Chapter 9
Control Systems with State Feedback

In Chap. 3. the description of processes in state-space was investigated. In many cases, this is the kind of description that is primarily available, and not the transfer function of the controlled system. This is the explanation, in part, for why there is a control design methodology directly based on the state-space description. For illustrative purposes, let us consider the state-space representation of an (*LTI*) process to be controlled,

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u \\ y &= \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \end{aligned} \tag{9.1}$$

which corresponds to (3.10) for the case of $d = 0$. This, as was mentioned earlier, does not impair generality, because it is a very rare case when the model contains proportional channel directly affecting the output. The block scheme of (9.1) is shown in Fig. 9.1.

Here u and y are the input and output signals of the process, respectively, and $\boldsymbol{x}$ is the state vector. According to the equivalent transfer function (3.17) we get

$$P(s) = \boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} = \frac{\mathcal{B}(s)}{\det(s\boldsymbol{I} - \boldsymbol{A})} = \frac{\mathcal{B}(s)}{\mathcal{A}(s)} = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}. \tag{9.2}$$

Figure 9.2 shows the so-called classical closed control system directly fitting the state-equation description, where r denotes the reference signal. In the closed-loop, the state vector is fed back with the linear proportional vector $\boldsymbol{k}^{\mathrm{T}}$ according to the expression below

$$u = k_{\mathrm{r}} r - \boldsymbol{k}^{\mathrm{T}}\boldsymbol{x} \tag{9.3}$$

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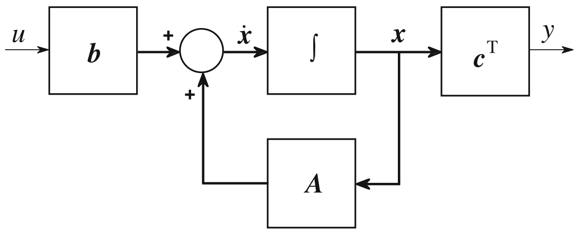

Fig. 9.1 Block scheme of the state-space equation of the *LTI* system

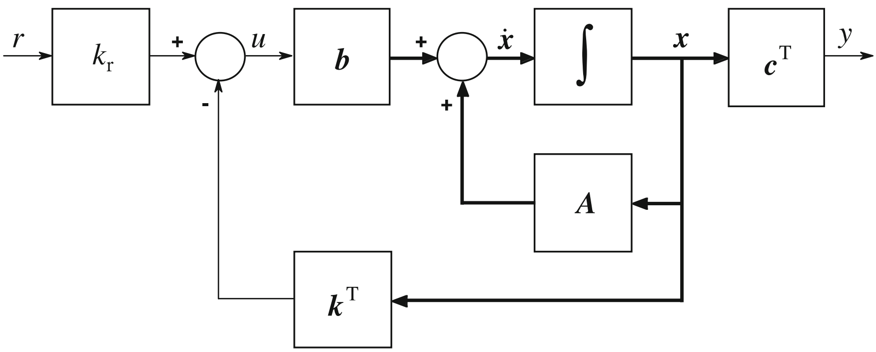

Fig. 9.2 Linear controller with state feedback

Based on Fig. 9.2, the state-equation of the complete closed system can be easily written as

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \left(\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)\boldsymbol{x} + k_{\mathrm{r}}\boldsymbol{b}r \\ y &= \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \end{aligned} \tag{9.4}$$

i.e. with the state feedback the dynamics represented by the original system matrix is modified by the dyadic product $\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}$ to $\left(\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)$.

The transfer function of the closed control loop is

$$\begin{aligned} T_{\mathrm{ry}}(s) = \frac{Y(s)}{R(s)} &= \boldsymbol{c}^{\mathrm{T}}\left(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)^{-1}\boldsymbol{b}k_{\mathrm{r}} = \frac{\mathbf{c}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}k_{\mathrm{r}}}{1 + \boldsymbol{k}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}} \\ &= \frac{k_{\mathrm{r}}}{1 + \boldsymbol{k}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}}P(s) = \frac{k_{\mathrm{r}}\mathcal{B}(s)}{\mathcal{A}(s) + \boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Psi}(s)\boldsymbol{b}} \end{aligned} \tag{9.5}$$

which comes from the comparison of the equations valid for the LAPLACE transforms $\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}U(s)$ [see (3.12)], $U(s) = k_{\mathrm{r}}R(s) - \boldsymbol{k}^{\mathrm{T}}\boldsymbol{X}(s)$ [see (9.3)] and

$Y(s) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{X}(s)$ [see (9.1)] using the matrix inversion lemma (for details, see A.9.1 in Appendix A.5). Note that the state feedback leaves the zeros of the process untouched and only the poles of the closed-loop system can be designed by $\boldsymbol{k}^{\mathrm{T}}$.

The so-called calibration factor k_{r} is introduced in order to make the gain of T_{ry} equal to unity $\left(T_{\mathrm{ry}}(0) = 1\right)$. The open-loop is obviously not of integrator type, it cannot provide zero error and unit static transfer gain. It can be assured only if the condition

$$k_{\mathrm{r}} = \frac{-1}{\boldsymbol{c}^{\mathrm{T}}\left(\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)^{-1}\boldsymbol{b}} = \frac{\boldsymbol{k}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b} - 1}{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}} \tag{9.6}$$

is fulfilled [see A.9.2 in Appendix A.5.]. The special control loop shown above is called *state-feedback*.

9.1 Pole Placement by State Feedback

The most natural design method for state feedback is the so-called pole placement. In this case the feedback vector $\boldsymbol{k}^{\mathrm{T}}$ has to be chosen to make the characteristic equation of the closed-loop equal to the prescribed (or design) polynomial $\mathcal{R}(s)$, i.e.,

$$\begin{aligned}\mathcal{R}(s) &= s^n + r_1 s^{n-1} + \cdots + r_{n-1}s + r_n = \prod_{i=1}^{n}(s - s_i) = \det\left(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right) \\ &= \mathcal{A}(s) + \boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Psi}(s)\boldsymbol{b}\end{aligned} \tag{9.7}$$

A solution always exists if the process is controllable. (It is reasonable if the order of $\mathcal{R}$ is equal to that of $\mathcal{A}$.) In the exceptional case when the transfer function of the controlled system is known, then the canonical state-equations can be written directly. Based on the controllable canonical form (3.47) the system matrices are

$$\boldsymbol{A}_{\mathrm{c}} = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}; \quad \boldsymbol{c}_{\mathrm{c}}^{\mathrm{T}} = [b_1, b_2, \dots, b_n]; \quad \text{and} \\ \boldsymbol{b}_{\mathrm{c}} = [1,0, \dots 0]^{\mathrm{T}} \tag{9.8}$$

Considering the special forms of $\boldsymbol{A}_{\mathrm{c}}$ and $\boldsymbol{b}_{\mathrm{c}}$, it can be seen that according to the design equation

$$\boldsymbol{A}_c - \boldsymbol{b}_c\boldsymbol{k}_c^T = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \boldsymbol{k}_c^T$$
$$= \begin{bmatrix} -r_1 & -r_2 & \dots & -r_{n-1} & -r_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{9.9}$$

the choice

$$\boldsymbol{k}^T = \boldsymbol{k}_c^T = [r_1 - a_1, r_2 - a_2, \dots, r_n - a_n] \tag{9.10}$$

ensures the satisfaction of the characteristic equation (9.7), i.e., the prescribed poles. The choice of the calibration factor can be determined by simple calculations

$$k_r = \frac{a_n + (r_n - a_n)}{b_n} = \frac{r_n}{b_n} \tag{9.11}$$

Based on Eqs. (9.4) and (9.6) it can be seen that in the case of state feedback pole placement, the transfer function turns out to be

$$T_{ry}(s) = \frac{k_r \mathcal{B}(s)}{\mathcal{R}(s)} \tag{9.12}$$

as was shown at (9.5).

Example 9.1 Consider an unstable process with transfer function

$$P(s) = \frac{-8}{(s+2)(s-4)} = \frac{1}{(1+0.5s)(1-0.25s)} = \frac{-8}{s^2-2s-8} = \frac{-8}{\mathcal{A}(s)}$$

where $\mathcal{A}(s) = (s+2)(s-4) = s^2 - 2s - 8 = s^2 + a_1 s + a_2$. To stabilize the process we should mirror the right half-plane unstable pole $p_2^c = 4$ into the left plane, i.e. $p_2^c = -4$ is to be obtained. This can be arranged by the choice of the polynomial $\mathcal{R}(s) = (s+2)(s+4) = s^2 + 6s + 8 = s^2 + r_1 s + r_2$. So the necessary stabilizing feedback vector is

$$\mathbf{k}^T = [\, r_1 - a_1 \quad r_2 - a_2 \,] = [\, 6 - (-2) \quad 8 - (-8) \,] = [\, 8 \quad 16 \,]$$

■

The most frequent case of state feedback is when rather than the transfer function, the state-space form of the control system is given. In relation with

Eq. (3.67) it has already been discussed that all controllable systems can be described in controllable canonical form using the transformation matrix $\boldsymbol{T}_\mathrm{c} = \boldsymbol{M}_\mathrm{c}^\mathrm{c}(\boldsymbol{M}_\mathrm{c})^{-1}$. This linear transformation also refers to the feedback vector

$$\begin{aligned} \boldsymbol{k}^\mathrm{T} &= \boldsymbol{k}_\mathrm{c}^\mathrm{T}\boldsymbol{T}_\mathrm{c} = \boldsymbol{k}_\mathrm{c}^\mathrm{T}\boldsymbol{M}_\mathrm{c}^\mathrm{c}\boldsymbol{M}_\mathrm{c}^{-1} \\ \boldsymbol{k}^\mathrm{T} &= \boldsymbol{b}_\mathrm{c}^\mathrm{T}\boldsymbol{M}_\mathrm{c}^{-1}\mathcal{R}(\boldsymbol{A}) = [0, 0, \ldots, 1]\boldsymbol{M}_\mathrm{c}^{-1}\mathcal{R}(\boldsymbol{A}) \end{aligned} \tag{9.13}$$

The design relating to the controllable canonical form (9.10), together with the linear transformation relationship corresponding to the first row of the non-controllable form (9.13) is called Bass-Gura algorithm. The algorithm in the second row of (9.13) is called Ackermann method after its developer (see the details in the A.9.3 of Appendix A.5).

In the Bass-Gura algorithm, the inverse of the controllability matrix $\boldsymbol{M}_\mathrm{c}$ has to be determined by the general system matrices $\boldsymbol{A}$ and $\boldsymbol{b}$, on the one hand, and the controllability matrix $\boldsymbol{M}_\mathrm{c}^\mathrm{c}$ of the controllable canonical form [see (3.61)], on the other. Since this latter term depends only on the coefficients a_i in the denominator of the process transfer function, then the denominator has to be calculated: $\mathcal{A}(s) = \det(s\boldsymbol{I} - \boldsymbol{A})$. Since $[0, 0, \ldots, 1]\boldsymbol{M}_\mathrm{c}^{-1}$ is the last row of the inverse of the controllability matrix, and besides this $\mathcal{R}(\boldsymbol{A})$ has to be also calculated, in the Ackermann method it is not necessary to calculate $\mathcal{A}(s)$.

It can be easily seen that state feedback formally corresponds to a serial compensation $R_\mathrm{s} = \mathcal{A}(s)/\mathcal{R}(s)$ (Fig. 9.3a). The real operation and effect of state feedback can be easily understood by the equivalent block schemes using the transfer functions shown in Fig. 9.3. The "controller" $R_\mathrm{f}(s)$ of the closed-loop is in the feedback line (Fig. 9.3b). The transfer function of the closed-loop (9.12) is

$$T_\mathrm{ry}(s) = \frac{k_\mathrm{r}\mathcal{B}(s)}{\mathcal{R}(s)} = \frac{k_\mathrm{r}\mathcal{B}(s)}{\mathcal{A}(s) + \mathcal{B}(s)} = \frac{k_\mathrm{r}P(s)}{1 + K_\mathrm{k}(s)P(s)} = \frac{k_\mathrm{r}\mathcal{A}(s)}{\mathcal{R}(s)}\frac{\mathcal{B}(s)}{\mathcal{A}(s)} = k_\mathrm{r}R_\mathrm{s}(s)P(s) \tag{9.14}$$

where

$$R_\mathrm{f} = K_\mathrm{k}(s) = \frac{\mathcal{K}(s)}{\mathcal{B}(s)} = \frac{\mathcal{R}(s) - \mathcal{A}(s)}{\mathcal{B}(s)} = \frac{\boldsymbol{k}^\mathrm{T}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}}{\mathbf{c}^\mathrm{T}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}} \tag{9.15}$$

and the calibration factor is

$$k_\mathrm{r} = \frac{\boldsymbol{k}^\mathrm{T}\boldsymbol{A}^{-1}\boldsymbol{b} - 1}{\boldsymbol{c}^\mathrm{T}\boldsymbol{A}^{-1}\boldsymbol{b}} = \frac{1 + K_\mathrm{k}(0)P(0)}{P(0)}. \tag{9.16}$$

Based on the block schemes of Fig. 9.3 it can be stated that the state-feedback also stabilizes the unstable terms, since due to the effect of the polynomial $\mathcal{K}(s) = \mathcal{R}(s) - \mathcal{A}(s)$, there is a pole allocation for any process, so by choosing a

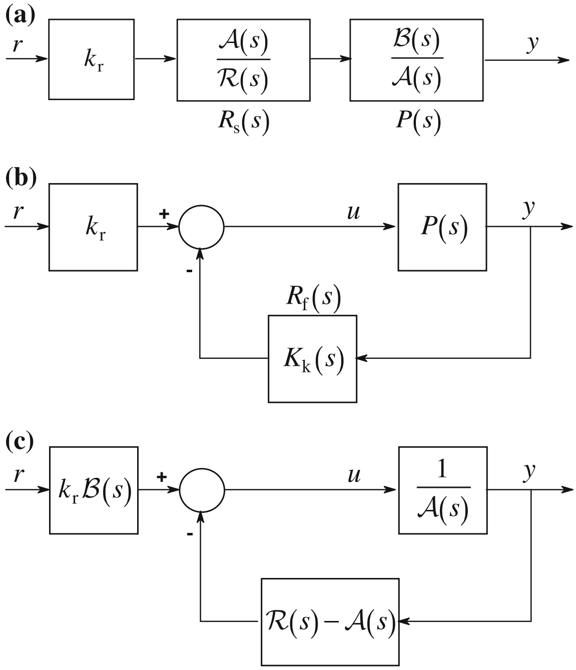

Fig. 9.3 Equivalent schemes of the state feedback design by transfer functions and polynomials

stable $\mathcal{R}(s)$, the stabilization is achieved. The feedback polynomial $\mathcal{K}(s)$ corresponds formally to $\boldsymbol{k}^{\mathrm{T}}$. The fact that the numerator $\mathcal{B}(s)$ of the process is present in the denominator of $K_{\mathrm{k}}(s)$ requires special consideration. It is used to be said in these cases that the controller can be applied only to minimum-phase (inverse stable) processes, where the roots of $\mathcal{B}(s)$ are stable. As a consequence of the special character of the state feedback, however, here $\mathcal{B}(s)$ is not replaced by its model $\widehat{\mathcal{B}}(s)$, but the method itself realizes the exact $1/\mathcal{B}(s)$.

Further methods have been developed for the calculation of the pole placement state feedback vector $\boldsymbol{k}^{\mathrm{T}}$. From among these, the so-called MAYNE-MURDOCH method is briefly shown here, on the basis of which useful statements can be made. In the BASS-GURA and ACKERMANN methods the controllable canonical form has a special role. A similarly important canonical form is the diagonal form. Let the diagonal form $\boldsymbol{A}_{\mathrm{d}} = \mathbf{diag}[\lambda_1, \ldots, \lambda_n]$ be built with the eigenvalues λ_i, i.e. the roots of $\mathcal{A}(s)$, and let the roots of the design polynomial $\mathcal{R}(s)$ be the prescribed values of $\{\mu_1, \ldots, \mu_n\}$. Assuming that the eigenvalues are single, the MAYNE-MURDOCH method gives the following closed form expression for the product $k_i^{\mathrm{d}} b_i^{\mathrm{d}}$,

$$k_i^{\mathrm{d}} b_i^{\mathrm{d}} = \frac{\Pi_{j=1}^{n}\left(\lambda_i - \mu_j\right)}{\Pi_{\substack{j=1 \\ i \neq j}}^{n}\left(\lambda_i - \lambda_j\right)} \quad i = 1, \ldots, n \tag{9.17}$$

from which k_i^{d} can be easily determined. Here the coefficient b_i^{d} is an element of the parameter vector $\boldsymbol{b}^{\mathrm{d}} = \left[b_1^{\mathrm{d}}, \ldots, b_n^{\mathrm{d}}\right]^{\mathrm{T}} = [\beta_1, \ldots, \beta_n]^{\mathrm{T}}$ of the diagonal form [see also (3.38)]. The most interesting consequence of (9.17) is that it clearly shows that the absolute value of the feedback gain k_i^{d} required by the pole placement increases directly proportionally to the "moving" distance between the poles of the open- and closed-loop.

9.2 Observer Based State Feedback

The method of state-feedback shown in the previous section requires the direct measurement of the state vector of the state-equation describing the process. Only very rarely can this be fulfilled: generally only in the case of low order dynamics (e.g., in mechanical systems measuring the values of the distance, velocity and acceleration). Thus the usefulness of the method depends on the possible measurement or estimation of the state vector. To determine the state vector the so-called observer principle has been developed. This method requires the knowledge of the system matrices $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}^{\mathrm{T}}$, by means of which an exact model of the process is realized and using the same excitation that is applied for the original process, this model (observer) provides estimated values $\hat{\boldsymbol{x}}$ and $\hat{y}$ of the variables $\boldsymbol{x}$ and y. The state-feedback is realized by using $\hat{\boldsymbol{x}}$. The principle is shown in Fig. 9.4.

More strictly the estimated values $\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{c}}^{\mathrm{T}}$ in the observer should have been used instead of $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}^{\mathrm{T}}$. The speciality of the observer, however, is that it applies not only a parallel model, but it calculates an error $\varepsilon = y - \hat{y}$ from the deviation of the original and estimated output values of the process, and has a feedback via a proportional feedback vector $\boldsymbol{l}$ to the input of the integrator of the observer. This feedback is in operation until the error exists, i.e., until the output of the process and the observer become equal. This operation can tolerate a rather large error in the knowledge of the system matrices.

It can be seen in the figure that now the state-feedback is

$$u = k_{\mathrm{r}} r - \boldsymbol{k}^{\mathrm{T}} \hat{\boldsymbol{x}} \tag{9.18}$$

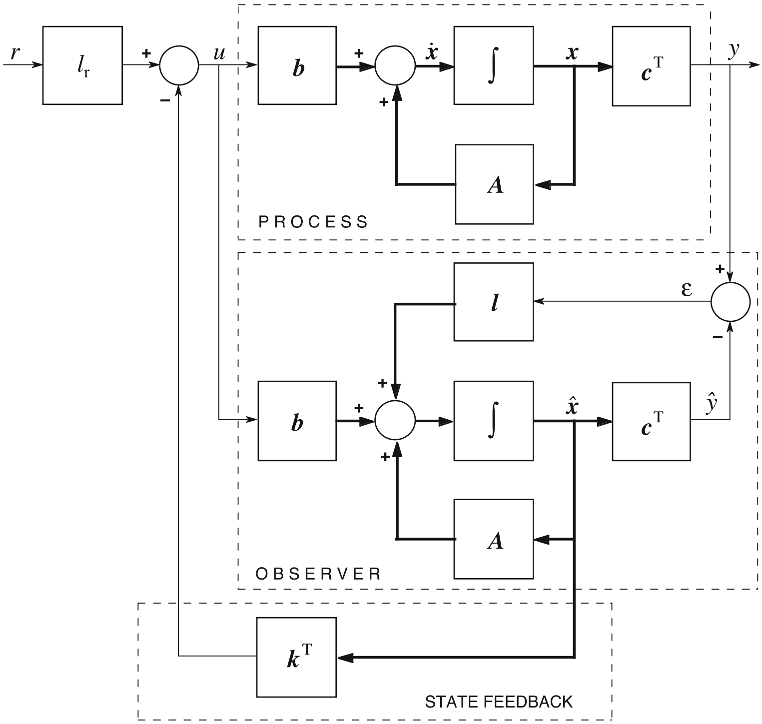

Fig. 9.4 Observer based state-feedback

thus simply $\hat{\boldsymbol{x}}$ is used instead of $\boldsymbol{x}$. Through a long and very complex deduction, whose details will not be discussed here, we get the overall closed-loop transfer function in the form

$$T_{ry}(s) = \frac{k_r P(s)}{1 + \boldsymbol{k}^T (s\boldsymbol{I} - \boldsymbol{A})^{-1} \boldsymbol{b}} = \frac{k_r \mathcal{B}(s)}{\mathcal{R}(s)}, \tag{9.19}$$

which, perhaps surprisingly, is exactly equal to (9.12), i.e., to the case of state-feedback without an observer. (A detailed proof can be seen in A.9.5 of Appendix A.5.) This means that the tracking property of the closed-loop does not depend on the choice of the vector $\boldsymbol{l}$. (The theoretical explanation for this phenomenon is that the observer is the non-controllable part of the whole closed-loop.) The feedback "controller" introduced in Fig. 9.3 can also be determined now as

$$R_f = \boldsymbol{k}^T \left(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}^T + \boldsymbol{l}\boldsymbol{c}^T\right)^{-1} \boldsymbol{l} = \frac{\boldsymbol{k}^T \left(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}^T\right)^{-1} \boldsymbol{l}}{1 + \boldsymbol{c}^T \left(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}^T\right)^{-1} \boldsymbol{l}} \tag{9.20}$$

which has a more complex form than in (9.15).

To investigate the operation of the observer, let us define a new state vector error as

$$\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}} \tag{9.21}$$

which can also be written as

$$\frac{\mathrm{d}\tilde{\boldsymbol{x}}}{\mathrm{d}t} = \left(\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}^{\mathrm{T}}\right)\tilde{\boldsymbol{x}} \tag{9.22}$$

which is very similar to (9.4) without excitation. For the design of observers, a method very similar to what was used in the case of the state-feedback, is applied, where by the choice of $\boldsymbol{l}$ our goal is to ensure the dynamics of (9.21) by the second characteristic polynomial

$$\det\left(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{l}\boldsymbol{c}^{\mathrm{T}}\right) = \mathcal{F}(s) = s^n + f_1 s^{n-1} + \cdots + f_{n-1}s + f_n \tag{9.23}$$

A solution always exists if the process is observable. (It is reasonable to assume that the order of $\mathcal{F}$ is equal to that of $\mathcal{A}$.) It is an exceptional case when the transfer function of the process to be controlled is known, by means of which the canonical state-equations can be directly written. Based on the observable canonical form of (3.53), the system matrices are

$$\boldsymbol{A}_{\mathrm{o}} = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix}; \quad \boldsymbol{c}_{\mathrm{o}}^{\mathrm{T}} = [1, 0, \dots, 0]; \quad \boldsymbol{b}_{\mathrm{o}} = [b_1, b_2, \dots, b_n]^{\mathrm{T}} \tag{9.24}$$

Considering the special form of $\boldsymbol{A}_{\mathrm{o}}$ and $\boldsymbol{c}_{\mathrm{o}}^{\mathrm{T}}$ it can be easily seen, that according to the design equation

$$\boldsymbol{A}_{\mathrm{o}} - \boldsymbol{l}_{\mathrm{o}}\boldsymbol{c}_{\mathrm{o}}^{\mathrm{T}} = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix} - \boldsymbol{l}_{\mathrm{o}}[1, 0, \dots, 0] =$$

$$= \begin{bmatrix} -f_1 & 1 & 0 & \dots & 0 \\ -f_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -f_{n-1} & 0 & 0 & \dots & 1 \\ -f_n & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{9.25}$$

the choice

$$\boldsymbol{l} = \boldsymbol{l}_{\mathrm{o}} = [f_1 - a_1, f_2 - a_2, \ldots, f_n - a_n]^{\mathrm{T}} \tag{9.26}$$

ensures the satisfaction of the characteristic equation of (9.23), i.e. the prescribed poles.

The general case now is that the state-space form of the process to be controlled is given instead of its transfer function. Referring to Eq. (3.79), it has been discussed that all observable systems can be written in observable canonical form by using the transformation matrix $\boldsymbol{T}_{\mathrm{o}} = \left(\boldsymbol{M}_{\mathrm{o}}^{\mathrm{o}}\right)^{-1}\boldsymbol{M}_{\mathrm{o}}$. This similarity transformation has an effect also on the feedback vector

$$\boldsymbol{l} = (\boldsymbol{T}_{\mathrm{o}})^{-1}\boldsymbol{l}_{\mathrm{o}} = \boldsymbol{M}_{\mathrm{o}}^{-1}\boldsymbol{M}_{\mathrm{o}}^{\mathrm{o}}\boldsymbol{l}_{\mathrm{o}} \tag{9.27}$$

To calculate (9.27) the inverse of the observability matrix $\boldsymbol{M}_{\mathrm{o}}$ is required using the system matrices $\boldsymbol{A}$ and $\boldsymbol{c}^{\mathrm{T}}$. Similarly the observability matrix $\boldsymbol{M}_{\mathrm{o}}^{\mathrm{o}}$ of the observable canonical form has to be formed [see (3.73)]. Since this latter one depends only on the coefficients a_i in the denominator of the transfer function of the process, so the denominator has to be calculated: $\mathcal{A}(s) = \det(s\boldsymbol{I} - \boldsymbol{A})$. This method of calculating the observer vector is called the Ackermann method, after its developer.

There is an interesting similarity in the design methods of the dynamics of the observer and the state-feedback, often called duality, i.e., they correspond to each other under the conditions: $\boldsymbol{A} \leftrightarrow \boldsymbol{A}^{\mathrm{T}}$, $\boldsymbol{b} \leftrightarrow \boldsymbol{c}^{\mathrm{T}}$, $\boldsymbol{k} \leftrightarrow \boldsymbol{l}^{\mathrm{T}}$, $\boldsymbol{M}_{\mathrm{c}}^{\mathrm{c}} \leftrightarrow \left(\boldsymbol{M}_{\mathrm{o}}^{\mathrm{o}}\right)^{\mathrm{T}}$.

Based on the equations of the error (9.21) and the process (9.1), the joint equations of the state-feedback and the observer are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \boldsymbol{x} \\ \tilde{\boldsymbol{x}} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^{\mathrm{T}} & \boldsymbol{b}\boldsymbol{k}^{\mathrm{T}} \\ \boldsymbol{0} & \boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \tilde{\boldsymbol{x}} \end{bmatrix} + \begin{bmatrix} k_{\mathrm{r}}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix} r \\ e &= y - \hat{y} = \boldsymbol{c}^{\mathrm{T}}\tilde{\boldsymbol{x}} \end{aligned} \tag{9.28}$$

Since the system matrix of the right hand side is block diagonal, the characteristic equation of the closed-loop is

$$\det\left(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)\det\left(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{l}\boldsymbol{c}^{\mathrm{T}}\right) = \mathcal{R}(s)\mathcal{F}(s) \tag{9.29}$$

Thus the polynomial is the product of two terms: the first term relates to the state-feedback, the other one to the observer. It is important to note, that $\mathcal{F}(s)$, in spite of (9.29), does not appear in the transfer function $T_{\mathrm{ry}}(s)$ of the closed-loop of (9.5). This interesting fact can be explained by the re-definition of the whole system given in the block diagram of Fig. 9.4, applying appropriate transfer functions.

Equation (9.29) of the observer based state-feedback, according to which the state-feedback and the characteristic equation of the observer are independent, is called the *separation principle*.

9.3 Observer Based State Feedback Using Equivalent Transfer Functions

The block scheme containing transfer functions has already been applied in the Fig. 9.3. A further generalized form of the approach used there can also be applied, which is shown in Fig. 9.5.

It follows from Fig. 9.5 that the resulting equivalent serial compensator is now again

$$R_s = \frac{1}{1+R_f P} = \frac{1}{1+K_k P} = \frac{\mathcal{A}(s)}{\mathcal{A}(s)+\mathcal{K}(s)} = \frac{\mathcal{A}(s)}{\mathcal{R}(s)} \tag{9.30}$$

It must be stated that R_s is a fictitious term: it is used only for demonstrating the final signal formation, i.e., $k_r R_s P$ ensures the same T_{ry} as (9.14). If the pole cancellation represented by R_s is intended to be performed by a serial compensator, then it cannot be applied to unstable processes, since the unstable zeros and poles cannot be eliminated by cancellation. The signal $\bar{x}$ (which is not the same as $\boldsymbol{x}$) introduced in Fig. 9.4 represents that finally both the state-feedback and the observer are *SISO* subsystems which can be performed by transfer functions, i.e., it is always possible to find equivalent representations for the input and output. Applying this approach and based on Fig. 9.4, the block scheme using transfer functions can be drawn as shown in Fig. 9.6.

After a long transformation procedure and block manipulations the block scheme of Fig. 9.6 can be traced back to the very simple, unit feedback closed-loop shown

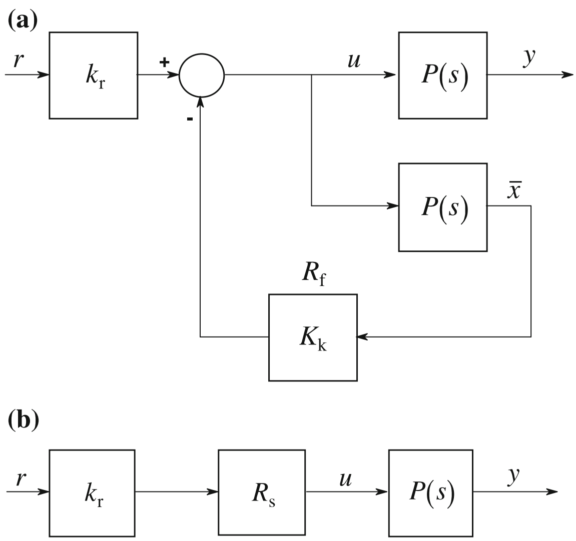

Fig. 9.5 The further equivalent schemes of the state feedback with transfer functions

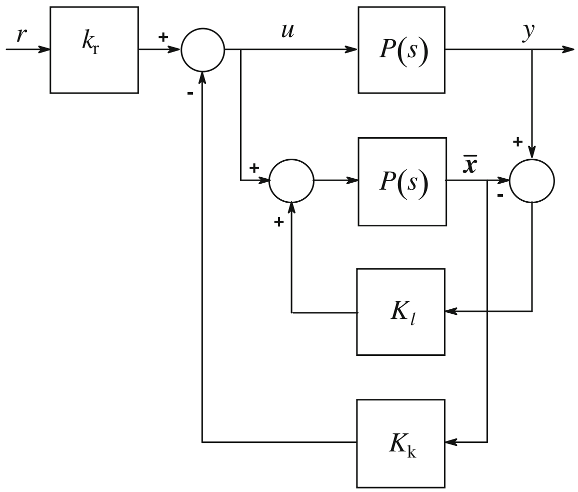

Fig. 9.6 State-feedback and observer using transfer functions

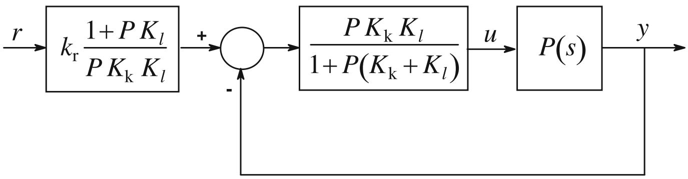

Fig. 9.7 The reduced block scheme of the state-feedback and observer

in Fig. 9.7. Here the relationship (9.15) defining K_k is also used, and K_l is introduced in a similar way

$$K_k(s) = \frac{\mathcal{K}(s)}{\mathcal{B}(s)}; \quad K_l(s) = \frac{\mathcal{L}(s)}{\mathcal{B}(s)}, \tag{9.31}$$

where the design polynomial equations

$$\mathcal{K}(s) = \mathcal{R}(s) - \mathcal{A}(s) \quad \text{and} \quad \mathcal{L}(s) = \mathcal{F}(s) - \mathcal{A}(s) \tag{9.32}$$

result from the conditions of the two kinds of pole placements.

It is easily seen that the resulting transfer function of the inner closed-loop

$$\frac{P^2 K_k K_l}{1 + P(K_k + K_l) + P^2 K_k K_l} = \frac{PK_k}{1 + PK_k} \frac{PK_l}{1 + PK_l} = \frac{\mathcal{K}}{\mathcal{A} + \mathcal{K}} \frac{\mathcal{L}}{\mathcal{A} + \mathcal{L}} = \frac{\mathcal{K}}{\mathcal{R}} \frac{\mathcal{L}}{\mathcal{F}} \tag{9.33}$$

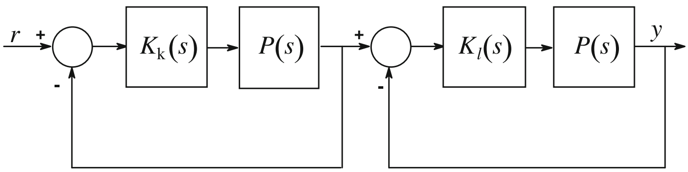

Fig. 9.8 Equivalent observer block schemes of the inner system

has a special form, but its denominator completely corresponds to the characteristic equation (9.29), i.e., represents two serially connected independent closed-loops (see Fig. 9.8). This fact is called the separation principle of the state-feedback and the observer. To ensure stability, both loops must be stable. This can be arranged by proper pole placement design.

At the same time, the transfer function of the whole system is

$$T_{ry}(s) = k_r \frac{1+PK_k}{PK_l K_k} \frac{PK_l}{1+PK_l} \frac{PK_k}{1+PK_k} = \frac{k_r P}{1+PK_l} = \frac{k_r \frac{\mathcal{B}}{\mathcal{A}}}{1+\frac{\mathcal{B}}{\mathcal{A}}\frac{\mathcal{K}}{\mathcal{B}}} = \frac{k_r \mathcal{B}}{\mathcal{A}+\mathcal{K}} = \frac{k_r \mathcal{B}(s)}{\mathcal{R}(s)}, \tag{9.34}$$

which is completely the same as (9.19). As expected, the poles of the observer do not appear in T_{ry}. The inner character of the whole system can be better seen from the final block scheme shown in Fig. 9.9 for the tracking properties.

This simple structure is not valid for the disturbance rejection capabilities of the closed-loop. This can be simply seen if the sensitivity function of the closed-loop is constructed,

$$\frac{1}{1+\dfrac{P^2 K_k K_l}{1+P(K_k+K_l)}} = \frac{1+P(K_k+K_l)}{1+P(K_k+K_l)+P^2 K_k K_l} = \left(1+\frac{\mathcal{L}}{\mathcal{R}}\right)\left(1-\frac{\mathcal{L}}{\mathcal{F}}\right), \tag{9.35}$$

which shows that both $\mathcal{R}$ and $\mathcal{F}$ appear in the transfer function of the disturbance rejection according to (9.29). Equation (9.35) has a special form, since formally it is the product of the output noise rejection transfer functions of two serially connected closed-loops, while it is known, that the tracking properties are indeed the result of a product of the transfer functions, but this phenomenon is not valid for the

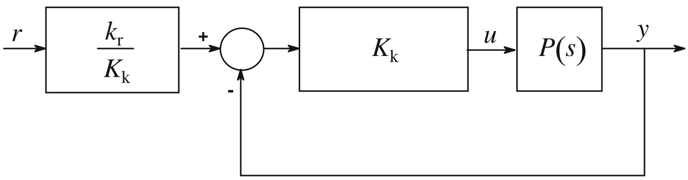

Fig. 9.9 The reduced block scheme of the state-feedback and the observer for the tracking properties

sensitivity functions. Note that the resulting noise rejection properties are not independent of the tracking ones, therefore the joint application of the state-feedback and the observer is not appropriate to realize an actual *TDOF* control loop.

9.4 Two-Step Design Methods Using State Feedback

It has been already seen in the discussion of the state-feedback based control that the most advantageous features of that method are:

- the applicability of the method does not depend on whether the process is stable or not
- the tracking property does not depend on the applied observer, thus it can be directly designed
- the method is not very sensitive to the exact knowledge of the parameter matrices of the state-equation.

(This last feature is usually demonstrated by experimental and simulation examples, but it can be proved that the error, using an observer, can be reduced by the $[1+K_l(s)P(s)]$ part of the original one, compared to the modeling error obtained by the simple parallel model of the state-equation of the process, thus being like that which would be obtained via a closed-loop $1/[1+K_l(s)P(s)]$. So it can be reduced by the feedback $K_l(s)$ of the observer in a specific frequency region. If the model of the process is applied, which is quite conventional practice, then both loops of the Fig. 9.8 must be robust stable.)

The unfavorable (unwanted) features are:

- the state feedback is basically a zero-type control, therefore the remaining error can be eliminated by the calibration factor, which, in the case of using a process model, never provides a precise result
- the state feedback can not change the zeros of the process
- the disturbance rejection property can not be designed directly.

Mostly because of these latter features, usually further steps are applied to the state-feedback based control systems. The necessity of the calibration factor can be eliminated in the simplest way by using a cascade integrating controller, as shown in Fig. 9.10.

Instead of (9.4), the joint state-equation of the closed-loop can be written as

$$\begin{aligned}\dot{\boldsymbol{x}}^*(t) &= \begin{bmatrix}\dot{\boldsymbol{x}}(t)\\ \dot{\delta}(t)\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{c}^{\mathrm{T}} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t)\\ \delta(t)\end{bmatrix} + \begin{bmatrix}\boldsymbol{b}\\ 0\end{bmatrix}u(t) + \begin{bmatrix}\boldsymbol{0}\\ -1\end{bmatrix}r(t)\\ &= \left(\boldsymbol{A}^* - \boldsymbol{b}^*\boldsymbol{k}_*^{\mathrm{T}}\right)\boldsymbol{x}^*(t) + \boldsymbol{v}^*r(t)\end{aligned} \qquad (9.36)$$

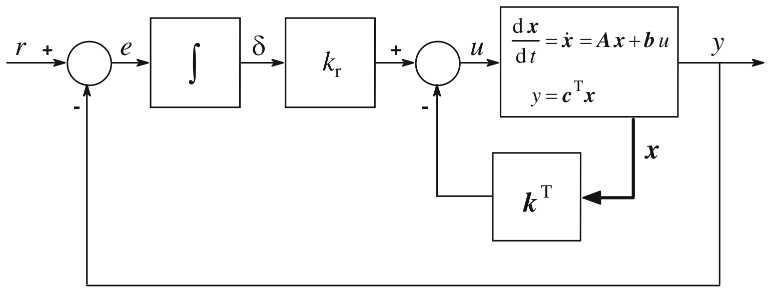

Fig. 9.10 Joint state-feedback and integrating controller

by introducing the new state variable $\delta(t)$, which is the integral of the error $e(t) = r(t) - y(t)$ in the outer loop. In this extended state-equation, the notation

$$\boldsymbol{A}^* = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{c}^{\mathrm{T}} & 0 \end{bmatrix}; \quad \boldsymbol{b}^* = \begin{bmatrix} \boldsymbol{b} \\ 0 \end{bmatrix}; \quad \boldsymbol{v}^* = \begin{bmatrix} \boldsymbol{0} \\ -1 \end{bmatrix} \tag{9.37}$$

and the new extended feedback equation

$$u(t) = -\begin{bmatrix} \boldsymbol{k}^{\mathrm{T}} & k_{\mathrm{r}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \delta(t) \end{bmatrix} = -\boldsymbol{k}_*^{\mathrm{T}} \boldsymbol{x}^*(t) = k_{\mathrm{r}} \int_0^t e(\tau)\,\mathrm{d}\tau - \boldsymbol{k}^{\mathrm{T}} \boldsymbol{x}(t) \tag{9.38}$$

are employed. Equation (9.38) clearly shows the integrating effect. The term $\boldsymbol{k}^{\mathrm{T}}\boldsymbol{x}(t)$, however, can be considered as a generalization of the differentiating effect.

Thus the closed control loop including an integrator can be formulated by a state-equation of order greater by one, where besides the coefficient $\boldsymbol{k}^{\mathrm{T}}$, now k_{r} has to be also determined. To design the extended system, the characteristic polynomial $\mathcal{R}^*(s)$ of order $(n+1)$ has to be required, and then the design Eq. (9.10) of the Ackermann method can be directly applied here too. If the process is not presented in the transfer function form, then first the general state-equation has to be transformed into the controllable canonical form, as was already shown in (9.13).

Note that the extended task can not be solved sequentially, i.e., in such a way that first the $\boldsymbol{k}^{\mathrm{T}}$ relating to $\mathcal{R}(s)$ is determined, then k_{r} based on $\mathcal{R}^*(s) = \mathcal{R}(s)(s - s_{n+1})$ is calculated. The task must be solved in one step for $\boldsymbol{k}_*^{\mathrm{T}}$ by $\mathcal{R}^*(s)$.

An integrating effect can also be included by the design of the state-feedback for a modified process $P^*(s) = P(s)/s$ instead of the transfer function $P(s)$. Note that the two state feedback vectors, obtained for the previous case and for this approach, are not equal!

Obviously beside the *I*-controller, other—higher order—controllers can be also applied, but the pole placement is not always automatically given by the Ackermann method, and can result in complicated systems of non-linear equations.

In the case of observer based state-feedback, at the feedback of the observer error, not only zero-type, but one-type or higher-type controllers can also be applied by the methods shown above.

The untouched zeros of the process can be modified by a serial compensator

$$K_s(s) = G_s(s)\frac{\mathcal{N}(s)}{\mathcal{B}_+(s)} \tag{9.39}$$

too, where the numerator of the process $\mathcal{B}(s) = \mathcal{B}_+(s)\mathcal{B}_-(s)$ is assumed according to the method applied in the Chap. 7. Here $\mathcal{B}_+$ is stable, $\mathcal{B}_-$, however, contains the unstable zeros. For realizability, $\mathcal{N}(s)/\mathcal{B}_+(s)$ must be proper, thus only as many zeros can be placed in the transfer function of the closed-loop as many stable zeros are in the process. Finally the resulting transfer function has the form

$$T_{ry}(s) = \frac{\mathcal{N}(s)}{\mathcal{R}(s)} k_r G_s(s)\mathcal{B}_-(s) \tag{9.40}$$

where the effect of the invariant $\mathcal{B}_-(s)$ can be optimally attenuated by the filter $G_s(s)$. In many cases, however, the simple, but not optimal, choice $G_s(s) = 1$ is used.

An acceptable design of the disturbance rejection feature can be reached by the application of the Youla-parameterized controller in the outer cascade loop. It can be done because by the state-feedback any process, even an unstable one, can be stabilized. The qualitative control of the unstable processes has two steps in general. In the first step the process is stabilized by the controller, then the required qualitative goals can be reached by a second outer control loop or even in *TDOF* structures.

The state-feedback based stabilizing controller can only be applied to processes without dead-time. If the process has considerable time-delay, then one possibility is to approach the dead-time by rational fractions [see Sect. 2.5]. The other solution is to use computer based sampled data control [see Chap. 15].

9.5 The *LQ* Controller

The method shown in the previous sections of this chapter could perform arbitrary (stabilizing) pole placement by the so-called state feedback from the state vector of the process. By this state feedback technique further optimization tasks can also be solved. The goal of this task is to optimally control the *LTI* process (9.1) by the minimization of a complex optimality criterion

$$I = \frac{1}{2}\int_0^\infty \left[\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{W}_{\mathrm{x}}\boldsymbol{x}(t) + W_{\mathrm{u}}u^2(t)\right]\mathrm{d}t. \tag{9.41}$$

Here $\boldsymbol{W}_{\mathrm{x}}$ is a real symmetrical positive semi-definite matrix weighting the state vector, and W_{u} is a positive constant weighting the excitation. The solution minimizing the criterion is a state-feedback

$$u(t) = -\boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}}\boldsymbol{x}(t) \tag{9.42}$$

[see (9.3)], where the feedback vector $\boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}}$ has the form

$$\boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}} = \frac{1}{W_{\mathrm{u}}}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{P}. \tag{9.43}$$

Here the symmetrical positive semi-definite matrix $\boldsymbol{P}$ comes from the solution of the algebraic Riccati matrix equation

$$\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} - \frac{1}{W_{\mathrm{u}}}\boldsymbol{P}\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{P} = -\boldsymbol{W}_{\mathrm{x}}. \tag{9.44}$$

Since this Riccati equation is non-linear in $\boldsymbol{P}$, it has no explicit algebraic solution. The *CAD* systems frequently used in the control technique, however, generally provide several numerical algorithms for the solution of this equation. This controller is called *Linear Quadratic* (*LQ*) controller. This stands for: *linear regulator—quadratic criterion*.

The state-equation of the *LQ* controller based closed-loop is

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \left(\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}}\right)\boldsymbol{x}; \quad \overline{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}}. \tag{9.45}$$

The details of the *LQ* based method are given in A.9.6 of Appendix A.5. (The above controller is very simple, but its derivation is quite time consuming.)

If the transfer function of the process is known, then the controllable canonical form can be easily given. For special $\boldsymbol{A}_{\mathrm{c}}$ and $\boldsymbol{b}_{\mathrm{c}}$, Eq. (9.10) gives the classical state feedback design algorithm. In the *LQ* method the feedback vector $\boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}}$ is obtained from the design (from the solution of the Riccati equation). So turning back the derivation of (9.10) the characteristic polynomial $\mathcal{R}(s)$ of the resulting closed-loop system can be given by its coefficients as

$$[r_1, r_2, \ldots, r_n]^{\mathrm{T}} = \boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}} + [a_1, a_2, \ldots, a_n]^{\mathrm{T}}. \tag{9.46}$$

It is also possible to employ an observer for constructing the state vector in *LQ* control.

In engineering practice it is simpler to solve the stabilizing task by pole allocation state-feedback, since there the prescribed poles are directly known. It is evident, however, that in this case the quality of the transient processes are less known. The *LQ* controller, beside the stabilization, also makes it possible to design even the quality of the transient processes, but it needs long term practice to determine the proper weighting matrix $\boldsymbol{W}_{\mathrm{x}}$ and weighting factor W_{u}, usually through a trial-and-error method.

A simpler version of the *LQ* controller is when, instead of the states, only the square of the output is weighted, similarly to the input, i.e., instead of (9.41) the criterion

$$I = \frac{1}{2}\int_0^{\infty} \left[W_{\mathrm{y}} y^2(t) + W_{\mathrm{u}} u^2(t)\right] \mathrm{d}t \tag{9.47}$$

is used. This task (in the case of $d = 0$), after some identical manipulations, can be traced back to the original *LQ* controller

$$W_{\mathrm{y}} y^2 = y W_{\mathrm{y}} y = \boldsymbol{x}^{\mathrm{T}} \boldsymbol{c} W_{\mathrm{y}} \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}} \left(\boldsymbol{c} W_{\mathrm{y}} \boldsymbol{c}^{\mathrm{T}}\right) \boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}} \left(W_{\mathrm{y}} \boldsymbol{c}\boldsymbol{c}^{\mathrm{T}}\right) \boldsymbol{x} \tag{9.48}$$

by a special choice of the weighting matrix like

$$\boldsymbol{W}_{\mathrm{x}} = W_{\mathrm{y}} \boldsymbol{c}\boldsymbol{c}^{\mathrm{T}}. \tag{9.49}$$

Observe that the state-feedback $\boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}}$ leaves the process zeros untouched.