful. Using  $A = 4\pi R^2$  and inverting the Stefan-Boltzmann law yields

$$T_{\text{eff}} = \left( \frac{L}{4\pi R^2 \sigma} \right)^{1/4} \quad (13.2)$$

The numbers for the Sun give

$$T_{\odot} = \left[ \frac{3.84 \times 10^{26} \text{ J s}^{-1}}{4\pi \times (6.96 \times 10^8 \text{ m})^2 \times (5.67 \times 10^{-8} \text{ kg s}^{-3} \text{ K}^{-4})} \right]^{1/4} = 5,780 \text{ K} \quad (13.3)$$

### 13.1.2 Spectrum

Moving beyond the total luminosity, Max Planck discovered a formula that describes the full spectrum of blackbody radiation. First, let's think about how it is expressed. We define the wavelength spectrum to be  $B_{\lambda}(\lambda; T)$  such that the luminosity can be written as

$$L = \int_0^{\infty} d\lambda \int dA \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \cos \theta \sin \theta B_{\lambda}(\lambda; T) \quad (13.4)$$

This expression involves two surfaces. The integral over  $A$  covers the entire surface from which light is *emitted*. The integrals over  $\theta$  and  $\phi$  cover an imaginary surface around the blackbody at which light is *received*. For each point on the emitting surface, we can define the polar angles relative to the vector perpendicular to  $dA$ , and we consider only  $0 \leq \theta \leq \pi/2$  because the light must travel outward. The integrand contains the standard spherical volume element  $\sin \theta d\theta d\phi$  along with an extra factor of  $\cos \theta$  for geometric reasons: an observer viewing from angle  $\theta$  would

see the surface element subtend a *projected* area of  $dA \cos \theta$ . As defined through the integral,  $B_\lambda(\lambda; T)$  has dimensions of energy per unit area per unit time per unit wavelength per unit solid angle,<sup>1</sup> and standard units<sup>2</sup> of  $\text{J m}^{-2} \text{s}^{-1} \text{m}^{-1} \text{sr}^{-1}$ .

Planck considered a box filled with electromagnetic radiation that has a small hole in one side through which light can escape. He postulated that light energy can exist only in discrete packets, called “quanta” (plural of “quantum”),<sup>3</sup> with energy

$$E = h\nu = \frac{hc}{\lambda} \quad (13.5)$$

where  $\nu$  and  $\lambda$  are the frequency and wavelength of light, respectively, and

$$h = 6.63 \times 10^{-34} \text{ J s} \quad (13.6)$$

is a new constant of nature now known as **Planck’s constant**. We sometimes rewrite it as

$$\hbar = \frac{h}{2\pi} = 1.05 \times 10^{-34} \text{ J s} \quad (13.7)$$

With the quantum assumption, Planck derived the spectrum

$$B_\lambda(\lambda; T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \quad (13.8)$$

This function, which is shown in Fig. 13.1, agrees very well with observed blackbody spectra. The most precise blackbody spectrum ever measured actually comes from astrophysics: the Cosmic Microwave Background (CMB) radiation, which was produced by hot glowing gas in the early universe. As the universe has expanded, the radiation has effectively cooled such that the spectrum today is accurately described by a Planck spectrum with a temperature of 2.73 K. (We will study the CMB in Chap. 20.)

The Planck spectrum can be expressed in terms of frequency if we rewrite Eq. (13.4) as

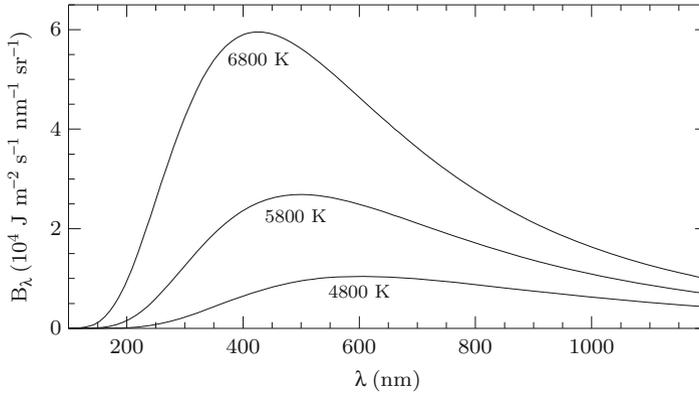
$$L = \int_0^\infty d\nu \int dA \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \cos \theta \sin \theta B_\nu(\nu; T) \quad (13.9)$$

Thinking of this as a change of integration variables, we can identify the frequency spectrum as

<sup>1</sup>Solid angle is like an angular area,  $d\Omega = \sin \theta d\theta d\phi$ , and it is measured in steradians. There are  $4\pi$  steradians on a sphere.

<sup>2</sup>In principle, this could be reduced to  $\text{kg m}^{-1} \text{s}^{-3} \text{sr}^{-1}$ , but that would make the physical meaning much less clear.

<sup>3</sup>We now use “quantum” as a general term, and “photon” when speaking specifically of light.



**Fig. 13.1** Examples of the Planck spectrum for different temperatures

$$B_\nu(\nu; T) = B_\lambda \left| \frac{d\lambda}{d\nu} \right|_{\lambda=c/\nu} = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \quad (13.10)$$

We can use the Planck spectrum to derive the Stefan-Boltzmann constant  $\sigma$  in terms of fundamental constants. In Eq. (13.9), the integrals over  $A$ ,  $\theta$ , and  $\phi$  can be evaluated:

$$\int dA = A \quad \int_0^{2\pi} d\phi = 2\pi \quad \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta = \frac{1}{2}$$

The  $\nu$  integral can be computed by changing variables to  $x = h\nu/kT$  and using the result

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15} \quad (13.11)$$

Thus, we find

$$L = A \times \frac{2\pi^5 k^4}{15 c^2 h^3} \times T^4$$

Comparing with Eq. (13.1) reveals that

$$\sigma = \frac{2\pi^5 k^4}{15 c^2 h^3} = \frac{\pi^2 k^4}{60 c^2 \hbar^3} \quad (13.12)$$

### 13.1.3 Color

Early studies of blackbody radiation noted that an object's color shifts from red to orange to yellow (and so on) as the temperature rises. While the blackbody spectrum contains a mixture of wavelengths (the function is continuous), we can get a sense of the color by considering the location of the peak in the spectrum. To find the peak, let's begin by computing the derivative:

$$\frac{dB_\lambda}{d\lambda} = -\frac{10hc^2}{\lambda^6} \frac{1}{e^{hc/\lambda kT} - 1} + \frac{2h^2c^3}{\lambda^7 kT} \frac{e^{hc/\lambda kT}}{(e^{hc/\lambda kT} - 1)^2}$$

Set  $dB/d\lambda = 0$ , substitute  $x = hc/(\lambda_{\text{peak}}kT)$ , and simplify:

$$0 = \frac{2k^6 T^6 x^6}{h^5 c^4} \frac{5 - 5e^x + xe^x}{(e^x - 1)^2} \Rightarrow \frac{xe^x}{e^x - 1} = 5$$

This equation is solved for  $x = 4.965$ , or

$$\begin{aligned} \lambda_{\text{peak}} T &= \frac{hc}{4.965k} \\ &= \frac{(6.626 \times 10^{-34} \text{ kg m}^2 \text{ s}^{-1}) \times (3.0 \times 10^8 \text{ m s}^{-1})}{4.965 \times (1.38 \times 10^{-23} \text{ kg m}^2 \text{ s}^{-2} \text{ K}^{-1})} \\ &= 2.90 \times 10^{-3} \text{ m K} \end{aligned}$$

Equivalently, we can write

$$\lambda_{\text{peak}} = \frac{2.90 \times 10^{-3} \text{ m K}}{T} \quad (13.13)$$

This relation is known as **Wien's displacement law** after Wilhelm Wien. It quantifies the connection between temperature and color: as temperature increases, the peak wavelength decreases, so the spectrum shifts toward bluer colors. Notice that we did not specify anything about the size or shape of the object; Wien's law is universal for blackbodies.

#### Example: What Is the Peak Wavelength of Sunlight?

Using the effective temperature of the Sun from Eq. (13.3) gives

$$\lambda_{\text{peak}} = \frac{2.90 \times 10^{-3} \text{ m K}}{5,780 \text{ K}} = 5.02 \times 10^{-7} \text{ m} = 502 \text{ nm} = 5,020 \text{ \AA}$$

This is green light. It is no accident that our eyes are most sensitive to light around these wavelengths; they evolved to operate in the portion of the electromagnetic spectrum in which the Sun emits most of its light.

### Example: In What Portion of the Spectrum Does Earth Radiate?

Using  $T = 288 \text{ K}$  as the average temperature of Earth's surface yields

$$\lambda_{\text{peak}} = \frac{2.90 \times 10^{-3} \text{ m K}}{288 \text{ K}} = 10^{-5} \text{ m} = 10 \mu\text{m}$$

This is in the infrared region of the electromagnetic spectrum.

### 13.1.4 Pressure

According to Planck, each photon carries energy  $E = h\nu = hc/\lambda$ . According to relativity, then, we can think of a photon as a massless particle with momentum  $p = E/c$  (see Eq. 10.26), or equivalently  $p = h\nu/c = h/\lambda$ . Particles that carry momentum can exert pressure (recall Sect. 12.1.3), so we infer that there must be some pressure associated with light.

We can compute the pressure by thinking of blackbody radiation as a “gas” of photons. The number density of photons with momentum between  $p$  and  $p + dp$  is

$$n(p) dp = \frac{2}{h^3} \frac{1}{e^{pc/kT} - 1} 4\pi p^2 dp$$

This basically comes from expressing the Planck spectrum (13.10) in terms of momentum, except that we have switched to number density; note that the factor of  $4\pi p^2 dp$  is the spherical volume element. Using the pressure integral (Eq. 12.10), we can write the pressure of the photon gas as

$$P = \frac{1}{3} \int c p n(p) dp = \frac{8\pi c}{3h^3} \int_0^\infty \frac{p^3}{e^{pc/kT} - 1} dp = \frac{8\pi k^4}{3c^3 h^3} T^4 \int_0^\infty \frac{x^3}{e^x - 1} dx$$

where we use  $v = c$  for photons, and we change variables to  $x = pc/kT$ . Again using the integral (13.11), we find

$$P = \frac{8\pi^5 k^4}{45c^3 h^3} T^4 = \frac{4\sigma}{3c} T^4 \quad (13.14)$$

where we use Eq. (13.12) to replace some of the constants with the Stefan-Boltzmann constant. Plugging in numbers yields

$$P = 2.52 \times 10^{-16} \text{ kg m}^{-1} \text{ s}^{-2} \times \left( \frac{T}{\text{K}} \right)^4$$

Comparing Eqs. (13.14) and (13.1) shows that photon pressure is proportional to luminosity, which makes sense.

## 13.2 Predicting Planet Temperatures

We can now use the properties of blackbody radiation to make a simple model for the temperature of a planet. The logic is as follows: light from the Sun heats the planet, causing the planet to radiate. The planet heats up until it reaches an equilibrium state in which the energy it radiates exactly balances the energy it receives from the Sun. The model is simplistic because it assumes the temperature and composition are uniform across the planet, which we know is not true, but using average quantities yields a model that works surprisingly well. Plus, the model illuminates the basic physics, which is ultimately the purpose of a simple model.

In Sect. 13.1.1 we already defined the effective temperature of the Sun,  $T_{\odot}$ , through the relation  $L_{\odot} = 4\pi R_{\odot}^2 \sigma T_{\odot}^4$ . By the inverse square law, the energy flux (energy per unit area per unit time) at a distance  $D$  from the Sun is

$$f = \frac{L_{\odot}}{4\pi D^2} = \frac{\sigma R_{\odot}^2 T_{\odot}^4}{D^2}$$

The total power (energy per unit time) incident on a planet of radius  $R_p$  is the flux times the area of the planet. From the perspective of the Sun, the planet appears to subtend a circle of area  $\pi R_p^2$ . Planets are not perfect blackbodies; they reflect part of the incident light. We define the **albedo**,  $a$ , to be the fraction of the incident light that is reflected. Thus the fraction of the incident power that is *absorbed* is  $(1 - a)$ , and the total power that acts to heat the planet is<sup>4</sup>

$$P_{\text{abs}} = (1 - a) \times \pi R_p^2 \times \frac{\sigma R_{\odot}^2 T_{\odot}^4}{D^2}$$

If the planet has (effective) temperature  $T_p$ , then from the Stefan-Boltzmann law the total luminosity (energy per unit time) it emits is

$$L_{\text{em}} = 4\pi R_p^2 \sigma T_p^4$$

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<sup>4</sup>Here  $P$  denotes power, not pressure.

As we said above, the planet will reach an equilibrium in which  $P_{\text{abs}} = L_{\text{em}}$ , or

$$(1 - a)\pi R_p^2 \frac{\sigma R_{\odot}^2 T_{\odot}^4}{D^2} = 4\pi R_p^2 \sigma T_p^4$$

Solving for  $T_p$  gives

$$T_p = T_{\odot}(1 - a)^{1/4} \left(\frac{R_{\odot}}{2D}\right)^{1/2} \quad (13.15)$$

Notice that the radius of the planet has dropped out; in this simple model, a planet's temperature depends only on its distance from the Sun and its albedo (and the properties of the Sun, of course).

We have constructed the model using the average temperature and albedo of the planet. In reality, the absorption and emission properties vary with latitude and from the day-side to the night-side (not to mention smaller-scale features), and the variation depends on whether there is much atmosphere to retain and circulate warmth. We do not attempt to incorporate such details into the model, though. At this point we just want to see whether this simple model of planet heating can help us gain a basic understanding of why some planets are hot and others are cold.

### Example: Earth

Earth lies  $D = 1 \text{ AU} = 1.50 \times 10^{11} \text{ m}$  from the Sun and has an average albedo of  $a = 0.306$ . What does this model predict for its average temperature?

$$T_{\oplus} = 5,780 \text{ K} \times (1 - 0.306)^{1/4} \times \left(\frac{6.96 \times 10^8 \text{ m}}{2 \times 1.50 \times 10^{11} \text{ m}}\right)^{1/2} = 254 \text{ K} = -19 \text{ deg C} = -2 \text{ deg F}$$

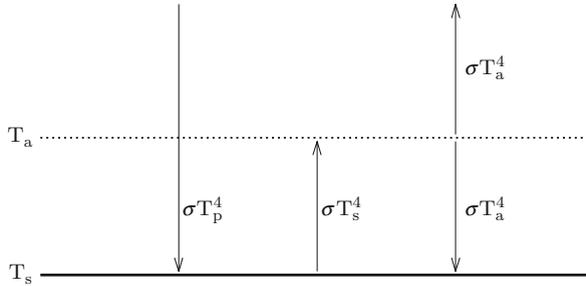
Not bad for a simple model, but a bit colder than reality. Why? Our analysis actually applies to the outermost layer of a planet; the situation may be somewhat different on the surface, if the planet has an atmosphere.

## 13.3 Atmospheric Heating

In order to predict the temperature on the *surface* of a planet, we need to account for the fact that an atmosphere can trap heat near the surface. In Sect. 13.4 we will consider the physical processes by which light can interact with gas in an atmosphere; for now, let's estimate how much atmospheric heating can warm the surface.<sup>5</sup>

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<sup>5</sup>This analysis is inspired by Problems 19.13 and 20.7 in the book by Carroll and Ostlie [1].



**Fig. 13.2** Setup for the one-layer model of atmospheric heating. The atmosphere (*dotted line*) is treated as a thin layer at temperature  $T_a$ , while the surface (*solid line*) has temperature  $T_s$ . Energy flux (in units of energy per unit time per unit area) is denoted with *arrows*. The incident flux is quantified with  $T_p$  from Eq. (13.15). The atmosphere is transparent to the visible light incident from the Sun, but it absorbs the infrared radiation emitted by the surface and then re-radiates that energy both upward and downward

### 13.3.1 One Layer

To start, let's make a toy model in which the atmosphere is a thin layer hovering above the surface of the planet. Suppose the atmosphere is transparent to the visible light that comes from the Sun, but it absorbs 100% of the infrared radiation emitted by the planet and then re-radiates that energy. (Again, we will see below how this happens at the level of atoms and molecules.) Consider the energy flow, quantified in terms of the energy flux (power per unit area), as sketched in Fig. 13.2:

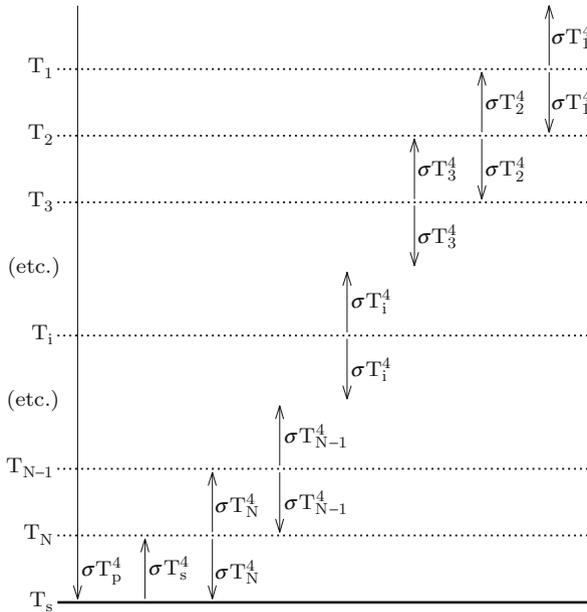
- The incident flux from the Sun comes down through the atmosphere with no effect, and then hits the surface. We can use the analysis from Sect. 13.2 to quantify the incident flux as  $\sigma T_p^4$  where  $T_p$  is given by Eq. (13.15).
- The surface has temperature  $T_s$  (which is to be determined), so it radiates a flux  $\sigma T_s^4$  upward. That flux is absorbed by the atmosphere.
- The atmosphere has temperature  $T_a$  (which is to be determined), so it radiates a flux  $\sigma T_a^4$  both upward and downward.

The net flux incident on Earth is  $\sigma T_p^4$ , while the net flux leaving Earth is  $\sigma T_a^4$ . In equilibrium, we must therefore have  $T_a = T_p$ . Thus, what Eq. (13.15) predicts is the temperature of the *atmosphere*.

What about the surface? In order for the incident and emitted radiation to balance, we must have

$$\sigma T_s^4 = \sigma T_p^4 + \sigma T_a^4 = 2\sigma T_p^4 \quad \Rightarrow \quad T_s = 2^{1/4} T_p \quad (13.16)$$

In this simple model, the atmosphere raises the surface temperature by a factor of  $2^{1/4} = 1.19$ . An increase of 19% may not seem like a lot, but remember that it applies to the Kelvin temperature.



**Fig. 13.3** Setup for the many-layer model of atmospheric heating. Atmospheric layers are labeled  $1, \dots, N$  from top down (most intermediate layers are not shown here). Energy flux is denoted with arrows. The layers are transparent to the visible light incident from the Sun, but each layer absorbs 100 % of the infrared radiation incident upon it and then re-radiates that energy both upward and downward

**Example: Earth**

Using  $T_p = 254 \text{ K}$  as above, the one-layer model would predict a surface temperature of  $T_s = 302 \text{ K}$ . This is closer to Earth’s average surface temperature, but a little high. The problem, presumably, is the assumption that 100 % of the infrared light emitted by the surface is trapped by the atmosphere.

**13.3.2 Many Layers**

To improve the model we need to handle what is effectively a partial layer. While it may seem counterintuitive, our best bet is actually to understand what happens when we add more layers, and then to generalize the notion of layers, so we can finally circle back to the case of a partial layer.

Now suppose the atmosphere is made of  $N$  layers that transmit visible light but absorb infrared light. Number the layers from top to bottom and then consider the energy flow as sketched in Fig. 13.3. Again, we write the incident flux as  $\sigma T_p^4$ . Now

consider the energy balance for each layer; on the left of each equation is the output flux, and on the right is the input<sup>6</sup>:

$$\begin{aligned}
 2T_1^4 &= T_2^4 \\
 2T_2^4 &= T_1^4 + T_3^4 \\
 2T_3^4 &= T_2^4 + T_4^4 \\
 &\vdots \\
 2T_i^4 &= T_{i-1}^4 + T_{i+1}^4 \\
 &\vdots \\
 2T_{N-1}^4 &= T_{N-2}^4 + T_N^4 \\
 2T_N^4 &= T_{N-1}^4 + T_s^4 \\
 T_s^4 &= T_N^4 + T_p^4
 \end{aligned}$$

We also have the condition that the net input to the planet must equal the net output from the planet. This yields:

$$T_1^4 = T_p^4$$

Now that we know  $T_1$ , we can work from the top down:

$$\begin{aligned}
 T_1^4 &= T_p^4 \\
 T_2^4 &= 2T_1^4 = 2T_p^4 \\
 T_3^4 &= 2T_2^4 - T_1^4 = 3T_p^4 \\
 &\vdots
 \end{aligned}$$

The pattern is clear: the temperature of layer  $i$  satisfies

$$T_i^4 = i T_p^4 \quad \Rightarrow \quad T_i = i^{1/4} T_p$$

Finally, the last energy balance equation tells us what happens at the surface:

$$T_s^4 = (1 + N)T_p^4 \quad \Rightarrow \quad T_s = (1 + N)^{1/4} T_p \quad (13.17)$$

Adding more layers creates more heating at the surface,<sup>7</sup> with the scaling  $(1 + N)^{1/4}$ .

<sup>6</sup>To get true energy flux we need a factor of  $\sigma$  multiplying each  $T^4$ , but those all factor out.

<sup>7</sup>This is why we wear more layers of clothing in the winter.

### 13.3.3 Optical Depth

The preceding analysis becomes more applicable to Earth when we generalize the notion of “layers” in the atmosphere. What actually matters is the number of times a photon is absorbed and reemitted between the surface and free space. If the mean free path of light in the atmosphere is  $\ell = (n\sigma)^{-1}$ , then the number of interactions over some distance  $dx$  is  $n\sigma dx$ . The total number of interactions as light travels through the atmosphere is then (cf. Eq. 12.19)

$$\tau = \int n\sigma dx \quad (13.18)$$

We call this the **optical depth** because it gives a sense of how far light penetrates into a gas: roughly speaking, we can see fairly clearly up until the point where  $\tau \sim 1$ .

Since  $\tau$  counts the number of interactions, we can interpret it as the effective number of layers in the atmosphere. Then we can write our model for atmospheric heating as

$$T_s = (1 + \tau)^{1/4} T_p \quad (13.19)$$

This is a nice generalization because  $\tau$  no longer needs to be an integer. In particular, we can use a value  $\tau < 1$  to model an atmosphere that absorbs infrared radiation partially but not perfectly. In Problem 13.7 you can see how to apply this model to Earth’s atmosphere.

## 13.4 Interaction of Light with Matter

In the previous section we postulated that Earth’s atmosphere can absorb infrared light. How does this happen on a microscopic level? More generally, how do light and matter interact? At the moment we are most interested in how matter *absorbs* light, and there are four phenomena that may be relevant:

- Photoionization: a photon can give enough energy to an electron to knock it out of its atom/molecule.
- Electron excitation: a photon can give energy to an electron and cause it to jump to a higher energy level.
- Molecular vibration: a photon can cause the atoms in a molecule to vibrate relative to one another.
- Molecular rotation: a photon can cause an entire molecule to spin.

Let's consider each in turn. For now we want to figure out which phenomena affect infrared light, but we will encounter all of them in various contexts in coming chapters.

### 13.4.1 Photoionization

To analyze photoionization, we need to determine what wavelengths of light have enough energy to unbind an electron. Let's use the Bohr model to estimate the energy levels of electrons in atoms. In this model we picture the electron in a classical circular orbit, but we say the angular momentum must be quantized. While the physical picture is not strictly correct, the resulting energy levels turn out to be accurate for hydrogen and reasonable for some other elements.

Consider a single electron orbiting a nucleus with atomic number  $Z$ . The electric force between the nucleus and electron is  $Ze^2/r^2$ , while the force needed to keep the electron in a circular orbit with speed  $v$  is  $m_e v^2/r$ . Equating these lets us determine the orbital speed  $v$  and angular momentum  $L$ :

$$v = \left( \frac{Ze^2}{m_e r} \right)^{1/2} \quad \text{and} \quad L = m_e r v = (Ze^2 m_e r)^{1/2}$$

We then quantize the angular momentum by setting  $L_n = n\hbar$  where  $n$  is an integer. This yields the orbital radius for level  $n$ :

$$r_n = \frac{n^2 \hbar^2}{Ze^2 m_e}$$

The total energy of the electron in this orbit is then

$$E_n = -\frac{Ze^2}{r_n} + \frac{1}{2} m_e v_n^2 = -\frac{Ze^2}{2r_n} = -\frac{Z^2 e^4 m_e}{2n^2 \hbar^2} \quad (13.20)$$

In order to eject such an electron, a photon must have energy  $E > |E_n|$  or wavelength

$$\lambda < \lambda_n \quad \text{where} \quad \lambda_n = \frac{hc}{E_n} = \frac{4\pi \hbar^3 cn^2}{Z^2 e^4 m_e}$$

Plugging in numbers yields

$$\lambda_n = 911 \text{ \AA} \times \frac{n^2}{Z^2} \quad (13.21)$$

Photoionization mainly involves ultraviolet light.

### 13.4.2 Electron Excitation

If the photon does not have sufficient energy to unbind the electron, it might still be able to excite the electron into a higher energy level. In the Bohr model, the energy required to raise an electron from level  $n$  to level  $m$  is

$$\Delta E_{nm} = E_m - E_n = \frac{Z^2 e^4 m_e}{2\hbar^2} \left( \frac{1}{n^2} - \frac{1}{m^2} \right)$$

The corresponding wavelength is

$$\lambda_{nm} = \frac{4\pi\hbar^3 c}{Z^2 e^4 m_e} \left( \frac{1}{n^2} - \frac{1}{m^2} \right)^{-1} = \frac{911 \text{ \AA}}{Z^2} \left( \frac{1}{n^2} - \frac{1}{m^2} \right)^{-1} \quad (13.22)$$

As an example, in hydrogen the  $1 \rightarrow 2$  transition has  $\lambda_{12} = 1,216 \text{ \AA}$ , while the  $2 \rightarrow 3$  transition has  $\lambda_{23} = 6,563 \text{ \AA}$ . These spectral lines will play an important role when we study stars in Chap. 14. For now, the key point is that electron excitation occurs at discrete wavelengths of visible and ultraviolet light.

### 13.4.3 Molecular Vibration

In a molecule, light can cause the chemical bonds to vibrate. We can use our tools of dimensional analysis and toy models to estimate the range of wavelengths that can excite vibrational motion.

Let's begin with dimensional analysis. The force involved is the electric force, so we ought to use  $e$ . Since the nuclei themselves move, the mass scale is that of a proton or neutron. The important length scale is the typical distance  $r_0$  between atoms in the molecule. Note that  $r_0$  encodes the physics (including quantum mechanics) that governs the chemical bond and determines the size of the molecule; we can avoid those details by working with  $r_0$  directly. Let's write

$$\begin{aligned} \omega &\sim e^\alpha m_p^\beta r_0^\gamma \\ [T^{-1}] &\sim [M^{\alpha/2} L^{3\alpha/2} T^{-\alpha} \times M^\beta \times L^\gamma] \end{aligned}$$

This is solved with  $\alpha = 1$ ,  $\beta = -1/2$ , and  $\gamma = -3/2$ , yielding

$$\omega \sim \frac{e}{(m_p r_0^3)^{1/2}}$$

The angular frequency  $\omega$  corresponds to a linear frequency  $\nu = \omega/(2\pi)$ . The corresponding wavelength of light is

$$\lambda = \frac{c}{\nu} \sim \frac{2\pi c}{e} (m_p r_0^3)^{1/2}$$

The most important vibrating molecule in Earth's atmosphere is water, which has  $r_0 \approx 10^{-10}$  m, and hence

$$\lambda \sim \frac{2\pi \times (3.0 \times 10^8 \text{ m s}^{-1})}{1.52 \times 10^{-14} \text{ kg}^{1/2} \text{ m}^{3/2} \text{ s}^{-1}} [(1.67 \times 10^{-27} \text{ kg}) \times (10^{-10} \text{ m})^3]^{1/2} \sim 5 \mu\text{m}$$

This is in the infrared portion of the electromagnetic spectrum.

To make a fully realistic model, we would need to consider the structure of the bond between atoms, handle quantum mechanics properly, and account for different vibrational modes. Without getting into all of the details, we can go one step further by thinking about the bond. In water, the hydrogen and oxygen share a covalent bond: each atom contributes one electron to a sort of cloud surrounding the two nuclei.<sup>8</sup> Let's make a toy model in which the oxygen and hydrogen ions (each with charge  $+e$ ) are enclosed in a spherical cloud with total charge  $-2e$  and radius  $R$ . The oxygen ion is heavier so we imagine it sits motionless at the center of the cloud while the hydrogen ion moves. Let's suppose the hydrogen oscillates radially. What is its equation of motion? There is a repulsive force between the two positive ions of  $e^2/r^2$ . There is also an attractive force due to the portion of the electron cloud interior to the hydrogen ion's position.<sup>9</sup> If the electron cloud has a uniform charge density, the charge contained within  $r$  is  $-2er^3/R^3$ . Thus, the equation of motion is

$$m \frac{d^2 r}{dt^2} = \frac{e^2}{r^2} - \frac{2e^2 r}{R^3}$$

How do we analyze this equation? Consider: there is some equilibrium position  $r_0$  where the attractive and repulsive forces exactly balance. This is the place where  $d^2 r/dt^2 = 0$ , or

$$r_0 = \frac{R}{2^{1/3}} \tag{13.23}$$

Now we imagine the ion makes small excursions around this position. Let's write

$$r = r_0 + \delta r = r_0 \left( 1 + \frac{\delta r}{r_0} \right)$$

and imagine that  $\delta r \ll r_0$ . Then we can write the equation of motion as

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<sup>8</sup>We can picture a cloud for two reasons: according to quantum mechanics the electron wavefunctions are shells; even in classical mechanics, if we took a long-exposure photograph the electrons would look smeared out due to their motion.

<sup>9</sup>In analogy with gravity, the portion of the cloud outside the hydrogen ion's position produces no net force.

$$m \frac{d^2(\delta r)}{dt^2} = \frac{e^2}{r_0^2} \left(1 + \frac{\delta r}{r_0}\right)^{-2} - \frac{e^2}{r_0^2} \left(1 + \frac{\delta r}{r_0}\right)$$

In the second term we use Eq. (13.23) to substitute for  $R$ . What do we do now? If  $\delta r/r_0 \ll 1$  then we can make a Taylor series expansion of the factor in the first term:  $(1 + \delta r/r_0)^{-2} \approx (1 - 2\delta r/r_0)$ . This gives

$$\begin{aligned} m \frac{d^2(\delta r)}{dt^2} &\approx \frac{e^2}{r_0^2} \left(1 - 2\frac{\delta r}{r_0}\right) - \frac{e^2}{r_0^2} \left(1 + \frac{\delta r}{r_0}\right) \\ &\approx -\frac{3e^2}{r_0^3} \delta r \end{aligned}$$

This is the equation for simple harmonic motion. Thus, in our cloud model the hydrogen ion will experience sinusoidal oscillations with angular frequency

$$\omega = \left(\frac{3e^2}{mr_0^3}\right)^{1/2}$$

This matches what we obtained from dimensional analysis, up to a factor of  $\sqrt{3}$ . Now, the numerical factor is not necessarily precise, because our toy model does not account for all the details of the real chemical bond. Nevertheless, the model does contain some real physics (even if simplified), so perhaps it helps you believe the dimensional analysis.

### 13.4.4 Molecular Rotation

A molecule can also rotate, but perhaps at a different frequency than it vibrates. Here, dimensional analysis will not involve the electric force, but it will explicitly involve  $\hbar$  because that is the natural scale of angular momentum for atoms and molecules. In this case, the usual dimensional analysis yields

$$\omega \sim \frac{\hbar}{mr_0^2} \quad \Rightarrow \quad \lambda \sim 2\pi \frac{cmr_0^2}{\hbar}$$

Again the most important molecule in Earth's atmosphere is water, which has  $m = 18m_p$  and  $r_0 \approx 10^{-10}$  m, so the wavelength scale is

$$\lambda \sim 2\pi \times \frac{(3.0 \times 10^8 \text{ m s}^{-1}) \times (18 \times 1.67 \times 10^{-27} \text{ kg}) \times (10^{-10} \text{ m})^2}{1.05 \times 10^{-34} \text{ kg m}^2 \text{ s}^{-1}} \sim 5 \text{ mm}$$

This is in the microwave portion of the electromagnetic spectrum.<sup>10</sup>

<sup>10</sup>Incidentally, microwave ovens operate using molecular rotation. Microwave radiation induces water molecules in food to rotate; friction then disperses the rotational energy as heat.

### 13.4.5 Recap

As a rule of thumb, we can say that the four phenomena we have considered are important in different parts of the electromagnetic spectrum:

- Photoionization: ultraviolet
- Electron excitation: near-UV and visible
- Molecular vibration: infrared
- Molecular rotation: microwave

Molecular vibration is the main phenomenon that drives atmospheric absorption of infrared radiation, with molecular rotation kicking in at the long-wavelength end.

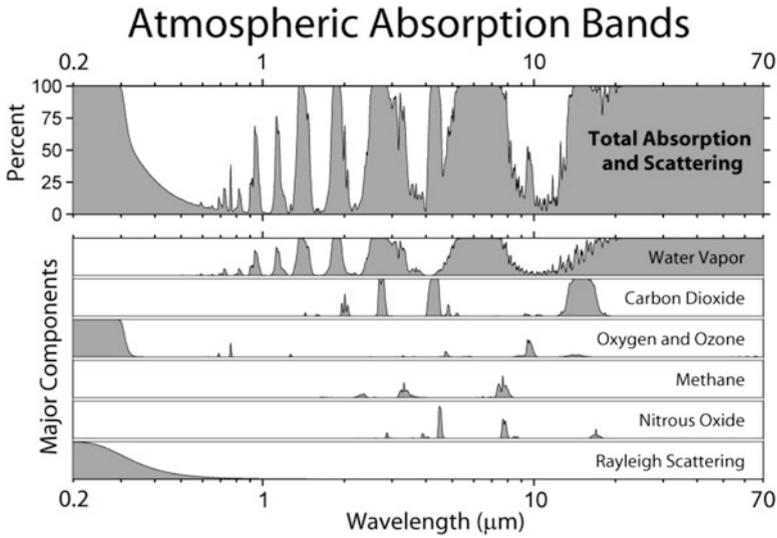
## 13.5 Greenhouse Effect and Climate Change

Atmospheric heating is commonly known as the **greenhouse effect**. We hear a lot about the greenhouse effect in connection with global warming and climate change, but the three phenomena are not exactly identical. The greenhouse effect is a well-understood physical effect that we know takes place on Earth. (As we have seen, Earth's surface would be measurably cooler without it.) Global warming occurs if the greenhouse effect strengthens with time, especially due to changes caused by humans. An increase in (average) temperature is part of a broader set of changes to the climate that can develop when the greenhouse effect is increased. Notwithstanding any political controversy about how we should respond to climate change, there is no doubt that the greenhouse effect is real.

### 13.5.1 Earth

Detailed studies of molecular rotation lead to the absorption spectra shown in Fig. 13.4. In Earth's atmosphere, water is responsible for most of the absorption between 1–8  $\mu\text{m}$  and beyond about 20  $\mu\text{m}$ . Carbon dioxide is important beyond 14  $\mu\text{m}$ , and in a band around 4  $\mu\text{m}$ . There is little natural greenhouse effect in an "atmospheric window" at 8–14  $\mu\text{m}$ . Using information like this, we can make a table showing how much different gases contribute to the greenhouse effect on Earth, and the degree to which people are concerned about their role in climate change. (Here ppm=parts per million) [2]

Notice that the contribution to the greenhouse effect is not dictated by abundance alone: molecular nitrogen and oxygen are very common, but they are poor absorbers at infrared wavelengths, so they contribute little to the greenhouse effect.



**Fig. 13.4** The *top panel* shows the percentage of light that is absorbed by Earth’s atmosphere as a function of wavelength, running from the visible into the infrared. The *bottom panels* show contributions from different gases (Credit: Robert A. Rohde/Global Warming Art)

Molecule	Abundance in Earth’s atmosphere	Contribution to greenhouse effect	Concern?
N <sub>2</sub>	78.1 %	–	No
O <sub>2</sub>	20.1 %	–	No
Ar	0.9 %	–	No
H <sub>2</sub> O	<1 %	60 %	Indirect
CO <sub>2</sub>	0.04 %	26 % ↗	Yes
CH <sub>4</sub> , N <sub>2</sub> O	2 ppm	6 %	Yes
O <sub>3</sub>	0.1 ppm	8 %	Yes

Public discussion of climate change focuses on carbon dioxide, whose atmospheric abundance is increasing due to human activity. The fact that atmospheric CO<sub>2</sub> is within our control means we can and should be concerned about it. Industrial gases containing carbon and fluorine and/or chlorine are also important. While they are not very abundant, they are efficient absorbers in the 8–14 μm atmospheric window so even trace amounts can strengthen the greenhouse effect.

What about water vapor? It contributes the majority of the greenhouse effect, but we have little direct control over its atmospheric abundance because of natural evaporation from the oceans. Water vapor plays a complicated role in climate change because of **feedback loops**. If the temperature rises, more water can evaporate, which would strengthen the greenhouse effect and thus raise the temperature even further. This would be an example of *positive feedback*, which reinforces any change that takes place. But there may be another effect: evaporating water can form

more clouds, which can reflect more sunlight, which would increase the albedo and thereby reduce the net energy absorbed from the Sun. That would create *negative feedback*, which acts against the prevailing trend and tries to regulate the temperature.

Additional factors include snow and ice (whose formation removes water vapor from the atmosphere and increases the albedo), and forests (which consume carbon dioxide and have low albedo), to name just two. In detail, climate change is a complicated process with many connected pieces that climatologists are working hard to understand fully. Nevertheless, there is solid evidence that the average temperature of Earth's surface and the amount of carbon dioxide in the atmosphere are on the rise. Where there are legitimate scientific questions, they mainly involve detailed predictions about exactly how the climate will change in the next century. Even there, though, the question is not *whether* the climate is changing; the question is really, "how much?"

### 13.5.2 Venus

Venus offers an example of what can happen when an atmosphere has strong positive feedback. Today the planet has a thick atmosphere dominated by carbon dioxide that warms the surface to about 740 K, far above the temperature of 185 K that we would expect if there were no atmospheric heating.<sup>11</sup> Also, the atmospheric pressure at the surface of Venus is about 90 times higher than at the surface of Earth (and comparable to the water pressure almost a kilometer down in the oceans on Earth).

Yet Venus and Earth were probably alike when they were young: the planets are similar in size, and they presumably formed by similar processes (see Sect. 19.4.2). What happened? Venus is closer to the Sun, so young Venus would have been a little warmer than young Earth. That would have caused any water on Venus to evaporate quickly, loading the atmosphere with water vapor and creating a strong greenhouse effect. As the temperature rose, carbon dioxide began to "bake out" of rocks,<sup>12</sup> which created yet more positive feedback and led to a **runaway greenhouse effect**. As the atmosphere filled with CO<sub>2</sub>, the lighter H<sub>2</sub>O molecules rose to the upper atmosphere, where they could be dissociated by ultraviolet light from the Sun. Then the light hydrogen atoms escaped from the atmosphere (see Sect. 12.3.3), leaving mostly carbon dioxide.

A tangential aspect of this story makes a testable prediction. A small fraction of the hydrogen in Venus's atmosphere was actually the isotope deuterium (D, whose nucleus contains one proton and one neutron). Being heavier, deuterium would evaporate less quickly than hydrogen, so the ratio of deuterium to hydrogen would

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<sup>11</sup>Venus's cloud cover creates a high albedo, so the predicted temperature is actually lower than Earth's even though Venus is closer to the Sun.

<sup>12</sup>On Earth, most of the carbon is locked up in the crust.

rise with time. Today the D/H ratio is about 100 times higher on Venus than on Earth, supporting the notion that Venus was once wet but the runaway greenhouse effect has made it such an inhospitable place [3,4].

## Problems

*In several problems, the albedo is relevant but not specified. Recall it must be between 0 and 1, and think about whether the specific value affects your conclusions. If it does, explain any assumptions you make.*

**13.1.** Is a blackbody actually black? Explain. How would you determine if a given object emits electromagnetic radiation like a blackbody?

**13.2.** In common usage, we sometimes call things “red hot” or “white hot.” Which is hotter? Explain.

**13.3.** Rank the following stars by luminosity:

Star	$R (R_{\odot})$	$T (T_{\odot})$
A	0.5	0.5
B	1.0	2.0
C	1.5	1.0
D	20.0	1.0

**13.4.** In the Sirius binary system (see Problem 4.4b), star A has a luminosity of  $25.4 L_{\odot}$  and the peak in its spectrum is at 292 nm, while star B has a luminosity of  $0.026 L_{\odot}$  with peak emission at 115 nm. Treating these stars as blackbodies, what physical properties of the stars can you determine from the information given? Calculate at least two properties for each star.

**13.5.** The star HD 209458 has mass  $M_s = 1.13 M_{\odot}$ , luminosity  $L_s = 1.61 L_{\odot}$ , and surface temperature  $T = 6,000$  K. It is orbited by a planet with mass  $M_p = 0.69 M_J$  and radius  $R_p = 1.35 R_J$  that lies 0.045 AU from the star (see Sect. 4.3.2). The size and mass of the planet suggest it is a gaseous planet like Jupiter, but it is so close to its star that it must be hotter than Jupiter. Would you expect hydrogen to be able to evaporate from the planet? Explain, and be quantitative.

**13.6.** Consider the planet orbiting the star HD 209458 (see data in Problem 13.5). Treating both objects as blackbodies, compute the brightness of the planet relative to the star at the following wavelengths:  $\lambda = 450$  nm (blue light),  $\lambda = 700$  nm (red light), and  $\lambda = 2.2 \mu\text{m}$  (infrared light). Consider both starlight reflected off the planet and blackbody light emitted by the planet itself. Explain your reasoning. If you want to detect light from this exoplanet, which of these three wavelengths is the best choice?

**13.7.** If Earth were much farther from the Sun, it would be too cold for water to remain liquid on the surface. If Earth were much closer, it would be too hot. The region in which water can remain liquid is called the “habitable zone.”

- (a) Taking Earth’s mean surface temperature to be 288 K, estimate the optical depth of Earth’s atmosphere. (Don’t worry about the detailed absorption spectrum; use our simple atmospheric heating model to obtain the average optical depth.)
- (b) Assuming the optical depth and albedo remain fixed, compute the inner and outer edges of the habitable zone for Earth around the Sun.

**13.8.** In 2005 astronomers discovered an object in the outer Solar System now known as the dwarf planet Eris (which played a major role in Pluto’s demotion to dwarf planet status). Here is a way to understand how Eris’s size was first determined.

- (a) Using the model from Sect. 13.2, find the luminosity  $L_p$  reflected from a planet with radius  $R_p$  at a distance  $D$  from the Sun.
- (b) When astronomers first discovered Eris they found it to be  $D = 97$  AU from the Sun and measured its reflected luminosity to be  $L_p = 5.8 \times 10^{11} \text{ J s}^{-1}$ . Use this information to derive a bound on Eris’s size.
- (c) More recently, astronomers measured the infrared light emitted by Eris and found that its spectrum peaks at a wavelength of 116 microns. Use this information to calculate the effective temperature, and state any assumptions that you have to make.
- (d) Use your results from (b) and (c) to determine the size of Eris. How does Eris compare in size with Pluto ( $R_{\text{Pluto}} = 1,153$  km)?

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