# Chapter 17
# Stellar Remnants

White dwarfs and neutron stars are dense objects left behind when low- and high-mass stars die (respectively). These objects have no ongoing fusion to generate the heat and pressure that normally counteract gravity, so the gas gets crushed to incredibly dense states that are quite unfamiliar to us on Earth. Essentially, gravity squeezes the gas until quantum mechanics pushes back. In this chapter we study white dwarfs in some detail and discuss neutron stars briefly.[1]

## 17.1 Cold, Degenerate Gas

In physics, the term **degeneracy** describes a situation in which multiple states have the same energy. If the lowest energy states in a gas are completely filled, we say the gas is degenerate. Our first task is to determine the equation of state for such a system. In Sect. 12.1.3 we derived a general expression for the pressure,

$$P = \frac{1}{3} \int p\, v\, n(\mathbf{p})\, \mathrm{d}\mathbf{p}$$

where $n(\mathbf{p})\, \mathrm{d}\mathbf{p}$ is the number density of particles with momentum between $\mathbf{p}$ and $\mathbf{p} + \mathrm{d}\mathbf{p}$ (now in vector form). Let's introduce the concept of **phase space** as an abstract space in which each possible state of a system is represented as a unique point. For a particle, phase space has six dimensions; in Cartesian coordinates the six dimensions are $(x, y, z, p_x, p_y, p_z)$, but it is possible to use other coordinate systems as well. Then $n$ is the number of particles per unit phase space volume. This quantity is more generally known as the **phase space distribution function**,

[1] This presentation follows part of the book *Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects* by Shapiro and Teukolsky [1], which gives considerably more detail.

C. Keeton, *Principles of Astrophysics: Using Gravity and Stellar Physics to Explore the Cosmos*, Undergraduate Lecture Notes in Physics, DOI 10.1007/978-1-4614-9236-8_17,

$$\mathscr{F} = \text{number of particles per unit phase space volume}$$

Thus, we can write a generalized version of the pressure integral as

$$P = \frac{1}{3} \int p\, v\, \mathscr{F}\, \mathrm{d}\mathbf{p} \tag{17.1}$$

In classical mechanics we treat phase space as continuous, but in quantum mechanics we think of it being discretized into cells whose 6-d volume is $h^3$, where $h$ is Planck's constant.

Let's consider gas that is "cold." We will specify what this means in a moment; for now it lets us say the system will arrange itself to minimize the total energy. In an ideal gas the energy is all kinetic (there is no interaction potential), so the lowest energy state has the particles packed as close as possible to the origin of momentum space. The fundamental limitation comes from the Pauli exclusion principle: for spin-1/2 particles like protons, neutrons, or electrons, the maximum number of particles in each phase space cell is two (spin-up and spin-down). The configuration that minimizes the total energy is a sphere centered on $\mathbf{p} = 0$ that has two particles in every cell out to some momentum threshold $p_F$, which is referred to as the **Fermi momentum**. In other words, the distribution function for the **Fermi sphere** is

$$\mathscr{F} = \begin{cases} 2h^{-3} & p < p_F \\ 0 & p > p_F \end{cases} \tag{17.2}$$

Since white dwarfs and neutron stars emerge from the cores of dying stars, it may not be clear whether they qualify as "cold." The key is how the thermal energy ($E_T$, Sect. 12.1.2) compares with the kinetic energy at the surface of the Fermi sphere ($E_F$). If $E_T \gg E_F$, then random thermal motions prevent particles from settling into the Fermi sphere, and the gas is not degenerate. If $E_T \ll E_F$, by contrast, then particles can settle into the Fermi sphere with little or no thermal fluctuations above $E_F$. In scenarios where $E_F$ is large, particles can have what we might consider to be a high temperature—they can even be relativistic—yet still qualify as "cold" in terms of the criterion for degeneracy. (See Problem 17.2 for quantitative examples.)

Given the distribution function, we can obtain the number density by integrating over all momenta:

$$n \equiv \int \mathscr{F}\, \mathrm{d}\mathbf{p} = \int_0^{p_F} \frac{2}{h^3}\, 4\pi p^2\, \mathrm{d}p = \frac{8\pi p_F^3}{3h^3} \tag{17.3}$$

Turning this around, we can express the Fermi momentum in terms of the density:

$$p_F = \left(\frac{3h^3 n}{8\pi}\right)^{1/3} = \left(3\pi^2 \hbar^3 n\right)^{1/3} \tag{17.4}$$

Finally, combining the phase space distribution function (17.2) with the relativistic expression (10.27) for $v$ lets us write the pressure integral as

$$P = \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^4 c^2}{(m^2 c^4 + p^2 c^2)^{1/2}} \, \mathrm{d}p \tag{17.5}$$

This is the general expression for the pressure of a cold, degenerate gas.

**Non-relativistic Case**

If the gas is non-relativistic, having $v \ll c$ and hence $p \ll mc$ lets us simplify the integrand to $p^4/m$. Then we can evaluate the integral:

$$P = \frac{8\pi}{3h^3 m} \int_0^{p_F} p^4 \, \mathrm{d}p = \frac{8\pi}{15h^3 m} p_F^5 = \frac{8\pi}{15h^3 m} \left( \frac{3h^3 n}{8\pi} \right)^{5/3} \tag{17.6}$$

where we use Eq. (17.3) for $p_F$. Collecting constants yields

$$P = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m} n^{5/3} \tag{17.7}$$

This is the *exact* equation of state for a non-relativistic, cold, degenerate gas. You may recall that we obtained this expression—up to the numerical factors—in Chap. 1 using dimensional analysis. Now we see where it comes from in detail.

**Ultra-relativistic Case**

If the gas is ultra-relativistic, having $p \gg mc$ modifies the analysis. In Problem 17.1 you can work through this case to derive the equation of state

$$P = \frac{(3\pi^2)^{1/3}}{4} \hbar \, c \, n^{4/3} \tag{17.8}$$

Again, we found this expression using dimensional analysis in Chap. 1, but now we see the details (and the numerical factors).

## 17.2 White Dwarfs

We have seen that the pressure of a degenerate gas depends on the number density of particles, and in the non-relativistic case it depends inversely on the particle mass. White dwarfs are typically composed of ionized carbon and/or oxygen (the byproducts of the helium burning that is the last stage of fusion in a low-mass star; see Sect. 16.3). Electrons outnumber nuclei and have smaller masses, so the pressure comes mainly from electrons even though the mass is mostly in nuclei. In this section we develop a model for a star composed of a cold, degenerate electron gas and compare the model predictions with the properties of observed white dwarfs.

### 17.2.1 Equation of State

The analysis in Sect. 17.1 gives the pressure in terms of the number density of particles, but in astrophysics we find it more convenient to work in terms of mass density. To relate the electron number density to the mass density (which is dominated by protons and neutrons, collectively known as nucleons), we use:

$$n_e = \left(\frac{\text{\# electrons}}{\text{nucleon}}\right)\left(\frac{\text{\# nucleons}}{\text{volume}}\right) = \frac{Z}{A}\,\frac{\rho}{m_p} \tag{17.9}$$

where $Z$ and $A$ are the atomic number and atomic mass, respectively, and for our purposes here it is adequate to say that all nucleons have mass $m_p$. In the non-relativistic case we can combine Eqs. (17.7) and (17.9) to write the pressure in terms of $\rho$ as

$$P = \frac{(3\pi^2)^{2/3}}{5}\,\frac{\hbar^2}{m_e}\left(\frac{Z}{A}\,\frac{\rho}{m_p}\right)^{5/3} \tag{17.10}$$

while in the relativistic case we instead use (17.8) to obtain

$$P = \frac{(3\pi^2)^{1/3}}{4}\,\hbar c\left(\frac{Z}{A}\,\frac{\rho}{m_p}\right)^{4/3} \tag{17.11}$$

In both cases, the pressure has the form of a **polytropic equation of state**,

$$P = K\,\rho^{1+1/n} \tag{17.12}$$

where the constant $K$ depends on the gas composition through $Z$ and $A$, and the **polytropic index** is

$$n = \begin{cases} 3/2 & \text{non-relativistic} \\ 3 & \text{relativistic} \end{cases}$$

(Please do not confuse this $n$ with number density. The notation is unfortunate, but it is so common that we will stick with it. It should be clear from context whether $n$ represents number density or polytropic index.)

### 17.2.2 Polytropic Stars

Let's return to the stellar structure equations in Sect. 16.2 and consider a star that has gravity and polytropic pressure, but no energy production or transport. This is a simplified but useful model for a white dwarf. (You can test the assumptions in Problem 17.2.) In this case, the key equation of stellar structure is the equation of hydrostatic equilibrium,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho \frac{GM(r)}{r^2} \quad \Rightarrow \quad \frac{r^2}{\rho} \frac{\mathrm{d}P}{\mathrm{d}r} = -GM(r)$$

Take the derivative of both sides, and use the mass equation $\mathrm{d}M/\mathrm{d}r = 4\pi r^2 \rho$:

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left( \frac{r^2}{\rho} \frac{\mathrm{d}P}{\mathrm{d}r} \right) = -4\pi G\rho \tag{17.13}$$

Now use the polytropic equation of state. It is convenient to introduce some new variables. Let $\theta$ be a dimensionless density variable defined by

$$\rho = \rho_c \theta^n \tag{17.14}$$

where $\rho_c$ is the central density. Then the polytropic equation of state is

$$P = K \rho_c^{1+1/n} \theta^{n+1}$$

Also, let $\xi$ be a dimensionless radial coordinate defined by

$$r = a\xi \quad \text{where} \quad a = \left[ \frac{n+1}{4\pi G} K \rho_c^{1/n-1} \right]^{1/2} \tag{17.15}$$

It is useful to invert the last equation and solve for $\rho_c$:

$$\rho_c = \left[ \frac{(n+1)K}{4\pi G} \right]^{n/(n-1)} a^{-2n/(n-1)} \tag{17.16}$$

The motivation for these choices becomes clear when we use the new variables in Eq. (17.13):

$$\frac{1}{a^2\xi^2} \frac{1}{a} \frac{\mathrm{d}}{\mathrm{d}\xi} \left[ \frac{a^2\xi^2}{\rho_c \theta^n} K \rho_c^{1+1/n} (n+1)\, \theta^n \frac{1}{a} \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right] = -4\pi G \rho_c \theta^n$$

Simplifying, and substituting for $a$ using Eq. (17.15), yields

$$\frac{1}{\xi^2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left( \xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right) = -\theta^n \tag{17.17}$$

This is a well-known differential equation known as the **Lane-Emden equation**. For most values of $n$ the solutions must be found numerically, but they are well studied. We can use simple boundary conditions: $\theta(0) = 1$ by construction from

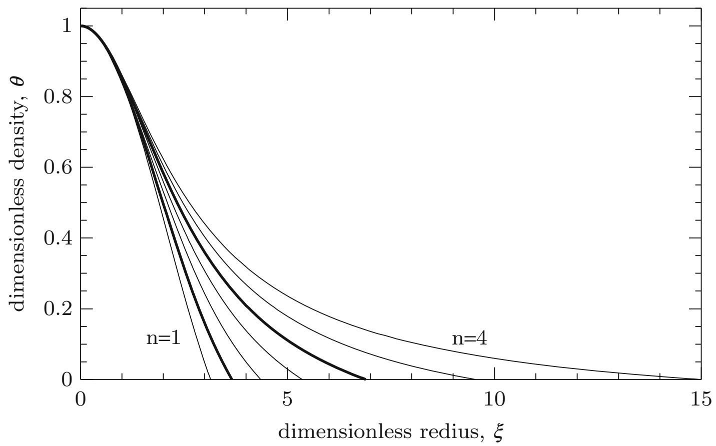


**Fig. 17.1** Solutions of the Lane-Emden equation. From left to right, the curves have $n = 1$–$4$ in steps of $1/2$. The solutions with $n = 3/2$ and 3 are highlighted

Eq. (17.14), and $\theta'(0) = 0$ because we want the density to be smooth at the center. Then there is a unique solution for each value of $n$, some of which are shown in Fig. 17.1. For all cases $n < 5$ the density goes to zero at some finite value of radius, which corresponds to the surface:

$$\text{surface:} \qquad \xi = \xi_1 \qquad R = a\xi_1 \tag{17.18}$$

We can tabulate some important properties of the solutions that are relevant for non-relativistic and relativistic white dwarfs:

| $n$ | $\xi_1$ | $\xi_1^2 \, \lvert\theta'(\xi_1)\rvert$ |
|---|---|---|
| 3/2 | 3.65 | 2.71 |
| 3 | 6.90 | 2.02 |

Let's use the Lane-Emden equation to understand some physical properties of the star. The total mass is

$$\begin{aligned} M &= \int_0^R 4\pi r^2 \rho \, \mathrm{d}r \\ &= 4\pi a^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n \, \mathrm{d}\xi \\ &= 4\pi a^3 \rho_c \int_0^{\xi_1} \xi^2 \left[ -\frac{1}{\xi^2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left( \xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right) \right] \mathrm{d}\xi \\ &= 4\pi a^3 \rho_c \, \xi_1^2 \, \lvert\theta'(\xi_1)\rvert \end{aligned}$$

In the third step we use Eq. (17.17) to replace $\theta^n$; then we evaluate the integral in terms of $\xi_1$ and $\theta'(\xi_1)$, which are tabulated above. Now replace $\rho_c$ using Eq. (17.16):

$$M = 4\pi a^{3-2n/(n-1)} \left[\frac{(n+1)K}{4\pi G}\right]^{n/(n-1)} \xi_1^2 \, |\theta'(\xi_1)|$$

Finally, use $a = R/\xi_1$ from Eq. (17.18):

$$M = 4\pi \, R^{(n-3)/(n-1)} \left[\frac{(n+1)K}{4\pi G}\right]^{n/(n-1)} \xi_1^{(3-n)/(n-1)} \, \xi_1^2 \, |\theta'(\xi_1)| \tag{17.19}$$

What was the point of all of this? We have obtained a relation between mass and radius for a polytropic star. The collection of constants looks a little messy, but the important scaling is

$$M \propto R^{(n-3)/(n-1)}$$

For the **non-relativistic** case we have $n = 3/2$ and hence

$$M \propto R^{-3} \quad \Leftrightarrow \quad R \propto M^{-1/3}$$

We found this scaling using dimensional analysis in Chap. 1, but now we have shown it rigorously. Physically, a star with more mass needs more pressure to balance gravity. If the star is supported by degeneracy pressure, electrons need to move closer together in order for $P$ to increase. Consequently, more massive stars must be smaller.

We can go further and fill in the constants. For the non-relativistic case we found

$$K = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m_e} \left(\frac{Z}{Am_p}\right)^{5/3} = 9.915 \times 10^{16}\,\text{kg}^{-2/3}\,\text{m}^4\,\text{s}^{-2} \times \left(\frac{Z}{A}\right)^{5/3}$$

We listed properties of the Lane-Emden solutions in the table above. Putting everything together, we obtain for the non-relativistic case

$$M = 21.5\,M_\odot \times \left(\frac{R}{10^4\,\text{km}}\right)^{-3} \times \left(\frac{Z}{A}\right)^5 \quad \text{(non-relativistic)} \tag{17.20}$$

A carbon/oxygen white dwarf has $Z/A = 0.5$, yielding

$$M = 0.67\,M_\odot \times \left(\frac{R}{10^4\,\text{km}}\right)^{-3}$$

For comparison, the **ultra-relativistic** case has $n = 3$ and hence

$$M \propto R^0 = \text{constant}$$

Filling in the numerical factors yields

$$K = \frac{(3\pi^2)^{1/3}}{4}\,\hbar c \left(\frac{Z}{Am_p}\right)^{4/3} = 1.232 \times 10^{10}\,\text{kg}^{-1/3}\,\text{m}^3\,\text{s}^{-2} \times \left(\frac{Z}{A}\right)^{4/3}$$

so the mass is

$$M = 5.75\,M_\odot \times \left(\frac{Z}{A}\right)^2 \quad \text{(ultra-relativistic)} \tag{17.21}$$

This analysis reveals that *all relativistic polytropic stars have the same mass*, up to the composition-dependent factor $Z/A$. For $Z/A = 0.5$ as appropriate for a carbon/oxygen white dwarf, the mass is

$$M = 1.44\,M_\odot \tag{17.22}$$

We can connect the non-relativistic and ultra-relativistic cases with the following physical picture. As a white dwarf becomes more massive, it shrinks according to $R \propto M^{-1/3}$. The rising density increases the Fermi momentum (Eq. 17.4) and makes the system increasingly relativistic. Once the star reaches $1.44\,M_\odot$ (for $Z/A = 0.5$) it is ultra-relativistic, and it cannot get any more massive and still be supported by electron degeneracy pressure. The upper limit on the mass of a white dwarf is called the **Chandrasekhar limit** after Subramanyan Chandrasekhar, who made the theoretical prediction in 1930.

If a white dwarf exceeds the Chandrasekhar limit, it will explode as a type Ia supernova. These are objects that cosmologists have used as standard candles to chart the expanding universe (see Chap. 18).

### *17.2.3 Testing the Theory*

To recap, our detailed analysis of white dwarfs has yielded three key conceptual points:

1. A white dwarf the mass of the Sun is about the size of Earth.
2. More massive white dwarfs are smaller, with $M \propto R^{-3}$.
3. There is a maximum allowed mass for an object supported by electron degeneracy pressure.

How can we test these predictions?

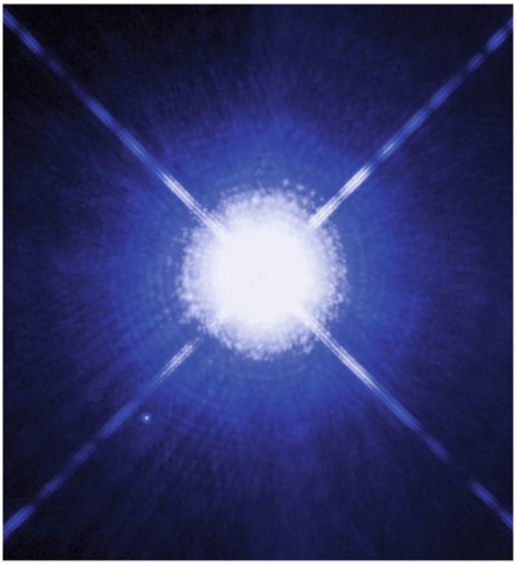

**Fig. 17.2** Hubble Space Telescope image of Sirius A (the bright one) and its white dwarf companion Sirius B (the faint spot in the lower left). Sirius A appears big because it is overexposed; its size is not actually resolved. The diagonal spikes are caused by diffraction within the telescope (Courtesy: NASA, ESA, H. Bond and E. Nelan (STScI), M. Barstow and M. Burleigh (Univ. of Leicester), and J. Holberg (Univ. of Arizona))

White dwarfs were seen long ago but not recognized as particularly unusual until the early twentieth century. One is a dim star discovered by William Herschel in 1783 in the triple star system 40 Eridani [2].[2] Another is in a binary system with Sirius, the brightest star in our night sky. Between 1834 and 1844, Friedrich Bessel observed that Sirius moves as if it has a companion [3]. Christian Peters used the motion to infer the orbit in 1851, and Alvan Clark identified the companion itself in 1862 [4]. Today the Hubble Space Telescope can easily resolve the two stars (see Fig. 17.2). The orbital motion reveals that the bright star has mass $M_A = 2.0\,M_\odot$ while the faint star has mass $M_B = 1.0\,M_\odot$ [5]. The small difference in mass is surprising given the large difference in luminosity (a factor of nearly 1,000).

Even more striking are the spectral properties of these stars. In 1910, Henry Norris Russell, Edward Pickering, and Williamina Fleming used spectral classification to realize that 40 Eridani B lies far below the main sequence in the HR diagram [6, 7]. In 1915, Walter Adams discovered that the spectrum of Sirius B is very similar to that of Sirius A despite the large difference in luminosity [8]. Now we know that the faint star is actually hotter than the bright star ($T_B \approx 25{,}000$ K vs. $T_A \approx 10{,}000$ K [5]). According to the Stefan-Boltzmann law (Eq. 13.1), an object with a high temperature but low luminosity must be very small (see Problem 13.4).

Today many more white dwarfs are known, although it is still challenging to measure masses and radii precisely enough to test theoretical predictions. (For a long time, one obstacle was knowing distances well enough to convert flux to luminosity.

[2]The brightest stars in our sky have individual names, but most stars are labeled by the name of the constellation in which they appear on the sky, and a letter or number that indicates how they rank among stars in that constellation.

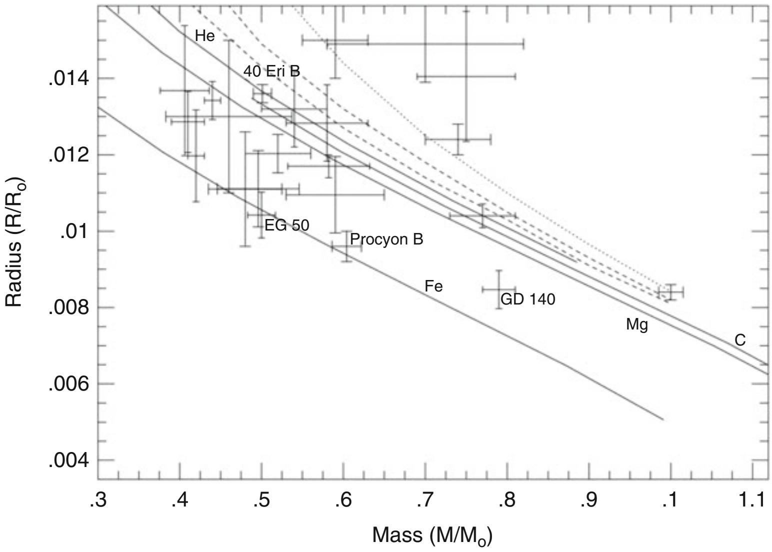


**Fig. 17.3** White dwarf mass/radius relation. The curves show theoretical relations for different compositions and assumptions about the star's atmosphere. The points with errorbars show measurements of real white dwarfs (Credit: Provencal et al. [9]. Reproduced by permission of the AAS)

In the 1990s the Hipparcos satellite measured precise distances for a large sample of nearby stars, including some white dwarfs.) Here is a brief summary of the analysis.

**Radius.** This is mainly determined from the star's luminosity (inferred from its flux and distance) and effective temperature (from a spectrum). Using the Stefan-Boltzmann law (Eq. 13.1), we can write

$$L = 4\pi R^2 \sigma T^4 \quad \Rightarrow \quad R = \frac{1}{T^2}\left(\frac{L}{4\pi\sigma}\right)^{1/2} \tag{17.23}$$

**Mass.** Three approaches have been used, depending on what information is available.

1. *Binary star.* If the white dwarf is in a binary system, the orbital motion yields reliable masses (see Sect. 4.2).
2. *Surface gravity.* The width of spectral lines depends on the strength of gravity at the star's surface, $g = GM/R^2$. Increasing $g$ raises the gas pressure and density, leading to more frequent collisions that perturb atomic and molecular energy levels. Measuring $g$ from the star's spectrum and $R$ from (17.23) makes it possible to infer the mass.
3. *Gravitational redshift.* Photons emitted by a star lose some energy as they climb out of the gravitational potential well, and thus shift to slightly longer

wavelengths (see Sect. 10.3.4). This gravitational redshift offers another way to determine the strength of gravity at the star's surface, which can be combined with the radius to find the mass.

Observational data are compared with theoretical predictions in Fig. 17.3. Many of the errorbars are fairly large, indicating the challenge in making the measurements. Nevertheless, the agreement with theory is quite good, especially for the predictions that white dwarfs are comparable in size to Earth and more massive white dwarfs are smaller. There is some scatter among the observed white dwarfs, which can be interpreted as evidence that the stars have different internal compositions and/or different atmospheres (see [9]). The bottom line is that objects made of the novel state of matter known as degenerate gas do exist, and we understand their properties.

## 17.3 Neutron Stars and Pulsars

Neutrons stars are also supported by degeneracy pressure, but from neutrons rather than electrons. The difference is important because degeneracy pressure depends on the *number* density of particles. Since they are so much more massive, neutrons must have a higher *mass* density than electrons to create a comparable number density and hence pressure. Neutron stars are therefore much smaller than white dwarfs—typically about 10 km in radius (see Problem 17.6).

Being so compact, neutron stars have very strong gravity, which means general relativity is needed for any detailed analysis. Furthermore, the neutrons are so close together that interactions between them cannot be ignored, which means the equation of state for dense nuclear matter plays a role as well. These two facts make neutron stars more complicated in detail than white dwarfs, but many of the key conceptual ideas are similar. (See [1] for details.)

Observationally, we study neutrons stars primarily as **pulsars**. Neutron stars tend to rotate very rapidly. (If a spinning star shrinks, it must spin faster to conserve angular momentum.) They also have strong magnetic fields (another consequence of having shrunk), which causes them to emit strong beams of radio waves from their magnetic poles. If the magnetic poles are not aligned with the spin axis, the radio beams sweep through space like beams from a lighthouse, and if one reaches Earth we detect periodic pulses of radio waves. Hence the name pulsar.

Pulsars are observed to have spin periods in the range of seconds down to milliseconds.[3] Moreover, they are extremely regular, and the periods can be measured incredibly precisely; for example, the binary pulsar system PSR J0737−3039 has an orbital period of 0.10225156248 day, with an uncertainty of $\pm 5$ in the last digit [10]. Such precision is rare in astronomy, and it makes pulsars important tools for testing

[3] A whole star spinning in a few milliseconds—wow!

general relativity. Pulsar discoveries have played a role in two Nobel Prizes: for Antony Hewish in 1974, and Russell Hulse and Joseph Taylor Jr. in 1993. Using pulsar timing to test relativity continues to be the focus of exciting research (e.g., [10, 11]).

## Problems

**17.1.** Evaluate the pressure integral equation (17.5) in the ultra-relativistic case to derive the equation of state (17.8).

**17.2.** We have assumed the gas in a white dwarf is cold and degenerate, while the gas inside the Sun is not degenerate. Measurements suggest the assumptions are reasonable (see Fig. 17.3), but we should check the numbers. As discussed in Sect. 17.1, the key is how the thermal and Fermi energies compare. For each of the following cases, calculate $E_T$ and $E_F$ and determine whether the gas is degenerate.

| | $T_c$ (K) | $\rho_c$ (kg m$^{-3}$) | Core composition |
|---|---|---|---|
| (a) Sun today | $1.6 \times 10^7$ | $1.5 \times 10^5$ | ~50/50 mix of H/He |
| (b) Sun on giant branch | $2.7 \times 10^7$ | $5.1 \times 10^7$ | He |
| (c) 5 $M_\odot$ star on giant branch | $1.1 \times 10^8$ | $7.7 \times 10^6$ | He |
| (d) 0.6 $M_\odot$ white dwarf | $1.1 \times 10^7$ | $1.1 \times 10^9$ | C/O |

**17.3.** It is possible to analyze an electron gas with a finite temperature. In this case the distribution function has the form

$$\mathscr{F} = \frac{2}{h^3} \frac{1}{e^{(E-E_F)/kT} + 1} \tag{17.24}$$

where $E_F$ is the Fermi energy. If the gas is non-relativistic, the number density and pressure are given by

$$n = \int \mathscr{F} \, d\mathbf{p} \quad \text{and} \quad P = \frac{1}{3m} \int p^2 \mathscr{F} \, d\mathbf{p} \tag{17.25}$$

(a) Plot $\mathscr{F}$ as a function of $E/E_F$ for $kT/E_F = 0.01, 0.1,$ and $0.2$.
(b) Explain qualitatively whether a gas with $T > 0$ will have *higher* or *lower* pressure than a gas with $T = 0$ and the same density.
(c) Change integration variables in Eq. (17.25) to show that

$$n = \frac{4\pi}{h^3} (2m)^{3/2} E_F^{3/2} \int \frac{x^{1/2} \, dx}{e^{(x-1)E_F/kT} + 1}$$

Find an analogous expression for $P$. Then show that the equation of state can be written as

$$P = \frac{(3\pi^2)^{2/3}}{5}\,\frac{\hbar^2}{m}\,n^{5/3} \times \frac{I_3}{I_1^{5/3}} \tag{17.26}$$

where

$$I_1 = \frac{3}{2}\int \frac{x^{1/2}\,\mathrm{d}x}{\mathrm{e}^{(x-1)E_F/kT}+1} \quad \text{and} \quad I_3 = \frac{5}{2}\int \frac{x^{3/2}\,\mathrm{d}x}{\mathrm{e}^{(x-1)E_F/kT}+1}$$

(d) Equation (17.26) differs from the zero temperature case (Eq. 17.7) by the factor $I_3/I_1^{5/3}$. Use numerical integration to compute this factor for $kT/E_F = 0.01$, 0.1, and 0.2. Is the result consistent with your answer to part (b)? Would you expect a small but finite temperature to significantly change the results from this chapter?

**17.4.** In Problem 12.6 we studied a uniform density star (also see Problems 14.3 and 16.4). The same model can be treated using the framework of Sect. 17.2.2.

(a) What is the appropriate value of the polytropic index $n$ for this model?
(b) Express $P(r)$ from Problem 12.6 in terms of the scaled variables as $\theta(\xi)$.
(c) Verify that your expression solves the Lane-Emden equation (17.17).

**17.5.** In the text we studied non-relativistic white dwarfs under the assumption $p \ll mc$. Now let's see whether the derived star properties are consistent with that assumption.

(a) Following our non-relativistic analysis in Sect. 17.2.2, find the Fermi momentum at the center of the star (i.e., using the central density), and compute the ratio $p_F/(m_e c)$. Work symbolically; express your answer in terms of the star's mass $M$ and composition factor $Z/A$, along with constants.
(b) Evaluate the ratio for a carbon/oxygen white dwarf with $M = 0.7\,M_\odot$, and again for $M = M_\odot$.

**17.6.** In the text we computed the mass of an ultra-relativistic degenerate star. Here is how to estimate the size.

(a) Following our relativistic analysis in Sect. 17.2.2, derive an expression for the star's radius in terms of the Fermi momentum $p_F$, the composition factor $Z/A$, and constants.
(b) For an ultra-relativistic white dwarf, we expect $p_F = \eta m_e c$ where $\eta \gg 1$. Compute the star's radius in terms of $\eta$, and evaluate the result using $\eta \sim 10$. Assume a carbon/oxygen composition.
(c) To analyze a neutron star we should really use general relativity, but let's forge ahead with our Newtonian approach. Repeat part (b) assuming $p_F = \eta m_n c$ and $\eta \sim 10$. (What should you use for $Z/A$?)

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