Chapter 5
Tidal Forces

Most of our analysis so far has used point masses. Now we ask whether the sizes of objects affect their gravitational interaction. For the *source* of gravity, size does not matter if the object is spherically symmetric (see Sect. 2.3). For the *target* of gravity, however, size does matter because gravity pulls harder on one side of the object than on the other. Newton studied this problem and realized that variations in the Moon's gravity across Earth's surface would "squeeze" the oceans and create the tides. This phenomenon is therefore known as the **tidal force**, and it has a variety of interesting consequences.

5.1 Derivation of the Tidal Force

Consider the gravitational force on an object of radius R from an object of mass M a distance r away (see Fig. 5.1). To use specific terminology, let's say the target of gravity is a planet and the source of gravity is a moon (later we will reverse the picture). Let's also say the moon lies in the planet's equatorial plane; while this is not quite correct for the Earth/Moon system, it allows us to use familiar geographic terms like equator, poles, and latitude. In this analysis we work in the plane containing the moon as well as the center and poles of the planet; everything else can be obtained by rotating around the line between the planet and moon.

Since the force of gravity scales as $1/r^2$, the side of planet that faces the moon feels a stronger force than the side of the planet away from the moon. The force on the center of planet causes the planet as a whole to move (orbiting the center of mass of the planet/moon system), so what creates the tidal force is the difference between the force at the surface and the force at the center. This differential force pulls "up" (relative to the planet's surface) near the equator and "down" near the poles.

Consider a small object of mass m on the surface at latitude θ. We draw the triangle shown in Fig. 5.1, and call s the length of the third side while α is the other

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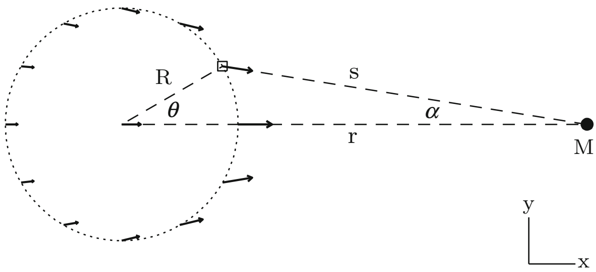

Fig. 5.1 The strength and direction of the gravitational force from the source on the right are denoted with *arrows*. Gravity varies across the surface of the planet, leading to a tidal force. The *dashed lines* indicate the geometry we use to analyze the tidal force at the position indicated by □

angle. These are important because s and α determine the strength and direction of the gravitational force, respectively. Specifically, the gravitational force from the moon on the small object m is

$$\mathbf{F}(\theta) = \frac{GMm}{s^2}\,(\cos\alpha\,\hat{\mathbf{x}} - \sin\alpha\,\hat{\mathbf{y}})$$

We would like to rewrite this in terms of coordinates on the planet (i.e., R and θ). As we saw with a similar analysis in Sect. 2.3, trigonometric identities let us write

$$\begin{aligned} s^2 &= r^2\left(1+\xi^2-2\xi\cos\theta\right) \\ \sin\alpha &= \frac{\xi\sin\theta}{(1+\xi^2-2\xi\cos\theta)^{1/2}} \\ \cos\alpha &= \frac{1-\xi\cos\theta}{(1+\xi^2-2\xi\cos\theta)^{1/2}} \end{aligned}$$

where we introduce $\xi = R/r$. We can then write the force as

$$\begin{aligned} \mathbf{F}(\theta) &= \frac{GMm}{r^2}\,\frac{(1-\xi\cos\theta)\hat{\mathbf{x}} - \xi\sin\theta\,\hat{\mathbf{y}}}{(1+\xi^2-2\xi\cos\theta)^{3/2}} \\ &\approx \frac{GMm}{r^2}\left[(1+2\xi\cos\theta)\hat{\mathbf{x}} - \xi\sin\theta\,\hat{\mathbf{y}} + \mathcal{O}\left(\xi^2\right)\right] \end{aligned}$$

where we do a Taylor series expansion in ξ because we expect this ratio to be small for many planet/moon systems. We have found the force at the *surface* of the planet. For comparison, the force at the center of the planet is

$$\mathbf{F}_0 = \frac{GMm}{r^2}\,\hat{\mathbf{x}}$$

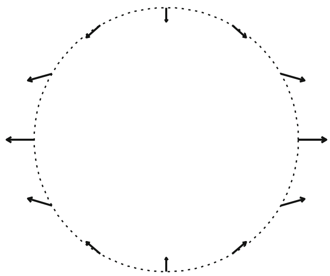

Fig. 5.2 The *arrows* indicate the direction and amplitude of the tidal force $\Delta\mathbf{F}$

The tidal force is the difference,

$$\Delta\mathbf{F}(\theta) \;=\; \mathbf{F}(\theta) - \mathbf{F}_0 \;\approx\; \frac{GMm}{r^2}\left[2\xi\cos\theta\,\hat{\mathbf{x}} - \xi\sin\theta\,\hat{\mathbf{y}} + \mathscr{O}\left(\xi^2\right)\right]$$

Since the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components are both linear in ξ, we can pull this factor out front and write

$$\Delta\mathbf{F}(\theta) \approx \frac{GMmR}{r^3}\left[2\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}} + \mathscr{O}\left(\frac{R}{r}\right)\right] \tag{5.1}$$

This is the general form of the tidal force on an object of size R that is a distance r away from the source of gravity (in the approximation $R/r \ll 1$). The geometry is shown in Fig. 5.2.

It is useful to consider the components of $\Delta\mathbf{F}$ relative to directions on the planet. The component perpendicular to the surface ("vertical") is

$$\begin{aligned}\Delta F_{\text{vert}} &= \Delta\mathbf{F}\cdot\hat{\mathbf{R}} \\ &= \frac{GMmR}{r^3}\Big(2\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}}\Big)\cdot\Big(\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}\Big) \\ &= \frac{GMmR}{r^3}\left(2\cos^2\theta - \sin^2\theta\right) \\ &= \frac{GMmR}{r^3}\left(3\cos^2\theta - 1\right) \end{aligned} \tag{5.2}$$

The component parallel to the surface ("horizontal") is

$$\begin{aligned}\Delta F_{\text{horiz}} &= \Delta\mathbf{F}\cdot\hat{\boldsymbol{\theta}} \\ &= \frac{GMmR}{r^3}\Big(2\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}}\Big)\cdot\Big(-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}\Big) \\ &= -3\frac{GMmR}{r^3}\sin\theta\cos\theta \end{aligned} \tag{5.3}$$

Given the sign convention that $\hat{\theta}$ points "north," the horizontal force ΔF_{horiz} always points toward the equator. Here are a few additional notes:

- The maximum "pull up" (at the equator) is twice the maximum "push down" (at the poles).
- The horizontal component of the tidal force is largest at midlatitudes.
- Relative to the surface (i.e., in terms of vertical and horizontal components), the tidal force is the same on the near and far sides of the planet.
- The maximum strength of the tidal force occurs along the line between the objects and is given by

$$\Delta F_{\text{max}} = \frac{2GMmR}{r^3}$$

5.2 Effects of Tidal Forces

Since the strength scales as $\Delta F \propto R/r^3$, the tidal force is important for *large* objects that are *near* the source of gravity. There are variety of systems in which tidal forces play an interesting role.

5.2.1 Earth/Moon

Like Newton, we first consider the Earth. We mentioned tidal forces from the Moon, but in principle there could be tidal forces from the Sun as well. Which are more important on Earth? The maximum tidal force from each is

$$\Delta F_{\text{Sun}} = \frac{2GM_{\odot}mR}{r_{\text{Sun}}^3} \quad \text{and} \quad \Delta F_{\text{Moon}} = \frac{2GM_{\text{Moon}}mR}{r_{\text{Moon}}^3}$$

The ratio is

$$\frac{\Delta F_{\text{Moon}}}{\Delta F_{\text{Sun}}} = \frac{M_{\text{Moon}}}{M_{\odot}} \left(\frac{r_{\text{Sun}}}{r_{\text{Moon}}} \right)^3 = \frac{7.35 \times 10^{22}\,\text{kg}}{1.99 \times 10^{30}\,\text{kg}} \left(\frac{1.50 \times 10^{11}\,\text{m}}{3.84 \times 10^{8}\,\text{m}} \right)^3 = 2.2$$

Although the Moon is much less massive than the Sun, it is so much closer that it exerts the stronger tidal force. Nevertheless, the Sun's effect is not negligible. It modulates the height of tides induced by the Moon, sometimes creating high tides that are higher than average (known as "spring tides") or low tides that are lower than average ("neap tides"; see Problem 5.1).

How does the tidal force from the Moon compare with Earth's own gravity? Let's consider the maximum vertical component of the tidal force:

$$\frac{\Delta F_{\text{Moon}}}{F_{\text{Earth}}} = \frac{2GM_{\text{Moon}} m R_{\text{Earth}} / r_{\text{Moon}}^3}{GM_{\text{Earth}} m / R_{\text{Earth}}^2} = \frac{2M_{\text{Moon}}}{M_{\text{Earth}}} \left(\frac{R_{\text{Earth}}}{r_{\text{Moon}}} \right)^3$$

$$= \frac{2 \times 7.35 \times 10^{22}\,\text{kg}}{5.97 \times 10^{24}\,\text{kg}} \left(\frac{6.38 \times 10^6\,\text{m}}{3.84 \times 10^8\,\text{m}} \right)^3 = 1.1 \times 10^{-7}$$

Another way to think about this is that the maximum tidal force from the Moon would create an acceleration of just $1.1 \times 10^{-6}\,\text{m s}^{-2}$ in the vertical direction. Thus, the vertical component of the tidal force would be very difficult to detect against the backdrop of the Earth's gravity.

The horizontal component of the tidal force is a different story, though. Earth's own gravity has a tangential component only to the extent that Earth is not a perfect sphere, which is a small effect. The horizontal tidal force therefore has little opposition beyond the rigidity of material on Earth's surface. It acts on water in the oceans (which, after all, is fluid rather than rigid) to create tidal "bulges" on the near and far sides of Earth (relative to the Moon). Analyzing ocean tides in detail requires advanced material, such as fluid dynamics on a rotating surface, but we can still understand several notable features.

As Earth rotates through the tidal bulges, we see two high tides and two low tides each day. Shore dwellers know the cycle of tides actually lasts longer than 1 day (almost 25 h) because a point on Earth's surface must complete a little more than one full rotation to "catch up" with the Moon moving in its own orbit. Also, friction between rock and water slows Earth's rotation; the length of the day is increasing at a rate of few milliseconds per century. While this effect is small, it can be measured using historical records of eclipse timing [1] as well as geological records of sedimentation that is influenced by tides [2].

As Earth's spin slows, its rotational angular momentum decreases; to keep total angular momentum conserved, the Moon drifts farther away at a rate of about 4 cm per year. We can measure this by timing how long it takes laser pulses to travel out to the Moon and back to Earth, reflecting off mirrors left on the Moon by Apollo astronauts [3]. The changes to Earth's rotation and the Moon's orbit will cease only when Earth's rotation period equals the Moon's orbital period, i.e., when the Earth and Moon are in **synchronous rotation**. At that point we would say Earth is **tidally locked** with the Moon (although we are unlikely to get there because it would take longer than the lifetime of the Sun).

So far we have considered the Moon's effect on Earth, but we can invert the picture and consider the tidal force on the Moon created by gravity from Earth. How do the forces compare? As we have seen, the maximum tidal force on Earth from the Moon is

$$\Delta F_{\text{on Earth}} = \frac{2GM_{\text{Moon}} m R_{\text{Earth}}}{r_{\text{Moon}}^3}$$

while the maximum tidal force on the Moon from Earth is

$$\Delta F_{\text{on Moon}} = \frac{2GM_{\text{Earth}} m R_{\text{Moon}}}{r_{\text{Moon}}^3} \tag{5.4}$$

The ratio is

$$\frac{\Delta F_{\text{on Moon}}}{\Delta F_{\text{on Earth}}} = \frac{M_{\text{Earth}} R_{\text{Moon}}}{M_{\text{Moon}} R_{\text{Earth}}} = \frac{(5.97 \times 10^{24}\,\text{kg}) \times (1.74 \times 10^{6}\,\text{m})}{(7.35 \times 10^{22}\,\text{kg}) \times (6.38 \times 10^{6}\,\text{m})} = 22.2$$

The tidal force on the Moon is strong enough to raise tidal bulges in the rock itself. The rotational deceleration has been so strong that the Moon is already tidally locked with Earth, which explains why we always see the same face of the Moon.

5.2.2 *Jupiter's Moon Io*

Another system that displays fascinating tidal phenomena is Jupiter and its moon Io. Let's examine the tidal force from Jupiter on Io,

$$\Delta F_{\text{on Io}} = \frac{2GM_{\text{Jup}} m R_{\text{Io}}}{r_{\text{Io}}^3}$$

and use the tidal force from Earth on our Moon (Eq. 5.4) as a reference point. Here are the numbers we need to make the comparison:

	M_{planet} (kg)	R_{moon} (km)	r (km)
Moon	5.97×10^{24}	1,737	3.84×10^{5}
Io	1.90×10^{27}	1,821	4.22×10^{5}

Io and the Moon are fairly similar in terms of their size and distance from the planet, but of course Io's planet is much more massive than the Moon's planet. That causes the ratio of tidal forces to be

$$\frac{\Delta F_{\text{on Io}}}{\Delta F_{\text{on Moon}}} = \frac{M_{\text{Jup}}}{M_{\text{Earth}}} \frac{R_{\text{Io}}}{R_{\text{Moon}}} \left(\frac{r_{\text{Moon}}}{r_{\text{Io}}} \right)^3 = 250$$

Tidal effects are *much* stronger on Io than on the Moon. They have caused Io to be tidally locked with Jupiter.

Io's orbit is slightly eccentric, with $e = 0.0041$. This may not seem like a lot, but it means Io's distance varies by 0.8 % between pericenter and apocenter, which translates into a 2.4 % change in the strength of the tidal force. This may not seem like a lot either, but 2.4 % of a strong tidal force is significant. Plus, the variation happens over the course of Io's orbital period, which is just 1.8 days. In essence, Io has been flexing every few days for more than 4 billion years, creating a significant amount of internal heat from friction[1] [4]. The cumulative heating has been strong enough to create volcanoes that have been observed by several spacecraft visiting Jupiter and its moons [5, 6]. The volcanoes on Io are perhaps the most striking manifestation of the amount of energy associated with tidal forces.

5.2.3 Extrasolar Planets

Tidal forces can be relevant for planets as well—especially hot Jupiters, since they are large and close to their stars. One interesting case is the planet that transits HD 209458 (see Sect. 4.3.2). Careful observations have shown that gas is escaping from the planet [7]. Heat from the star allows some of the gas molecules to exceed the planet's escape velocity (see Chap. 12 for more discussion), but the tidal force from the star contributes by helping to counteract the planet's gravity.

Many hot Jupiters are expected to be tidally locked to their stars. While direct evidence is difficult to obtain, there is indirect evidence for tidal locking from studies that examine how heat from a star is distributed across a planet by atmospheric circulation [8,9]. It is even conceivable that a star could be tidally locked to a planet. This may be the case for Tau Boötis: the rotation period of the star (measured from flux variability) is consistent with the orbital period of its planet [10, 11].

5.3 Tidal Disruption

When the tidal force from a planet pulls "up" on a moon's surface, it acts against the moon's self gravity. If the tidal force gets strong enough, it could actually tear the moon apart. To estimate when this occurs, let's adopt a simple criterion: if the tidal force "up" at the equator exceeds the gravitational force "down," the surface will be ripped off. (We remark on a more realistic criterion below.)

Consider a moon with mass M_m and radius R_m, which is orbiting a planet with mass M_p and radius R_p. Suppose the moon is a distance r from the planet. As we have seen, the tidal force up at the equator of the moon is

$$F_{\mathrm{tidal}} = \frac{2GM_p m R_m}{r^3}$$

while the gravitational force down is

$$F_{\mathrm{grav}} = \frac{GM_m m}{R_m^2}$$

[1]Think of repeatedly bending a paper clip back and forth.

According to the simple criterion we are using, tidal disruption will occur when $F_{\text{grav}} \lesssim F_{\text{tidal}}$, or

$$\frac{GM_m m}{R_m^2} \lesssim \frac{2GM_p m R_m}{r^3}$$

$$r^3 \lesssim 2\,\frac{M_p}{M_m}\,R_m^3$$

$$\lesssim 2\,\frac{\rho_p}{\rho_m}\,R_p^3$$

where we switch from mass to mean density using $M = (4/3)\pi R^3 \rho$. We find that the moon will be torn apart if it gets closer than

$$r \lesssim \left(2\,\frac{\rho_p}{\rho_m}\right)^{1/3} R_p \tag{5.5}$$

Notice that the threshold depends on the density of the moon, but not its size.

The preceding analysis applies to a moon that is rigid enough to maintain its shape as it is peeled away layer by layer. Edouard Roche considered a more general scenario in which the moon gets distorted even before it is disrupted. Tidal distortion stretches the moon in the radial direction (relative to the planet), which enhances the difference between the surface and center of the moon and causes disruption to occur at a somewhat larger distance from the planet. Roche found that a loosely bound moon would be disrupted when

$$r \lesssim 2.4\left(\frac{\rho_p}{\rho_m}\right)^{1/3} R_p \tag{5.6}$$

This condition is now called the **Roche limit**. The most conspicuous consequence of tidal disruption is Saturn's rings. You can explore this idea and some other interesting scenarios in the problems below.

Problems

5.1. Spring tides occur when the Sun is oriented in a way that reinforces the Moon's tidal force, while neap tides occur when the Sun partially cancels the Moon's effect. Sketch the arrangements of the Earth, Moon, and Sun that lead to spring and neap tides.

5.2. Saturn has mass 5.7×10^{26} kg and radius 60,300 km.

(a) Find an image of Saturn and estimate the radius of the outer edge of the rings, in units of Saturn's radius.
(b) Compute Saturn's average mass density.

(c) What is the minimum density that a moon of Saturn orbiting at the outer edge of the rings must have to resist tidal disruption?
(d) It is thought that Saturn's rings are composed of bodies made of water ice. Is this consistent with your answer from part (c)?

5.3. Neptune has mass 1.02×10^{26} kg and radius 24,764 km. Its moon Triton has mass 2.14×10^{22} kg, radius 1,353 km, and orbital period 5.88 day. Triton's orbit is "backwards" (retrograde) relative to Neptune's spin, so tidal forces are causing the orbit to shrink. Simulations predict that Triton will cross Neptune's Roche limit in a few billion years [12].

(a) Where is the Roche limit for the Neptune/Triton system?
(b) Assuming Triton will reach the Roche limit in 2 billion years, approximately how fast is its orbit shrinking?

5.4. You may have heard that a person falling feet-first into a black hole would be stretched out by the tidal force, in a process affectionately called "spaghettification." But would the effect actually be dramatic? Let's consider:

(a) Use scaling relations to determine whether the tidal force at the event horizon gets stronger or weaker as the black hole mass increases.
(b) It seems reasonable to say that we would "feel" the stretching only if the tidal acceleration exceeds the familiar acceleration due to gravity on Earth. Find the black hole mass that would produce such a tidal acceleration at the event horizon.
(c) Use your results from (a) and (b) to say whether we would be spaghettified by the black hole at the center of the Milky Way.
(d) What about by the black hole in the binary system M33-X7 ($M \approx 16\, M_{\odot}$)?

5.5. In 1994 the comet Shoemaker-Levy 9 collided with Jupiter. The comet was actually a set of fragments that hit Jupiter one after the other, producing a series of explosions that visibly scarred the planet. Why fragments? It is believed that the comet had been tidally disrupted during a previous close pass by Jupiter (probably in 1992). How close must the comet have come to Jupiter?

5.6. Suppose an asteroid is headed straight for Earth. People have talked about using a rocket or bomb to divert the asteroid. You need not only to prevent a collision, but also to avoid having the asteroid be tidally disrupted. (Creating a bunch of asteroid rubble around Earth would be no good!) If you reach the asteroid when it is 1 AU from Earth, how much "sideways" velocity would you need to impart? How about if you reach it when it is 384,000 km from Earth (the same distance as the Moon)?

You may assume the Earth and asteroid form an isolated system; in other words, you can neglect the effects of the Moon, the Sun, and everything else in the Solar System. You may assume the asteroid started from rest infinitely far from Earth.

Hint: think about the trajectory the asteroid will follow once you have diverted it, and about energy and angular momentum.

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