Chapter 16
Case Study

When analysing a process its model has to be established so that it describes correctly the static and dynamic behaviour of the outputs as responses to the given inputs and gives also the evolution in time of the state variables as the response to the initial values and the inputs.

To build the model the real operation of the system is taken into account. It has to be understood as deeply as possible. One analyses in which operating range of the input signal can the system be considered linear, or in which range of the operating points can it be linearized. The real operation is described then by mathematical equations (generally by differential equations or state equations). The values of the parameters in the equations have to be given. The parameters are known or have to be determined by measurements or by identification. Identification is a procedure where the values of the parameters are estimated from the data of input-output measurements.

On the basis of the model the output signals and the state variables of the system, as responses to the input signals and initial conditions can be calculated or simulated. Based on the model a regulator can be designed for the system to fulfil the quality specifications.

In the sequel the process of establishing the model of a heating process will be discussed (see also Sect. 2.6. in the textbook [1]).

16.1 Modelling and Analysing a Heat Process

Let us analyse the heat process of a system consisting of two heat sources. The arrangement is shown in Fig. 16.1. The temperature changes in several pieces of electrical equipment can be modelled by analysing the warming processes in two embedded bodies [5]. For example the warming processes in slots of electrical machines, where the copper winding is placed in the iron slots can be analysed on the basis of this model.

L. Keviczky et al., *Control Engineering: MATLAB Exercises*,
Advanced Textbooks in Control and Signal Processing,
https://doi.org/10.1007/978-981-10-8321-1_16

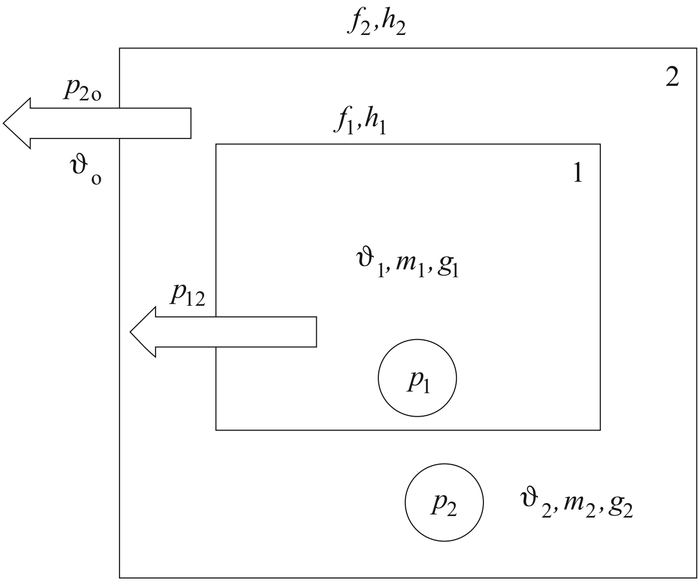


Fig. 16.1 A heat process

In the body of mass m_2 and specific heat g_2, power p_2 is converted to heat. This body encloses the body of mass m_1 and specific heat g_1, where the heat power is p_1. The surfaces where the two bodies are in contact with each other and with the external environment are f_1 and f_2, with heat transfer coefficients h_1 and h_2, respectively. Let us determine the change of temperatures ϑ_1 and ϑ_2 in the two bodies after switching on the heat generation, supposing that earlier the temperature of the system was equal to the environmental temperature ϑ_o. So the input signals of the system are the heating powers p_1 and p_2, the disturbance is the environmental temperature ϑ_o, and the output signals are ϑ_1 and ϑ_2, the temperatures of the two bodies.

The behaviour of the system can be described as follows. The temperature of the two bodies starts growing after switching on the heat. One part of the generated heat energy is stored in the heat capacity of the bodies, increasing their temperature, while the second part—as the effect of the temperature difference—leaves, entering the environment through the interfacial surface. It is supposed that the bodies are homogeneous, because of their good heat transfer properties a temperature difference does not take place inside the bodies.

In the inner body the heat generated over a time period of Δt partly increases the temperature of the body by $\Delta\vartheta_1$ degrees, and partly leaves for the outer body. The heat transfer depends on the difference of the temperatures in the two bodies, on the size of the interfacial surface, and on the heat transfer coefficient. In the outer body the heat transfer generated by the heat power p_2 is added to the amount of heat coming from the inner body. This resulting heat partly increases the temperature of the outer body by $\Delta\vartheta_2$ degrees, and partly goes into the environment.

The heating of the two bodies can be described by the continuity equations, $Q_{\text{in}} - Q_{\text{out}} = Q_{\text{change}}$, where Q denotes the quantity of heat. When heating a body this can be given as follows:

$$p\Delta t = mg\Delta\vartheta,$$

where p is the sum of the in- and out flow powers. $\Delta\vartheta$ is the temperature change in the body through Δt time, m is the mass of the body, and g is the specific heat. In the case of the two bodies

$$\begin{aligned}(p_1 - p_{12})\Delta t &= m_1 g_1 \Delta\vartheta_1\\(p_2 + p_{12} - p_{2\text{o}})\Delta t &= m_2 g_2 \Delta\vartheta_2\end{aligned}$$

where $p_{12} = h_1 f_1(\vartheta_1 - \vartheta_2)$ and $p_{2\text{o}} = h_1 f_1(\vartheta_2 - \vartheta_\text{o})$. The heat flow between the two bodies depends on the temperature difference, the surface f between the two bodies, and the heat transfer coefficient h. Replacing the small changes Δ by differentials, the following equations are obtained:

$$\begin{aligned}\frac{d\vartheta_1}{dt} &= -\frac{h_1 f_1}{m_1 g_1}\vartheta_1 + \frac{h_1 f_1}{m_1 g_1}\vartheta_2 + \frac{1}{m_1 g_1}p_1\\\frac{d\vartheta_2}{dt} &= \frac{h_1 f_1}{m_2 g_2}\vartheta_1 - \frac{h_1 f_1 + h_2 f_2}{m_2 g_2}\vartheta_2 + \frac{1}{m_2 g_2}p_2 + \frac{h_2 f_2}{m_2 g_2}\vartheta_\text{o}.\end{aligned}$$

To simplify the equations, introduce the following notation (analogous to the notation in electrical systems):

$$\begin{aligned}&R_1 = \frac{1}{h_1 f_1},\ R_2 = \frac{1}{h_2 f_2},\ G_1 = m_1 g_1,\ G_2 = m_2 g_2\\&\frac{d\vartheta_1}{dt} = -\frac{1}{G_1 R_1}\vartheta_1 + \frac{1}{G_1 R_1}\vartheta_2 + \frac{1}{G_1}p_1\\&\frac{d\vartheta_2}{dt} = \frac{1}{G_2 R_1}\vartheta_1 - \frac{1}{G_2}\frac{R_1 + R_2}{R_1 R_2}\vartheta_2 + \frac{1}{G_2}p_2 + \frac{1}{G_2 R_2}\vartheta_\text{o}\end{aligned}$$

The input signals, output signals and state variables of the system are as follows.

$$\begin{aligned}\text{Inputs}: & u_1 = p_1,\ u_2 = p_2,\ u_3 = \vartheta_0\\\text{Outputs}: & y_1 = \vartheta_1,\ y_2 = \vartheta_2\\\text{State variables}: & x_1 = \vartheta_1,\ x_2 = \vartheta_2\end{aligned}$$

Choosing values for the parameters $R_1 = 1$, $R_2 = 1$, $G_1 = 1$ and $G_2 = 5$, the state equation of the system is

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 + u_1\\\dot{x}_2 &= 0.2x_1 - 0.4x_2 + 0.2u_2 + 0.3u_3\\y_1 &= x_1\\y_2 &= x_2\end{aligned}$$

This is a system with three inputs and two outputs. A simplified case arises if only one input and one output is considered. Let $u = p_1$ be the input of the system (input heat power of the inner body) and let the output be $y = \vartheta_2$, the temperature of the outer body. Take the power p_2 as zero, so the outer body is not heated. The outer environmental temperature ϑ_o is considered as zero, which means the introduction of relative temperature values. With these assumptions the state equation of the system is

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 + u \\ \dot{x}_2 &= 0.2x_1 - 0.4x_2 \\ y &= x_2 \end{aligned}$$

Analyse the behaviour of the system with MATLAB™.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0.2 & -0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0\ u$$

```
A=[-1, 1;0.2, -0.4]
B=[1; 0]
C=[0, 1]
D=0
```

The characteristic equation of the system is

$$\det(s\boldsymbol{I} - \boldsymbol{A}) = (s+1)(s+0.4) - 1 \cdot 0.2 = s^2 + 1.4s + 0.2.$$

The coefficients of this polynomial can be obtained also with the command poly.

```
karpol=poly(A)
    1.0000 1.4000 0.2000
```

The roots of this polynomial are the poles of the system, which are also the eigenvalues of $\boldsymbol{A}$.

```
p=roots(karpol)
       -1.2385
       -0.1615
p=eig(A)
```

The transfer function of the system is calculated by using $H(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$. The polynomials that are its numerator and denominator can be obtained by the command ss2tf.

```
[num,den]=ss2tf(A,B,C,D)
    num = 0       -0.0000  0.2000
    den = 1.0000  1.4000  0.2000
```

or by using the *LTI sys* structure:

```
H=ss(A,B,C,D)
```

In transfer function form, this is

```
Htf=tf(H)
              0.2
     -----------------
      s^2 + 1.4 s + 0.2
```

In zero-pole form, this is

```
Hzpk=zpk(H)
              0.2
      -------------------
      (s+1.239)(s+0.1615)
```

Analyse the behaviour of the system in the time domain. The input signal is a unit step, $u(t) = 1(t)$ and calculate and plot the output signal (Fig. 16.2).

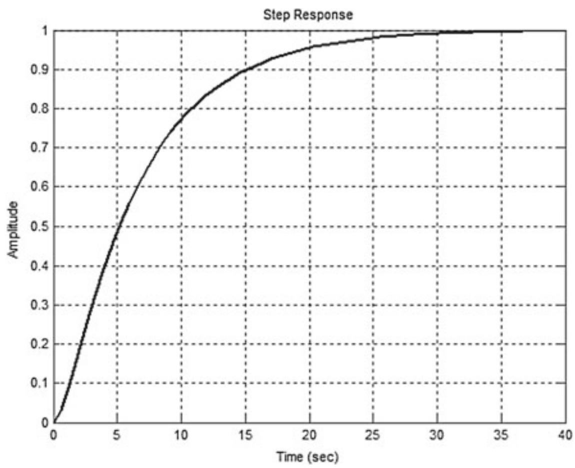


Fig. 16.2 Step response

```
step(H), grid
```

Give the output also in analytical form. The output can be calculated by using the LAPLACE transformation.

$$Y(s) = U(s)H(s)$$
$$y(t) = \mathcal{L}^{-1}\{U(s)H(s)\}$$

The LAPLACE transform of the unit step is $U(s) = 1/s$.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\frac{0.2}{(s+1.239)\,(s+0.1615)}\right\}$$

It is known that

$$\frac{r}{s+p} \xrightarrow{\mathcal{L}^{-1}} re^{-pt}$$

The inverse LAPLACE transform can be found based on the partial fractional representation.

```
s=zpk('s')
Us=1/s
Ys=Us*Hzpk
```

Determine $Y(s)$ in polynomial form.

```
[num,den]=tfdata(Ys,'v')
```

Expand in partial fractions:

```
[r,p,k]=residue(num,den)
```

```
r =    0.1499
      -1.1499
       1.0000
p =   -1.2385
      -0.1615
       0
k =    []
```

In the case of single roots,

$$Y(s) = \frac{r(1)}{s-p(1)} + \frac{r(2)}{s-p(2)} + \frac{r(3)}{s-p(3)} + k = \frac{0.1499}{s+1.2385} - \frac{-1.1499}{s+0.1615} + \frac{1}{s}$$

The output signal in the time domain is

$$y(t) = 0.1499\mathrm{e}^{-1.2385t} - 1.1499t\mathrm{e}^{-0.1615t} + 1(t) \text{ for } t \geq 0.$$

It can be seen that the output signal resulting as a response to the input signal consists of two parts, a transient and a stationary part.

The first two components give the transient response which depends on the poles of the system. The third component gives the stationary response which depends on the poles of the input signal. Compare the outputs calculated in the two different ways.

```
t=0:0.05:40;
y1=step(Hzpk,t);
y2=0.1499*exp(-1.2385*t)-1.1499*exp(-0.1615*t)+1;
plot(t,y1,'b',t,y2,'r'),grid
```

In Fig. 16.3 it can be seen that the two curves coincide. Analyse the behaviour of the system for the input signal $u(t) = t^2/2,\ t \geq 0$. Its LAPLACE transform is $U(s) = 1/s^3$. Determine the stationary and transient components of the output signal. Give an analytical expression for the output signal.

```
Us=1/(s^3)
Ys=Us*Hzpk
[num,den]=tfdata(Ys,'v')
[r,p,k]=residue(num,den)
```

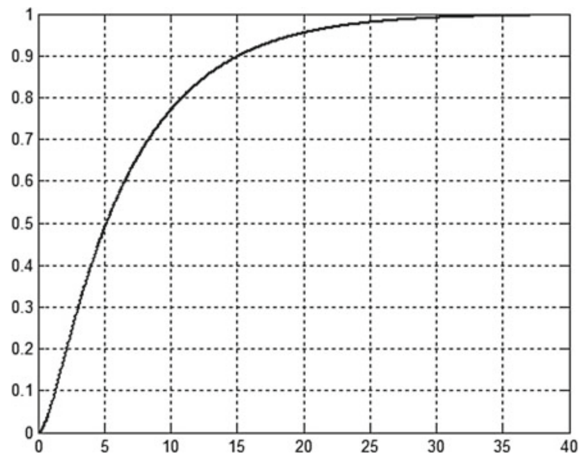


Fig. 16.3 Step response calculated in two ways

```
r =      0.0977
       -44.0977
        44.0000
        -7.0000
         1.0000
p =     -1.2385
         0.1615
         0
         0
         0
k =      []
```

The pole at $p = 0$ is a multiple pole of the system, therefore all of its powers have to be taken into consideration in the LAPLACE transform of the output signal.

$$Y(s) = X_2(s) = \frac{0.0977}{s+1.2385} - \frac{44.0977}{s+0.1615} + \frac{44}{s} - \frac{7}{s^2} + \frac{1}{s^3}$$
$$y(t) = x_2(t) = 0.0977e^{-1.2385t} - 44.0977e^{-0.1615t} + 44 - 7t + 0.5t^2$$

It can be seen that the first two components depend on the system dynamics (the poles of the system), and the last three components depend on the excitation. Separate these two components and plot them.

$$Y_{\text{tranz}}(s) = \frac{0.0977}{s+1.2385} - \frac{44.0977}{s+0.1615} \quad ; \quad Y_{\text{stac}}(s) = \frac{44}{s} - \frac{7}{s^2} + \frac{1}{s^3}$$

```
Ytranz=r(1)/(s-p(1))+r(2)/(s-p(2))
Ystac=r(3)/s+r(4)/(s*s)+r(5)/(s*s*s)
```

The entire signal is obtained as the sum of the two components

```
Ytranz+Ystac
```

The inverse LAPLACE transform can be found also with the impulse command, as the LAPLACE transform of the DIRAC delta is 1.

```
t=0:0.05:20;
y=impulse(Ys,t);
yt=impulse(Ytranz,t);
ys=impulse(Ystac,t);
plot(t,y,'r',t,yt,'b',t,ys,'g',t,yt+ys,'k')
```

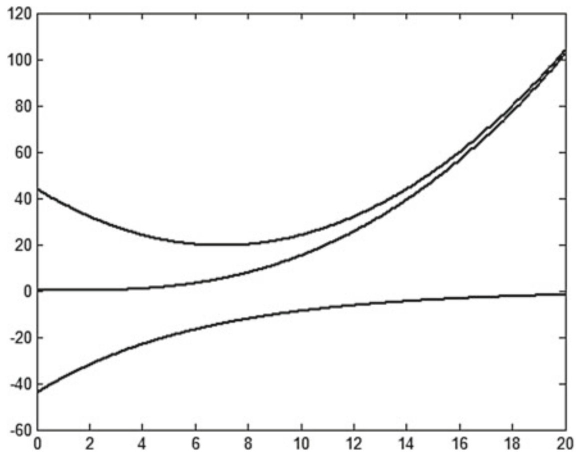


Fig. 16.4 The output signal is the sum of the stationary and transient responses

In Fig. 16.4 it can be seen that $y_{tranz}(t)$ is decreasing and finally reaches zero (blue curve). The course of the stationary curve $y_{stac}(t)$ depends on the input signal (green curve). The output signal $y(t)$ (red curve) is obtained as the sum of these two components.

Analyse the behaviour of the system for initial conditions. The system has two state variables. Set their initial values to be 10 and −10.

```
x0=[10, -10]
[y,t,x]=initial(H,x0);
```

The variable `y` contains the values of the output signal, `x` contains the state variables. Check the form of the state variables.

```
x
```

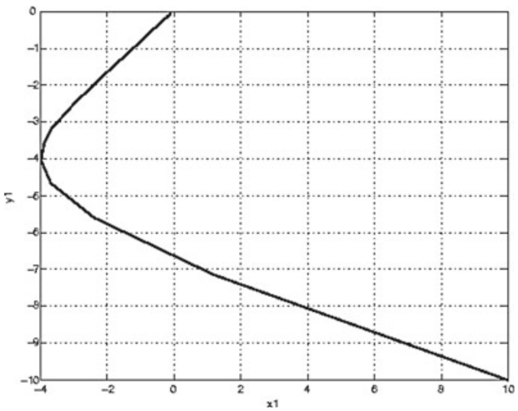
Separate the two columns. The colon: indicates all rows of the vector.

```
x1=x(:,1)
x2=x(:,2)
```

Plot the state trajectory.

```
plot(x1,x2),grid
```

Fig. 16.5 State trajectory



In Fig. 16.5 it is seen that the curve starts at (10, −10), the temperature increases in the colder body, while decreasing in the warmer one, then after the transients decay, both signals settle to the outer temperature, which is zero.

Problem Section 2.6 of the textbook [1] derives the models of a DC motor, of a tank and of the inverted pendulum. Simulate these models using MATLAB™.