Chapter 16
Stellar Structure and Evolution

In previous chapters we examined physical processes that occur near a star's surface (atomic excitation, ionization, absorption) and in the interior (nuclear fusion). Now we unite them in detailed models of stars. We use the models to analyze the structure of stars during the main stage of life when they burn hydrogen to generate the heat and pressure that balance gravity. We then consider what happens when the hydrogen fuel runs out. As we will see, old stars begin burning heavier nuclei and working their way up the periodic table of the elements. How far a star gets depends on its mass: stars with masses below about 8 $M_\odot$ reach carbon and oxygen before experiencing a relatively meek death; stars with masses above about 8 $M_\odot$, by contrast, create all the heavier elements and then literally go out with a bang.

16.1 Energy Transport

Before we can build stellar models, we need to think about how energy produced in the core can be transported to the surface, to be released as light. In Chap. 13 we saw one mechanism for moving energy: with **radiation**, energy is carried by light that can be absorbed by an atom or molecule and then reradiated in a different direction. Now we consider two other mechanisms that act in dense gas. With **conduction**, heat moves on microscopic scales by collisions between particles. With **convection**, heat travels across macroscopic scales by bulk motion of gas. Let's examine each in turn.

16.1.1 Conduction

Consider a box of gas (for now, one small enough that we can neglect the effects of gravity). In equilibrium, it has the same temperature throughout. Suppose we heat one end of the box just a little—not enough to substantially change the density

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and pressure, but enough to make the particles jiggle a little faster. As these faster particles fly around, they bump into other particles in the box and transfer some of their kinetic energy. In this way collisions can allow the extra energy (heat) to travel across the box. Can we find an equation to describe this process, specifically to describe how the temperature changes with time at different locations in the box?

The temperature can change only if T is not spatially uniform—heat can only "flow" from a hotter region to a colder region—so $\partial T/\partial t$ must be related to some spatial derivative of T. Can we have

$$\frac{\partial T}{\partial t} \propto \frac{\partial T}{\partial x} \quad ?$$

No, by symmetry. If this were true, the sign of $\partial T/\partial t$ would depend on whether we heat the left end or the right end of the box. That cannot be right! What about

$$\frac{\partial T}{\partial t} \propto \left|\frac{\partial T}{\partial x}\right| \quad ?$$

No again. Consider $T \propto x^2$. The temperature is non-uniform so there ought to be heat flow. But with this hypothesis we would have $\partial T/\partial t = 0$ at $x = 0$. That does not make sense.

If first derivatives do not work, what about

$$\frac{\partial T}{\partial t} \propto \frac{\partial^2 T}{\partial x^2} \quad ?$$

This makes sense physically: if we raise the temperature anywhere, $\partial^2 T/\partial x^2$ will be nonzero, and indeed it will be positive away from the source of heat. This is in fact the right dependence, so let's specify the proportionality constant, κ, and generalize to three dimensions:

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \kappa \nabla^2 T \tag{16.1}$$

This is the **heat equation**, and it has the general form of a **diffusion equation**. The coefficient κ is called the **thermal diffusivity**. By dimensional analysis, we must have

$$\kappa = [L^2 T^{-1}] \quad \Rightarrow \quad \kappa \sim \ell v \sim \frac{\ell^2}{\tau}$$

for some characteristic length ℓ and velocity v, which are presumably the mean free path and the typical particle velocity, respectively. Alternatively, we can write κ in terms of $\tau \sim \ell/v$, which is the typical time between collisions.

What is the time scale $\Delta t_{\rm diff}$ for heat to diffuse over some distance ΔL? If we suppose that some amount of heat ΔT moves, then we can approximate $\partial T/\partial t \sim \Delta T/\Delta t_{\rm diff}$, and we can approximate $\nabla^2 T \sim \Delta T/(\Delta L)^2$. Then the heat equation yields

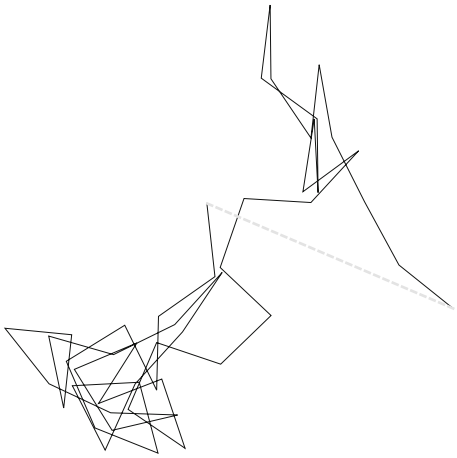

Fig. 16.1 The *solid line* shows a random walk with $N = 50$ steps of equal length. The *dashed gray line* connects the start and end points (the walk can proceed in either direction)

$$\frac{\Delta T}{\Delta t_{\text{diff}}} \sim \kappa \frac{\Delta T}{(\Delta L)^2} \quad \Rightarrow \quad \Delta t_{\text{diff}} \sim \frac{(\Delta L)^2}{\kappa} \sim \frac{(\Delta L)^2}{\ell^2/\tau} \sim \tau \left(\frac{\Delta L}{\ell}\right)^2 \tag{16.2}$$

For comparison, how long does it take for a particle to move *freely* across a distance ΔL? This is called the "crossing time":

$$\Delta t_{\text{cross}} \sim \frac{\Delta L}{v} \sim \frac{\Delta L}{\ell/\tau} \sim \tau \frac{\Delta L}{\ell}$$

If the step size (ℓ) is small compared with the size of the box (ΔL), the diffusion time may be *much* longer than the crossing time. Why? Collisions have random directions, so particles do not travel straight across the box. Rather, they follow meandering trajectories known as random walks.

Random Walk

Suppose a particle starts at the origin and takes a series of steps that have fixed length ℓ but random directions. An example of a random walk with 50 steps is shown in Fig. 16.1. In general, after N steps the particle's position is

$$\mathbf{X} = \mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_N = \sum_{i=1}^{N} \mathbf{x}_i$$

The square of the net distance from the starting point (the origin) is

$$X^2 = \sum_{i,j} \mathbf{x}_i \cdot \mathbf{x}_j = \sum_{i=j} |\mathbf{x}_i|^2 + \sum_{i \neq j} \mathbf{x}_i \cdot \mathbf{x}_j = N\,\ell^2 + \sum_{i \neq j} \mathbf{x}_i \cdot \mathbf{x}_j$$

Here we separate the sum into a piece in which the indices match and a piece in which they differ, and then use the fact that the step size is fixed so $|\mathbf{x}_i| = \ell$ for all i.

We are interested in the typical distance traveled after N steps, which we define to be the root mean square distance $X_{\rm rms} = \sqrt{\langle X^2 \rangle}$. The average has the form

$$\left\langle X^2 \right\rangle = N\,\ell^2 + \sum_{i \neq j} \left\langle \mathbf{x}_i \cdot \mathbf{x}_j \right\rangle$$

If the steps are independent of one another, then $\left\langle \mathbf{x}_i \cdot \mathbf{x}_j \right\rangle = \left\langle \mathbf{x}_i \right\rangle \cdot \left\langle \mathbf{x}_j \right\rangle$. If the directions are random, then $\left\langle \mathbf{x}_i \right\rangle = 0$ because the particle is equally likely to go right or left. The net result is

$$\left\langle X^2 \right\rangle = N\,\ell^2$$

Returning to the diffusion time, we can now estimate the number of steps needed to cross a distance ΔL. We set $X_{\rm rms} = \Delta L$ to find

$$N = \left(\frac{\Delta L}{\ell} \right)^2$$

This is the typical number of steps; the actual number may be somewhat larger or smaller for individual random walks. If each step takes time τ, then the overall diffusion time is

$$\Delta t_{\rm diff} \sim N\tau \sim \tau \left(\frac{\Delta L}{\ell} \right)^2$$

as in Eq. (16.2).

Example: What Is the Thermal Diffusivity of Earth's Atmosphere?

In Chap. 12 we estimated that nitrogen molecules in Earth's atmosphere have a mean free path of $\ell \sim 1.4 \times 10^{-7}$ m and a typical speed of $v \sim 5.1 \times 10^2\,{\rm m\,s^{-1}}$. Together these yield an estimate for the thermal diffusivity of

$$\kappa \sim \ell\, v \sim 7 \times 10^{-5}\,{\rm m^2\,s^{-1}}$$

For comparison, the laboratory value is [1]

$$\kappa = 1.9 \times 10^{-5}\,{\rm m^2\,s^{-1}}$$

What is the time scale for heat to diffuse across a room that is $\Delta L = 10$ m across?

$$\Delta t_{\rm diff} \sim \frac{(\Delta L)^2}{\kappa} \sim \frac{(10\,{\rm m})^2}{1.9 \times 10^{-5}\,{\rm m^2\,s^{-1}}} \sim 5.3 \times 10^6\,{\rm s} \sim 61\ {\rm days}$$

Heat does not diffuse very quickly! Apparently we need to find another mechanism that can transport heat more effectively.

16.1.2 Convection

Adding a lot of heat to the box can induce mechanical forces that generate *bulk motion* in the gas. To analyze convection, we imagine taking a small bubble of gas from height z, moving it to height $z + \Delta z$, and asking what will happen next (also see [2, 3]). If the bubble will fall back to its starting point, then small changes (such as the formation of a bubble) tend to damp out and the gas is stable. If the bubble will continue to rise, however, then small changes can grow into larger changes and convection can begin spontaneously.[1] Our goal in this section is to derive a condition under which gas is unstable to convection.

First, we need to recall some thermodynamics. **Specific heat** quantifies the amount of energy needed to raise the temperature of a substance. There are actually two specific heats: one if we increase the temperature while holding the pressure fixed, and another if we hold the volume fixed:

$$C_P = \left.\frac{\mathrm{d}Q}{\mathrm{d}T}\right|_P \qquad \text{and} \qquad C_V = \left.\frac{\mathrm{d}Q}{\mathrm{d}T}\right|_V$$

We define the ratio to be the **adiabatic index**,

$$\gamma = \frac{C_P}{C_V} \tag{16.3}$$

A non-relativistic ideal gas has $\gamma = 5/3$ for monatomic particles and $\gamma = 7/5$ for diatomic particles (such as N_2, O_2, etc.). A relativistic ideal gas has $\gamma = 4/3$. The adiabatic index characterizes a process in which no heat flows into or out of a system, which is a reasonable approximation for any "slow" thermodynamic process. Carroll and Ostlie [2] show that such a process is described by the **adiabatic equation of state**,

$$PV^\gamma = \text{constant} \quad \Rightarrow \quad P = K\rho^\gamma \tag{16.4}$$

where K is a constant.

Now consider an ideal gas whose density, temperature, and pressure vary with height. At height z the gas is described by T, P, and ρ, while at height $z + \Delta z$ it has $T + \Delta T$, $P + \Delta P$, and $\rho + \Delta\rho$. We can write

$$\Delta T = \frac{\mathrm{d}T}{\mathrm{d}z}\Delta z \qquad \Delta P = \frac{\mathrm{d}P}{\mathrm{d}z}\Delta z \qquad \Delta\rho = \frac{\mathrm{d}\rho}{\mathrm{d}z}\Delta z \tag{16.5}$$

From the ideal gas law, we know

[1] The situation is similar to water boiling, although that case is slightly more complicated because it involves the formation of *air* bubbles.

$$\rho = \bar{m} n = \frac{\bar{m} P}{k T}$$

where $\bar{m}$ is the average particle mass. Take the logarithm:

$$\ln \rho = \ln \bar{m} + \ln P - \ln k - \ln T$$

Take the derivative with respect to z, and multiply by Δz:

$$\frac{1}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}z} \Delta z = \frac{1}{P} \frac{\mathrm{d}P}{\mathrm{d}z} \Delta z - \frac{1}{T} \frac{\mathrm{d}T}{\mathrm{d}z} \Delta z$$

Using Eq. (16.5), we can write this as

$$\frac{\Delta \rho}{\rho} = \frac{\Delta P}{P} - \frac{\Delta T}{T} \tag{16.6}$$

Suppose we take a small bubble of gas from height z and move it to height $z + \Delta z$. It will adjust to some new density, pressure, and temperature $\rho + \delta\rho$, $P + \delta P$, and $T + \delta T$, which may or may not match the surrounding medium. The internal and external pressure forces must balance, so in fact $\delta P = \Delta P$. If we move the bubble quickly, there is no time for heat to flow into or out of the bubble and the enclosed gas must behave adiabatically:

$$\delta P = K \gamma \rho^{\gamma - 1} \delta\rho \quad \Rightarrow \quad \frac{\delta P}{P} = \gamma \frac{\delta \rho}{\rho} \tag{16.7}$$

If the density inside the bubble is lower than the density outside, the bubble will be buoyant and want to continue rising. The condition for buoyancy is therefore $\delta\rho < \Delta\rho$ or (using (16.6) and (16.7))

$$\frac{1}{\gamma} \frac{\delta P}{P} < \frac{\Delta P}{P} - \frac{\Delta T}{T}$$

Using $\delta P = \Delta P$ and rearranging yields

$$\Delta T < \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \Delta P \quad \Rightarrow \quad \frac{\mathrm{d}T}{\mathrm{d}z} < \frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{\mathrm{d}P}{\mathrm{d}z}$$

(To obtain the last expression we divide through by Δz and then turn the differentials into derivatives.) In general, T and P both decrease with height, so the derivatives are negative. Switching to absolute values so we work with positive quantities, we have the following condition for buoyancy:

$$\left| \frac{\mathrm{d}T}{\mathrm{d}z} \right| > \frac{\gamma - 1}{\gamma} \frac{T}{P} \left| \frac{\mathrm{d}P}{\mathrm{d}z} \right| \tag{16.8}$$

An ideal gas in hydrostatic equilibrium has $P = nkT$ and $\mathrm{d}P/\mathrm{d}z = -g\rho = -g\bar{m}n$ (see Eq. 12.13). In this case, Eq. (16.8) becomes

$$\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right| > \frac{\gamma - 1}{\gamma}\,\frac{\bar{m}}{k}\,g \tag{16.9}$$

We learn that gas is unstable to convection if the temperature gradient exceeds a threshold determined by the pressure gradient, which in turn depends on the acceleration due to gravity.

Example: Earth

At Earth's surface, $g = 9.80\,\mathrm{m\,s^{-2}}$. For an ideal gas of molecular nitrogen (N_2), $\gamma = 7/5$ and $m = 28\,m_p$. With these numbers, the right-hand side of Eq. (16.9) becomes

$$\frac{\gamma - 1}{\gamma}\,\frac{m}{k}\,g = \frac{2/5}{7/5}\,\frac{28 \times 1.67 \times 10^{-27}\,\mathrm{kg}}{1.38 \times 10^{-23}\,\mathrm{kg\,m^2\,s^{-2}\,K^{-1}}} \times 9.80\,\mathrm{m\,s^{-2}} = 9.5 \times 10^{-3}\,\mathrm{K\,m^{-1}}$$

In the lower part of Earth's atmosphere, the temperature varies roughly linearly with altitude, ranging from about 288 K at the ground to 210 K at an altitude of 15 km $= 1.5 \times 10^4$ m. This corresponds to a temperature gradient of

$$\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right| = \frac{78\,\mathrm{K}}{1.5 \times 10^4\,\mathrm{m}} = 5.2 \times 10^{-3}\,\mathrm{K\,m^{-1}}$$

The temperature gradient is smaller than the value required for convective instability, so Earth's atmosphere is not convectively unstable (at sea level). This does not mean that convection cannot occur; it just means that convection will not begin spontaneously.

16.2 Stellar Models

Now we are ready to assemble the pieces and write down a set of equations that describe a model star. For simplicity, we assume the star is spherically symmetric and static (time-independent), so the density, pressure, temperature, etc. are functions only of r. Stars may not be perfectly spherical, especially if they rotate rapidly, and they are not truly static, but during the bulk of their lives they change slowly; so a spherical, static model is not a bad place to start.

16.2.1 Equations of Stellar Structure

The first set of equations describe how mass, luminosity, pressure, and temperature vary with radius. The equation relating mass to density is one we have seen before:

$$\text{enclosed mass} \qquad \frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2 \rho \tag{16.10}$$

We can write an equation for luminosity that has a similar form if we define ϵ to be the energy generation rate per unit mass (in units of J s^{-1} kg^{-1}, for example). Then the energy output (or luminosity) from a spherical shell of radius r and thickness $\mathrm{d}r$ is $\mathrm{d}L = 4\pi r^2 \rho \epsilon \, \mathrm{d}r$, so the differential equation we need is

$$\text{energy generation} \qquad \frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho \epsilon \tag{16.11}$$

The equation for pressure comes from hydrostatic equilibrium (12.13):

$$\text{hydrostatic equilibrium} \qquad \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)\rho}{r^2} \tag{16.12}$$

Finally, the equation for temperature depends on how energy is transported. In a convective regime, the temperature gradient will match the right-hand side of Eq. (16.9)[2]:

$$\text{convection} \qquad \frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{\gamma - 1}{\gamma} \frac{\bar{m}}{k} \frac{GM(r)}{r^2} \tag{16.13}$$

In a radiative regime, the temperature gradient depends on the **opacity** of the gas, $\bar{\kappa} = 1/\ell\rho$. If the opacity is high, light cannot transport energy very efficiently, so heat remains trapped and the temperature gradient is large. Conversely, if the opacity is low, light is able to move heat energy and the temperature gradient remains small. Carroll and Ostlie [2] and Maoz [3] derive the differential equation for temperature in a radiative regime:

$$\text{radiation} \qquad \frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{16\sigma} \frac{\bar{\kappa}\rho}{T^3} \frac{L(r)}{4\pi r^2} \tag{16.14}$$

To supplement the four differential equations, we need equations of state that relate the pressure, energy generation rate, and opacity to the density, temperature, and composition of the gas. These are based on the gas physics, nuclear physics, thermodynamics, and other principles we have studied since Chap. 12. As an illustration, let's consider the pressure. The net gas pressure is the sum of partial pressures from all the constituents:

[2]Recall from Sect. 16.1.2 that this equation applies to an ideal gas in hydrostatic equilibrium.

$$P_{\text{gas}} = \sum_i n_i \, kT = n_{\text{tot}} kT$$

where $n_{\text{tot}} = \sum_i n_i$ and we are assuming an ideal gas. Pressure depends fundamentally on the *number* density of particles, but gravity depends on the *mass* density so our model is expressed in terms of ρ. Using the average particle mass, $\bar{m} \equiv \rho / n_{\text{tot}}$, we can write pressure in terms of mass density as

$$P_{\text{gas}} = \frac{\rho}{\bar{m}} kT \tag{16.15}$$

As a star evolves, its mass density remains nearly unchanged,[3] but its composition varies as hydrogen gets converted to helium. Consider how this affects $\bar{m}$:

- Pure neutral hydrogen: $\bar{m} = m_p + m_e \approx m_p$
- Pure neutral helium: $\bar{m} = 2m_p + 2m_n + 2m_e \approx 4m_p$
- Pure ionized hydrogen: $\bar{m} = (m_p n_p + m_e n_e)/(n_p + n_e) \approx m_p/2$, where we assume $n_p = n_e$ for charge neutrality

(Since $m_e \ll m_p$, we can neglect the mass in electrons.) Clearly $\bar{m}$ and P_{gas} depend on the composition of the gas, which can vary throughout the star.

In addition to gas pressure, there is **radiation pressure** from the light. In Sect. 13.1.4 we derived the equation of state for photon pressure,

$$P_{\text{rad}} = 2.52 \times 10^{-16} \, \text{kg m}^{-1} \, \text{s}^{-2} \times \left(\frac{T}{\text{K}} \right)^4$$

Radiation pressure is actually negligible in the Sun. We will see below that the temperature at the center of the Sun is about $T = 1.567 \times 10^7$ K, so the radiation pressure is

$$P_{\text{rad}} = 2.52 \times 10^{-16} \, \text{kg m}^{-1} \, \text{s}^{-2} \times \left(1.567 \times 10^7\right)^4 = 1.5 \times 10^{13} \, \text{kg m}^{-1} \, \text{s}^{-2}$$

This is small compared with the gas pressure $P = 2.357 \times 10^{16} \, \text{kg m}^{-1} \, \text{s}^{-2}$. Radiation pressure is subdominant in most normal stars, but it becomes increasingly important as the mass increases and helps determine the upper limit on the mass of normal stars (see Problem 16.6).

The differential equations shown above only describe how quantities change with r. In order to obtain a complete solution we must specify starting or ending values. There are some natural **boundary conditions** for stars. At the center, as $r \to 0$ there is no mass or luminosity enclosed, so we must have

$$M(0) = 0 \qquad L(0) = 0$$

[3] Strictly speaking, some matter gets converted to energy, but that is a small fraction of the total.

At the surface, the mass, luminosity, and temperature should take on their overall values for the star, and at least for simplicity we might imagine that the density and pressure vanish:

$$\rho(R) = 0 \qquad P(R) = 0 \qquad M(R) = M_{\text{tot}} \qquad L(R) = L_{\text{tot}} \qquad T(R) = T_{\text{eff}}$$

In reality the boundary conditions can be more complicated: stars have diffuse but extended atmospheres, and they can lose mass (due to a "stellar wind"). But the simple boundary conditions given here provide a good starting point.

The equations of stellar structure represent a set of coupled differential equations where some of the components may not even be known analytically. (Nuclear reaction rates are often empirically calibrated.) In general they cannot be solved by hand, but they are suitable for numerical integration as discussed in Sect. A.6.

16.2.2 The Sun

John Bahcall and his collaborators have used the stellar structure equations to develop a detailed model of the Sun known as the **Standard Solar Model** [4]. It predicts the following values at the center of the Sun:

Density	$1.529 \times 10^5\ \text{kg m}^{-3}$
Temperature	$1.567 \times 10^7\ \text{K}$
Pressure	$2.357 \times 10^{16}\ \text{kg m}^{-1}\ \text{s}^{-2}$
Hydrogen mass fraction	34.61 %
Helium mass fraction	63.37 %

The density, temperature, pressure, and composition change with radius as shown in Fig. 16.2. The composition curves suggest that fusion occurs mainly in a core region whose radius is 15–20 % of $R_\odot$. As we saw in Sect. 15.4, Bahcall et al. used the model to predict the abundance of neutrinos produced by the Sun, and subsequent detections have provided important evidence that the Standard Solar Model is accurate.

There are other ways to test the model as well. For example, some of the curves in Fig. 16.2 show a small "kink" around $0.7R_\odot$. As you can show in Problem 16.3, this corresponds to the point at which the Sun becomes convectively unstable. The idea that the outer 30 % of the Sun is convective is confirmed by **granulation** in the photosphere. Hot bubbles of gas rise to the surface, spread out, cool, and sink back down. Since the rising hot gas is brighter than the sinking cool gas (by the Stefan-Boltzmann law, Sect. 13.1.1), convection produces a patchwork of brighter and darker regions as shown in Fig. 16.3. Convection also causes the surface of the Sun effectively to oscillate. As with seismology on Earth, **helioseismology** uses

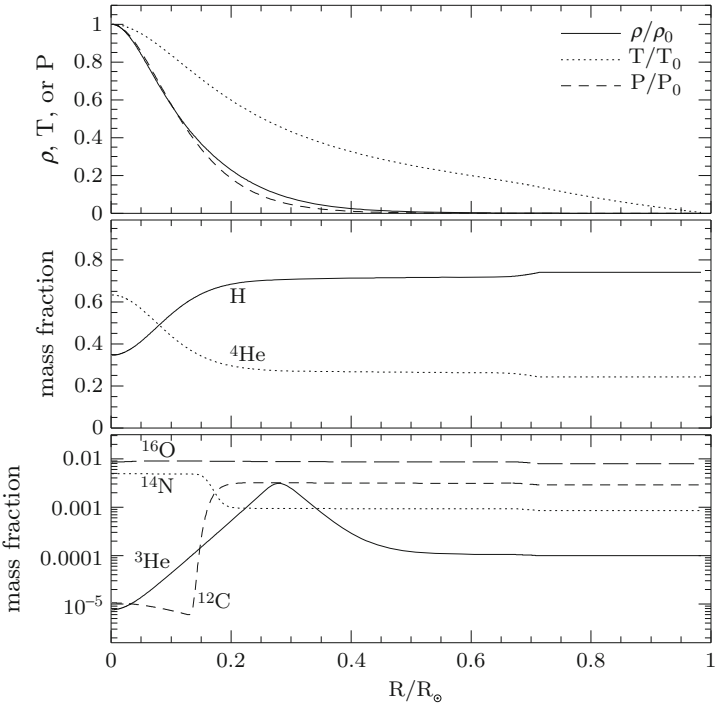

Fig. 16.2 Results from the Standard Solar Model. (*Top*) Density, temperature, and pressure as a function of radius, normalized by the values at the center: $\rho_0 = 1.529 \times 10^5\,\mathrm{kg\,m^{-3}}$, $T_0 = 1.567 \times 10^7\,\mathrm{K}$, and $P_0 = 2.357 \times 10^{16}\,\mathrm{kg\,m^{-1}\,s^{-2}}$. (*Middle*) Mass fraction of hydrogen and helium-4 as a function of radius. (*Bottom*) Mass fraction of helium-3, carbon, nitrogen, and oxygen as a function of radius (Data from Bahcall et al. [4])

motions of the Sun's surface to probe the depths and test our models of the Sun's interior.

16.2.3 Other Stars

When the equations are applied to different kinds of stars, a pattern known as the **Vogt-Russell theorem** emerges: "The mass and the composition structure throughout a star uniquely determine its radius, luminosity, and internal structure, as well as its subsequent evolution."[4] This is not a rigorous mathematical theorem because quantities beyond mass and composition (such as magnetic fields and

[4]This phrasing comes from [2]. Also see [5].

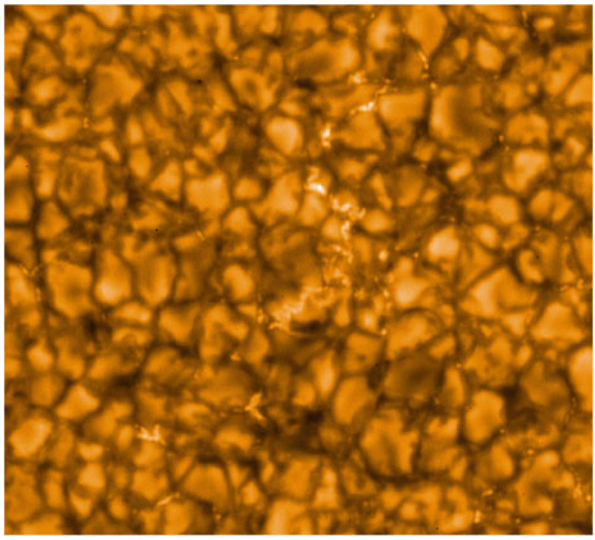

Fig. 16.3 Granulation in the Sun's photosphere, as seen by the Solar Optical Telescope on the Hinode mission. The brighter regions correspond to hotter gas rising by convection, while the darker regions correspond to cooler gas sinking back down (Credit: Hinode JAXA/NASA/PPARC)

rotation) can in fact influence stellar structure. Nevertheless, the Vogt-Russell theorem is a good rule of thumb that seems to describe real stars quite well.

Mass is the main factor that determines star properties, because compositions are fairly uniform at birth—mostly hydrogen and helium in a ratio that reflects cosmic abundances (see Chap. 20), plus trace amounts of heavier elements—and they change slowly with time. Using the models to predict the luminosity and effective temperature as a function of mass yields a curve that follows the main sequence in the HR diagram (see Fig. 16.4). In other words, the observed main sequence is actually a sequence in mass, running from cool, faint, low-mass stars in the lower right of the HR diagram to hot, luminous, massive stars in the upper left. Physically, a more massive star needs a higher temperature and pressure to balance the stronger gravity. Those conditions yield more fusion and thus a higher energy output (luminosity).

What sets the endpoints of the main sequence? At lower masses the internal temperature is cooler, and at some point it is too low to support fusion. This is the bottom end of the main sequence—the lowest mass object we call a star. Models indicate that the fusion limit occurs around $0.08\,M_\odot$. At the top end, above $\sim 100\,M_\odot$ the fusion is so intense that radiation pressure makes the core unstable. (You can estimate these limits in Problems 16.5 and 16.6.)

There are a couple of points at low masses that are worth noting. In Sect. 15.3.3 we remarked that fusion in stars with $M \lesssim 1.2\,M_\odot$ mainly follows the PP chain, whereas fusion in more massive stars mainly follows the CNO cycle. For CNO-driven stars, the fusion rate depends strongly on temperature and hence radius. Radiation cannot transport energy rapidly enough, so a large temperature gradient builds up and induces convection. As a result, stars with $M \gtrsim 1.2\,M_\odot$ have convective cores. For PP-driven stars, by contrast, the fusion rate varies less strongly with temperature and hence radius, so radiation is sufficient to transport energy and

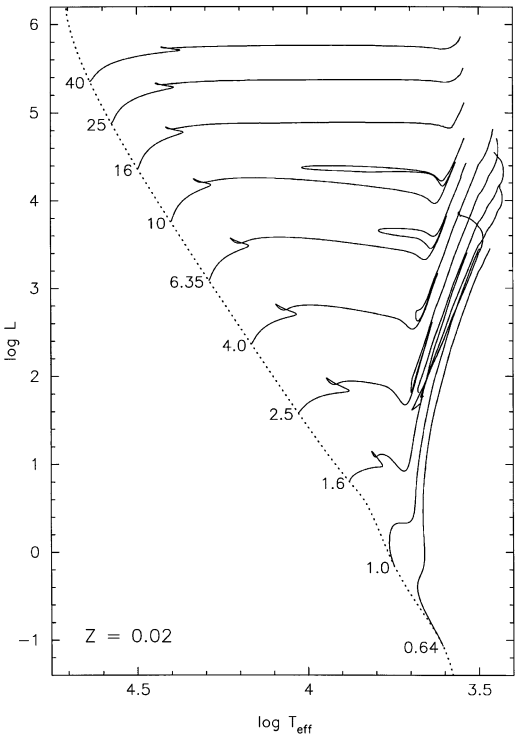

Fig. 16.4 A theory version of the HR diagram, based on stellar models. The *dotted line* shows luminosity and effective temperature as a function of mass (indicated in units of $M_\odot$); this corresponds to the main sequence in the observed HR diagram (see Fig. 14.6). *Solid lines* show evolutionary tracks once stars of different mass leave the main sequence. On the horizontal axis, temperature increases to the left to follow the convention in the observed HR diagram (Credit: Pols et al. [7], reproduced by permission of Oxford University Press on behalf of the Royal Astronomical Society)

there is no convection in the core. Radiation becomes inefficient near the surface, though, because the lower temperature leads to less ionization and excitation and hence greater opacity. As a result, stars with $M \lesssim 1.2\,M_\odot$ have convection in the outer layers. As the mass decreases, the bottom of the convection zone moves progressively downwards until at $\sim 0.4\,M_\odot$ it reaches the core and the star becomes fully convective (e.g., [6]).

To summarize, we can make a table like this:

Mass ($M_\odot$)	Fusion	Convection
$\lesssim$0.08	None	
0.08–0.4	PP	Fully convective
0.4–1.2	PP	Surface convection
1.2–100	CNO	Core convection
$\gtrsim$100	Unstable	

16.3 Evolution of Low-Mass Stars ($M \lesssim 8\,M_\odot$)

Stars are not truly static; they evolve as fusion modifies the internal composition and hence the equations of state. Stellar models can be extended to track the evolution and predict how stars move through the HR diagram once they leave the main sequence. Figure 16.4 shows examples of evolutionary tracks for a few different masses. In this section we consider the main stages in the life of a star whose mass is less than $\sim 8\,M_\odot$.[5]

16.3.1 Hydrogen, Helium, and Beyond

During most of a star's life it generates power by burning hydrogen to produce helium. Once a star exhausts its hydrogen fuel, it can burn helium to produce heavier elements. It turns out that the hydrogen and helium burning phases can each be divided into two sub-stages depending on where within the star the fusion occurs.

Hydrogen core burning. Stars that burn hydrogen in their cores lie on the main sequence in the HR diagram. Even during this stage there are some notable changes. As a star converts hydrogen into helium, its *internal* composition changes even if its *surface* composition (which determines its spectral type) does not. Recall from Eq. (16.15) that the pressure of a gas depends on the composition through the average particle mass, $\bar{m}$ (assuming the mass density is constant). As hydrogen changes to helium, $\bar{m}$ increases. In order to maintain the same pressure (to balance gravity), the temperature must likewise increase. That causes the fusion rate to increase, which in turn raises the star's luminosity. The net effect is that stars brighten a little as they age (by a factor of ~ 2 for the Sun [8]), so they have a small vertical movement in the HR diagram. This contributes to the thickness of the main sequence.

Hydrogen shell burning. Once the hydrogen in the core is exhausted, the star has a helium core surrounded by an envelope of hydrogen. The temperature and pressure can be high enough to ignite hydrogen fusion in a shell around the helium core. The temperature in the shell is actually higher than it was during the earlier stage of hydrogen core burning, so the energy production rate and luminosity are higher. Some of the energy goes into making the outer envelope expand, which causes the surface to cool and become redder. The star becomes luminous, large, and cool—a **red giant**. As the star continues to age, it moves up the red giant branch in the HR diagram.

Helium core burning. Once enough helium accumulates to cross a threshold in mass, the helium core begins to collapse due to its gravity. At some point the central

[5] Carroll and Ostlie [2] discuss stellar evolution at a similar technical level but in more detail.

temperature ($T \sim 10^8$ K) and density ($\rho \sim 10^7\,\mathrm{kg\,m^{-3}}$) become high enough to initiate fusion that converts three helium nuclei into carbon:

$$\begin{aligned} {}^4_2\mathrm{He} + {}^4_2\mathrm{He} &\rightleftharpoons {}^8_4\mathrm{Be} \\ {}^8_4\mathrm{Be} + {}^4_2\mathrm{He} &\rightarrow {}^{12}_6\mathrm{C} \end{aligned}$$

Helium-4 nuclei are known as "alpha particles," so this set of reactions is called the **triple alpha process**. Beryllium-8 can spontaneously decay (hence the reverse arrow above), so it needs to react quickly with another helium-4 nucleus to form carbon-12. Thus, the triple alpha process can proceed only if the temperature and density are high enough to achieve a sufficient reaction rate. Once some carbon exists, it can combine with more helium to form oxygen:

$${}^{12}_6\mathrm{C} + {}^4_2\mathrm{He} \rightarrow {}^{16}_8\mathrm{O}$$

With these new energy sources, the core is able to expand a little, which lowers the core temperature and reduces the luminosity. That, in turn, allows the envelope to shrink, causing the *surface* temperature to rise. Once helium ignites, in other words, the star shifts down and to the left in the HR diagram, moving onto the **horizontal branch**.

Helium shell burning. Once helium in the core is used up, the process above repeats itself, only at higher temperatures. Now the star burns hydrogen in an outer shell and helium in an inner shell, all surrounding a carbon/oxygen core. Such a star lies on the **asymptotic giant branch** (AGB) in the HR diagram.

AGB stars are large and have cool envelopes ($T \sim 3{,}000$ K) that contain carbon and oxygen dredged up from the core by convection. The conditions are right to generate dust that is rich in silicates and/or graphite. Such dust will eventually return to the interstellar medium, where it can be detected by the way it scatters light.

The energy production rate in AGB stars is high enough to drive a **stellar wind** that carries away some of the mass. The mass loss can be $\mathrm{d}M/\mathrm{d}t \sim -10^{-4}\,M_\odot\,\mathrm{yr}^{-1}$, which may not seem like a lot but is enough that a star could lose most of its mass in as little as 10,000 years.

Planetary nebula. Eventually the entire outer envelope is expelled to form an expanding gas cloud known as a planetary nebula. (These have nothing to do with planets; they got the name because in early telescopes they appeared as circular disks and thus resembled images of nearby planets.) The planetary nebula expands at a rate of 10–30 km/s, so after 10,000–50,000 years it will be so large and diffuse that it will have dispersed into the interstellar medium.

White dwarf. The remnants of the carbon/oxygen core settle into a hot, dense object known as a white dwarf. The gas has changed from an ideal gas in which the pressure is produced by motion of the gas particles to a "degenerate" gas in which the pressure arises from a quantum mechanical effect that prevents particles

from being in the same quantum state. We will study a gas supported by electron degeneracy pressure in Chap. 17. White dwarfs cool slowly, and they can be around for a long time.

To recap, here are the stages in the life of a low-mass star, the corresponding parts of the HR diagram, and the approximate duration of each stage (for a solar mass star [7, 9]):

Hydrogen core burning	Main sequence	10^{10} yr
Hydrogen shell burning	Red giant branch	10^{9} yr
Helium core burning	Horizontal branch	10^{8} yr
Helium shell burning	Asymptotic giant branch	10^{7} yr
Mass loss	Planetary nebula	10^{4} yr
Electron degeneracy	White dwarf	

Again, this evolutionary process creates elements up through carbon and oxygen.

16.3.2 Observations

We cannot observe all the stages of evolution for any single star; even the "short" stages are longer than a human lifetime. Nevertheless, it is possible to see the whole evolutionary pathway. The key idea is that all stars with $M \lesssim 8\,M_\odot$ follow the same set of steps, *but at different rates*. More massive stars pass through the sequence more quickly than less massive stars (they have more fuel but burn it faster).

Consider a set of stars with different masses that all formed at the same time. After a few billion years, the more massive stars will have progressed through the evolutionary sequence to reach, say, the asymptotic giant branch. Stars that are a little less massive will have reached the horizontal branch. Still smaller stars will be on the red giant branch. The lowest mass stars will remain on the main sequence. At a snapshot in time, the stars will trace out the full evolutionary track in the HR diagram (with position along the track determined by mass).

The universe kindly provides exactly what we need to see this. **Star clusters** are collections of stars that formed at approximately the same time (when a gas cloud collapsed and fragmented; see Chap. 19). Figure 16.5 shows HR diagrams for two observed star clusters. Stellar evolution theory can be used to predict curves showing the positions of stars that have different masses but the same age, known as **isochrones** (from *iso* = sam + *chrone* = time). Matching an isochrone to the observed HR diagram makes it possible to determine the age of a star cluster.

A particularly important point in the HR diagram of a cluster is the **main sequence turn-off (MSTO)**. Less massive stars have longer main sequence lifetimes than more massive stars, so as time passes the turn-off point moves down the main sequence. Matching observed MSTO points with theoretical predictions

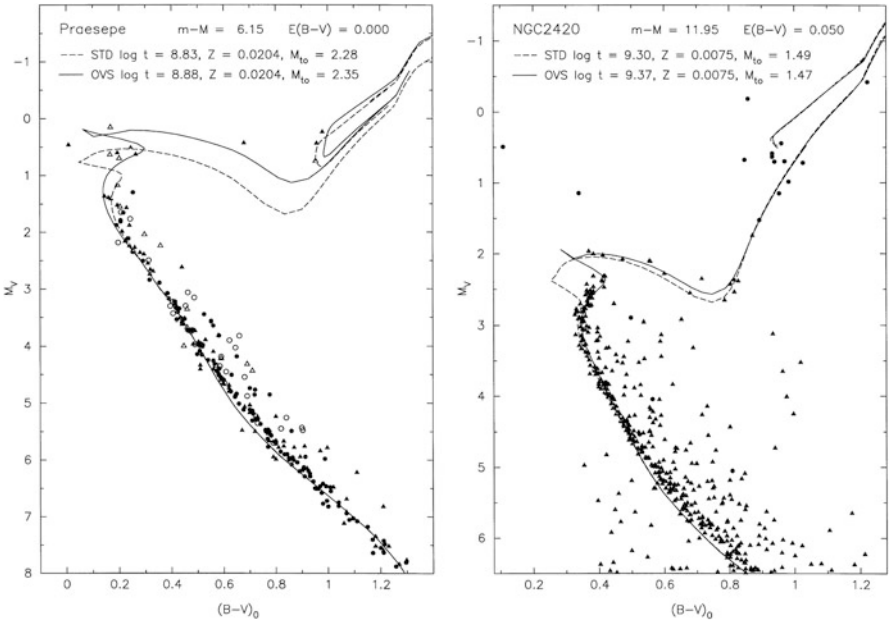

Fig. 16.5 HR diagrams for two star clusters. The horizontal axis is an index astronomers use to quantify color (see Fig. 14.6 for the corresponding spectral types). The vertical axis is $M_V = -2.5 \log_{10} L + \text{constant}$, with the minus sign causing brighter stars to have smaller values of M_V. In each panel, the points indicate individual stars while the curves represent isochrones from different evolution models. Praesepe (*left*) is a relatively young cluster with an age of about 0.7 Gyr, while NGC 2420 (*right*) is an older cluster with an age of about 2 Gyr (Credit: Pols et al. [7], reproduced by permission of Oxford University Press on behalf of the Royal Astronomical Society)

reveals how much time has passed since the cluster formed—the age of the cluster. Such "age dating" has revealed that the oldest known star clusters are around 11.5 Gyr old [10], whereas the universe is about 13.8 Gyr old [11]. (The two clusters shown in Fig. 16.5 are relatively young.)

16.4 Evolution of High-Mass Stars ($M \gtrsim 8\,M_\odot$)

High-mass stars also go through stages of core H burning, shell H burning, core He burning, and shell He burning. What makes them different from low-mass stars is that they do not stop at helium. The higher mass leads to higher temperatures and densities that can drive fusion further up the periodic table of the elements.

16.4.1 Beyond Carbon and Oxygen

After helium burning produces carbon and oxygen, the temperature in a high-mass star remains high enough ($T \gtrsim 10^8$ K) that carbon and oxygen can burn by reacting with helium:

$$\begin{aligned} {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He} &\rightarrow {}^{16}_{8}\mathrm{O} + \gamma \\ {}^{16}_{8}\mathrm{O} + {}^{4}_{2}\mathrm{He} &\rightarrow {}^{20}_{10}\mathrm{Ne} + \gamma \end{aligned}$$

As the temperature increases still further, carbon and oxygen can burn in new ways, producing a whole range of byproducts. Above 6×10^8 K, carbon/carbon reactions become possible:

$$\begin{aligned} {}^{12}_{6}\mathrm{C} + {}^{12}_{6}\mathrm{C} &\rightarrow {}^{16}_{8}\mathrm{O} + 2\,{}^{4}_{2}\mathrm{He} \\ &\rightarrow {}^{20}_{10}\mathrm{Ne} + {}^{4}_{2}\mathrm{He} \\ &\rightarrow {}^{23}_{11}\mathrm{Na} + p \\ &\rightarrow {}^{23}_{12}\mathrm{Mg} + n \\ &\rightarrow {}^{24}_{12}\mathrm{Mg} + \gamma \end{aligned}$$

Above 10^9 K, oxygen/oxygen reactions begin:

$$\begin{aligned} {}^{16}_{8}\mathrm{O} + {}^{16}_{8}\mathrm{O} &\rightarrow {}^{24}_{12}\mathrm{Mg} + 2\,{}^{4}_{2}\mathrm{He} \\ &\rightarrow {}^{28}_{14}\mathrm{Si} + {}^{4}_{2}\mathrm{He} \\ &\rightarrow {}^{31}_{15}\mathrm{P} + p \\ &\rightarrow {}^{31}_{16}\mathrm{S} + n \\ &\rightarrow {}^{32}_{16}\mathrm{S} + \gamma \end{aligned}$$

And above 3×10^9 K, even heavier elements can burn:

$$\begin{aligned} {}^{28}_{14}\mathrm{Si} + {}^{4}_{2}\mathrm{He} &\rightleftharpoons {}^{32}_{16}\mathrm{S} + \gamma \\ {}^{32}_{16}\mathrm{S} + {}^{4}_{2}\mathrm{He} &\rightleftharpoons {}^{36}_{18}\mathrm{Ar} + \gamma \\ &\vdots \\ {}^{48}_{24}\mathrm{Cr} + {}^{4}_{2}\mathrm{He} &\rightleftharpoons {}^{52}_{26}\mathrm{Fe} + \gamma \\ {}^{52}_{26}\mathrm{Fe} + {}^{4}_{2}\mathrm{He} &\rightleftharpoons {}^{56}_{28}\mathrm{Ni} + \gamma \end{aligned}$$

These reactions are known collectively as **silicon burning**.

Where does the process stop? As fusion creates heavier and heavier nuclei, it releases less and less energy because the available binding energy decreases (see Fig. 16.6). The burning must go faster and faster in order to supply the energy the

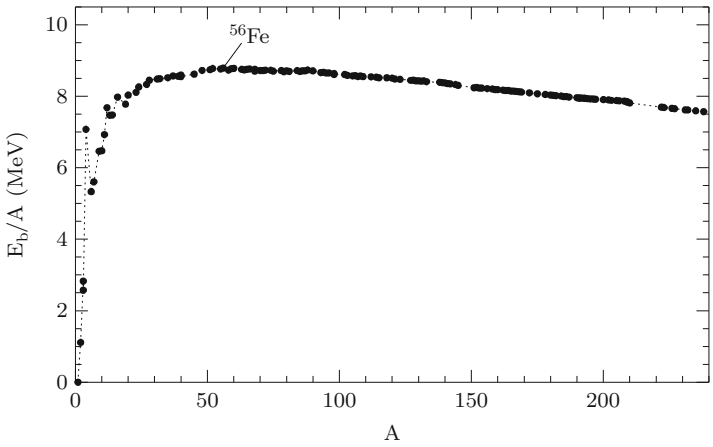

Fig. 16.6 Binding energy per nucleon, as a function of the atomic mass number (similar to Fig. 15.3, but better showing the high-mass end) (Data from [12])

star needs to avoid collapsing. Here are the durations of the various stages for a 25 $M_\odot$ star [3]:

Core hydrogen burning	5×10^6 yr
Core helium burning	5×10^5 yr
Core carbon burning	500 yr
Core silicon burning	1 day

Once silicon burning produces nickel-56, the core has crossed the peak in the binding energy curve. At that point there is no more energy to be released by fusion.[6] When the star loses its ability to create energy, things get really wild.

16.4.2 Explosion: Supernova

By the time the star is burning silicon, the temperature is so high that photons possess enough energy to destroy heavy nuclei; this process is called **photodisintegration** (and it is why the silicon burning reactions above have reverse arrows). Two reactions that absorb energy from the star are:

$$^{56}_{26}\mathrm{Fe} + \gamma \;\rightarrow\; 13\,^{4}_{2}\mathrm{He} + 4\,n$$

[6] A little energy is available from radioactive decay of nickel-56 into cobalt-56 and then into iron-56, but it is not enough to support the star.

$$^4_2\mathrm{He} + \gamma \;\rightarrow\; 2\,p + 2\,n$$

Another significant reaction is **inverse beta decay**,

$$p + e^- \;\rightarrow\; n + \nu_e$$

Essentially, the pressure is high enough to squeeze protons and electrons together to form neutrons and neutrinos. This process is important for two reasons. First, the neutrinos carry away huge amounts of energy. Second, electrons that had been helping to support the core (via electron degeneracy pressure) are removed. When they disappear, the core quickly collapses. How quickly? In Problem 16.9 you can derive the **freefall** time for a sphere of density ρ to collapse under its own gravity:

$$t_{\mathrm{ff}} = \left(\frac{3\pi}{32G\rho}\right)^{1/2} \tag{16.16}$$

The degenerate core of a massive star can be comparable to the mass of our Sun. In Sect. 1.3.2 we used dimensional analysis to estimate that an object of mass $M \sim M_\odot$ supported by electron degeneracy pressure has a radius of $R \sim 6\times10^6$ m. Thus the mean density is

$$\rho \;\sim\; \frac{3M}{4\pi R^3} \;\sim\; \frac{3\times(1.99\times10^{30}\,\mathrm{kg})}{4\pi\times(6\times10^6\,\mathrm{m})^3} \;\sim\; 2\times10^9\,\mathrm{kg\,m^{-3}}$$

The freefall time scale is then

$$t_{\mathrm{ff}} \;\sim\; \left[\frac{3\pi}{32\times(6.67\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})\times(2\times10^9\,\mathrm{kg\,m^{-3}})}\right]^{1/2} \;\sim\; 1.4\,\mathrm{s}$$

With no pressure support, it takes a core that is the mass of the Sun and the size of Earth only about a second to collapse.

Once the core shrinks to a size of ∼10 km, the density is comparable to that of an atomic nucleus. The gas develops **neutron degeneracy pressure** that prevents the core from collapsing further. (This is an example of a relativistic degenerate gas, which we will study in Sect. 17.1.) The collapse releases a tremendous amount of gravitational potential energy, going from

$$U_{\mathrm{before}} \;\sim\; -\frac{(6.67\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})\times(1.99\times10^{30}\,\mathrm{kg})^2}{6\times10^6\,\mathrm{m}} \;\sim\; -4\times10^{43}\,\mathrm{J}$$

to

$$U_{\mathrm{after}} \;\sim\; -\frac{(6.67\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})\times(1.99\times10^{30}\,\mathrm{kg})^2}{10^4\,\mathrm{m}} \;\sim\; -3\times10^{46}\,\mathrm{J}$$

Fig. 16.7 Images of M51, the Whirlpool Galaxy. The *left panel* shows a typical view of the galaxy. In the *middle* and *right panels*, the crosshairs mark different supernovae that were observed in 2005 and 2011, respectively (Image © 2011 R. Jay GaBany, Cosmotography.com, reproduced by permission)

Something in the ballpark of 10^{46} J of gravitational energy is released. Most of it is carried by the neutrinos produced when the protons and electrons combined to form neutrons. Only a fraction of the energy goes into kinetic energy, but it is enough to create a shock wave that blows apart the gaseous envelope of the star. The result is an enormous stellar explosion called a **type II (core collapse) supernova**.[7] An even smaller fraction of the energy goes into photons—but that is still a lot of light energy. At peak brightness a supernova can have a luminosity of around $10^9\,L_\odot$, so it stands out even against the background galaxy (see Fig. 16.7). Overall, the rough energy budget for the explosion is as follows [3]:

Neutrinos	$\sim 10^{46}$ J
Kinetic energy	$\sim 10^{44}$ J
Photons	$\sim 10^{42}$ J

As a rule of thumb, a typical massive star-forming galaxy has about one supernova per century. In the Milky Way, a type II supernova observed in 1054 left a remnant known as the Crab Nebula (Fig. 16.8).[8] Another one seen in 1987 in the Large Magellanic Cloud (a satellite galaxy of the Milky Way) has provided a wealth of information about core collapse supernovae. In particular, Supernova

[7]There is another class of supernova, called type Ia, that occurs when a white dwarf crosses a threshold in mass known as the Chandrasekhar limit (see Sect. 17.2.2). These are the supernovae that are used to study the expansion of the universe (see Chap. 18).

[8]In addition, there were supernovae that were probably type Ia seen in the years 1006, 1572 ("Tycho's supernova"), and 1604 ("Kepler's supernova").

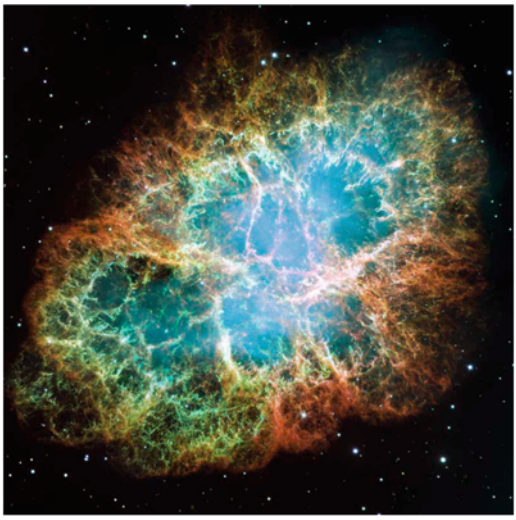

Fig. 16.8 The Crab Nebula is the remnant of a Type II supernova that was observed in the year 1054 (Credit: NASA, ESA, J. Hester and A. Loll) (Arizona State University)

1987A was close enough (cosmically speaking) that three neutrino detectors on Earth were able to record a total of 24 neutrinos from the explosion. While that may not seem like a large number, it was consistent with predictions given the distance to the supernova and the small cross section for neutrino interaction. This marked the first direct confirmation that neutrinos are produced in copious amounts as part of a core collapse event.

16.4.3 Beyond Iron

Where do elements heavier than iron come from? There are two sets of reactions that do not produce energy but can occur when neutrons are abundant. The first is **neutron capture**:

$$ {}^{A}_{Z}\mathrm{X} + n \rightarrow {}^{A+1}_{Z}\mathrm{X} + \gamma $$

The second is **beta decay**, which turns a neutron within a nucleus into a proton and a free electron:

$$ {}^{A+1}_{Z}\mathrm{X} \rightarrow {}^{A+1}_{Z+1}\mathrm{X} + e^{-} + \bar{\nu}_e + \gamma $$

There are two cases:

- If beta decay is more rapid than neutron capture, then heavy elements build up "slowly." This ***s*-process** tends to create stable nuclei.

- If neutrons are captured more rapidly than beta decay can eliminate them, then heavy, neutron-rich elements build up "rapidly." This is the ***r*-process**.

With all of this as background, we can now understand where all of the elements in the universe come from:

Hydrogen, helium, lithium	Big bang (see Chap. 20)
Beryllium through carbon and oxygen	Low- and high-mass stars
All heavier elements	High-mass stars

Nuclear processes in dying stars is responsible for all elements in the universe heavier than hydrogen, helium, and lithium.

Problems

16.1. Throughout most of the Sun the gas is fully ionized. In this problem, you may ignore elements heavier than helium, but remember to account for the free electrons.

(a) In intermediate layers of the Sun ($r \approx 0.5 R_\odot$), there are about 86 helium nuclei for every 1,000 hydrogen nuclei. What is the average particle mass $\bar{m}$ in this region, in units of the proton mass m_p?
(b) In the core of the Sun, the average particle mass approaches $0.84 m_p$. For every 1,000 hydrogen nuclei in the Sun's core, how many helium nuclei are there?

16.2. How long does it take a photon to escape from the center of the Sun? A photon has to random walk its way out because it scatters off free electrons with cross section $\sigma_T = (8\pi e^4)/(3m_e^2 c^4) = 6.65 \times 10^{-29}\,\mathrm{m}^2$ (the Thomson cross section). Make a rough estimate of the travel time using the average density of the Sun. (See [13] and references therein for a more detailed treatment.)

16.3. The data file for Bahcall's Standard Solar Model (Sect. 16.2.2, [4]) is available online. You may use any appropriate software to analyze the data and make plots. Note that in electronic files, we abbreviate scientific notation, so that 6.02×10^{23} becomes 6.02E+23 (or just 6.02E23) and 1.38×10^{-16} becomes 1.38E−16.

(a) Use Eq. (16.15) to compute and plot the average particle mass (in units of m_p) versus radius (in units of $R_\odot$). Make sure to label the axes with appropriate units. You should see three fairly distinct zones: (i) the inner 20 % of the Sun; (ii) the region between 20 and 95 % of the Sun's radius; (iii) the outer 5 % of the Sun. What is happening physically that distinguishes these three zones? Do your results agree with Problem 16.1?

(b) On a new graph, plot both of these quantities

$$\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right| \qquad \text{and} \qquad \frac{\gamma-1}{\gamma}\,\frac{\bar{m}}{k}\,g$$

as a function of radius. Show that the outer layer of the Sun is convective. Hint: to take a derivative numerically, use

$$\frac{\mathrm{d}T}{\mathrm{d}r} \approx \frac{T(r_{i+1})-T(r_i)}{r_{i+1}-r_i}$$

Use the appropriate adiabatic index γ for a non-relativistic ideal monatomic gas, and recall that the acceleration due to gravity is $g(r) = GM(r)/r^2$.

16.4. Problems 12.6 and 14.3 involve a model star with a uniform density of hydrogen gas in hydrostatic equilibrium. Would such a star be stable or unstable to convection?

16.5. The lower limit to the main sequence occurs when the core temperature of a star is not sufficient to fuse hydrogen into helium. In Sect. 1.3.2 we estimated the central temperature of a star of mass M and radius R. For main sequence stars, radius and mass are correlated: $R \propto M^{\alpha}$ with $\alpha \approx 0.7$ [14, 15]. In Sect. 15.2.2 we estimated the central temperature required to support fusion. Put the pieces together to estimate the mass (in $M_\odot$) of the smallest star whose core is hot enough to ignite fusion.

16.6. The upper end of the main sequence occurs where radiation pressure is strong enough to make a star unstable. In Sect. 1.3.2 we used dimensional analysis to derive scaling relations for the central pressure and temperature of a star with mass M and radius R. Filling in the constants of proportionality by working in reference to the Sun, we can write

$$P_c = P_\odot\,\frac{(M/M_\odot)^2}{(R/R_\odot)^4} \qquad \text{and} \qquad T_c = T_\odot\,\frac{(M/M_\odot)}{(R/R_\odot)}$$

This is the pressure required to counteract the pull of gravity. If a star is hot enough that the pressure from photons (see Sect. 13.1.4),

$$P_{\mathrm{rad}} = \frac{\pi^2(kT)^4}{45(\hbar c)^3}$$

is strong enough to counteract gravity, the star will be unstable. Find an expression (in terms of symbols) for the mass at which this occurs. Then plug in numbers to estimate the mass (in $M_\odot$) at the upper end of the main sequence.

16.7. Estimate the duration of the core hydrogen and helium burning phases in the life of a 4 $M_\odot$ star. In both phases, the star's luminosity is roughly 500 $L_\odot$ (see Fig. 16.4). Why is the helium burning phase much shorter than the hydrogen burning phase?

16.8. Supernova 1987A occurred in the Large Magellanic Cloud about 50 kpc from Earth. Models indicate that about 1.5×10^{44} J went into the kinetic energy of the explosion, with an ejected mass of about 20 $M_\odot$. Estimate the typical speed of the ejecta. About how long would it take for the ejecta to expand to subtend a radius of 0.1 arcsec on the sky such that we could resolve the debris as a supernova remnant? You may assume for simplicity that the shell expands at a constant speed, but please comment on how this assumption affects your answer.

16.9. Here is how you can derive the freefall time for gravitational collapse.

(a) Consider dropping a test particle from rest at a height r_0 above a mass M. Use conservation of energy to determine the speed v of the particle at any height r.
(b) With $v(r) = \mathrm{d}r/\mathrm{d}t$ we have a differential equation for r, which can be solved by writing

$$\frac{\mathrm{d}r}{v(r)} = \mathrm{d}t$$

and integrating both sides from the initial state ($t = 0$, $r = r_0$) to the final state ($t = t_{\mathrm{ff}}$, $r = 0$). Evaluate the integral to find an expression for t_{ff} in terms of M, r_0, and constants. Hint: to evaluate the r integral, you may find it helpful to change variables using $r = r_0 \cos^2\theta$.
(c) Suppose the mass M was initially spread out into a sphere of radius r_0 and density ρ_0. Rewrite t_{ff} in terms of ρ_0. (This is reasonable because the sphere collapses as the particle falls, so the preceding analysis remains valid.)

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