Chapter 3
Description of Continuous-Time Systems in State-Space

The so-called state-equations are widely used in the scientific and engineering fields for the description of dynamical systems. The necessity for this kind of description is explained in different ways. Perhaps the easiest way is the recognition that the operation of a wide class of complex dynamical systems can be modeled with relatively high precision by the first order vector differential equations

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} &= \dot{\boldsymbol{x}}(t) = \boldsymbol{f}[\boldsymbol{x}(t), u(t)] \\ y(t) &= g[\boldsymbol{x}(t), u(t)] \end{aligned} \tag{3.1}$$

The state variables of the system as scalar components are collected in a vector $\boldsymbol{x}$ called the state vector. The system input is u, and the output is y. The dimension of $\boldsymbol{x}$ is called the degree or the order of the system. The function $\boldsymbol{f}(\boldsymbol{x}, u)$ represents the varying "speed" of the state vector as the function of the states and the input signal. The function $g(\boldsymbol{x}, u)$ is called sensor or measurement function since it provides the output of the system. Let's call attention here to the fact that $\boldsymbol{f}(\boldsymbol{x}, u)$ and $g(\boldsymbol{x}, u)$ do not depend on time in an explicit way. (But here we emphasize that nevertheless the signals of the state-equations obviously depend on time!) This kind of system is called a time-invariant system. The state variables contain the information about the past of the system, and the future values of the signals can be predicted, therefore the state vector behaves like the memory of the system.

In engineering systems the state vectors are often related to the basic physical processes, where the relations necessary to describe the storage of the mass, flow, impulse, and power, have to be determined. (It has to be noted, however, that in certain fields, e.g. in chemistry, the definition of the state vector is different from the above general system-theoretical concept: it mostly reflects the variables—like pressure, temperature, composition, etc.—representing the physico-chemical state of the investigated material, mixture, compound, etc.)

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The state variables as coordinates define a space (*state-space*). The state vector $\boldsymbol{x}(t)$ is interpreted in this space. The motion of the end point of the vector represents the motion of the system. The curve described by the motion of the end point of the state vector gives the state-trajectory.

A special class of the non-linear dynamical systems is given by the Eq. (3.1), whose possible equilibrium state $(\boldsymbol{x}_o, u_o)$ (where $\dot{\boldsymbol{x}} = \boldsymbol{0}$) is obtained from the equation

$$\boldsymbol{f}(\boldsymbol{x}_o, u_o) = \boldsymbol{0}. \tag{3.2}$$

(Remark: In general, several equilibrium states can be obtained. These equilibrium states can provide different stable states. The performance of these states requires the investigation of the second order derivatives of $\boldsymbol{f}(\boldsymbol{x}, u)$.)

The static systems can be described by degenerate state-equations, since they do not have memory, or the corresponding states, so they can be described by the second equation of (3.1) by itself

$$y = g(u) \tag{3.3}$$

Taking the Taylor-expansion at the point u_o, we get

$$y = g(u_o) + \frac{dg(u_o)}{du}(u - u_o) + \cdots = g(u_o) + g'(u_o)(u - u_o) + \cdots \tag{3.4}$$

and the linearized model

$$y - y_o = \Delta y = y - g(u_o) = g'(u_o)(u - u_o) = g'(u_o)\Delta u \tag{3.5}$$

can be obtained from the first order term of (3.4).

The linearized model replaces the original curve with its tangent at the operating point u_o and establishes a static linear connection between the changes $(\Delta y, \Delta u)$ around the operating point.

Actually the linearization of the state-space Eq. (3.1) can also be given in a very similar way.

With the following notation, valid for changes around the equilibrium state $(\boldsymbol{x}_o, u_o)$, i.e.

$$\boldsymbol{x} = \boldsymbol{x}_o + \Delta\boldsymbol{x}; \quad u = u_o + \Delta u; \quad y = y_o + \Delta y \tag{3.6}$$

let us calculate the first order linearized approach of (3.1)

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{f}(\boldsymbol{x}_o + \Delta\boldsymbol{x}, u_o + \Delta u) \approx \boldsymbol{f}(\boldsymbol{x}_o, u_o) + \frac{d\boldsymbol{f}(\boldsymbol{x}_o, u_o)}{d\boldsymbol{x}^T}\Delta\boldsymbol{x} + \frac{d\boldsymbol{f}(\boldsymbol{x}_o, u_o)}{du}\Delta u \\ y &= g(\boldsymbol{x}_o + \Delta\boldsymbol{x}, u_o + \Delta u) \approx g(\boldsymbol{x}_o, u_o) + \frac{dg(\boldsymbol{x}_o, u_o)}{d\boldsymbol{x}^T}\Delta\boldsymbol{x} + \frac{dg(\boldsymbol{x}_o, u_o)}{du}\Delta u \end{aligned} \tag{3.7}$$

Let us use the fact that at the equilibrium point $\boldsymbol{f}(\boldsymbol{x}_\text{o}, u_\text{o}) = 0$ and let us introduce the notation $y_\text{o} = g(\boldsymbol{x}_\text{o}, u_\text{o})$, so the linearized model valid for small changes takes the form

$$\begin{aligned} \frac{\text{d}(\boldsymbol{x} - \boldsymbol{x}_\text{o})}{\text{d}t} &= \frac{\text{d}\Delta \boldsymbol{x}}{\text{d}t} = \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x}_\text{o}) + \boldsymbol{b}(u - u_\text{o}) = \boldsymbol{A}\Delta\boldsymbol{x} + \boldsymbol{b}\Delta u \\ y - y_\text{o} &= \Delta y = \boldsymbol{c}^\text{T}(\boldsymbol{x} - \boldsymbol{x}_\text{o}) + d(u - u_\text{o}) = \boldsymbol{c}^\text{T}\Delta\boldsymbol{x} + d\Delta u \end{aligned} \tag{3.8}$$

where the following notations are employed

$$\begin{aligned} \boldsymbol{A} &= \frac{\text{d}\boldsymbol{f}(\boldsymbol{x}_\text{o}, u_\text{o})}{\text{d}\boldsymbol{x}^\text{T}}; \quad \boldsymbol{b} = \frac{\text{d}\boldsymbol{f}(\boldsymbol{x}_\text{o}, u_\text{o})}{\text{d}u} \\ \boldsymbol{c}^\text{T} &= \frac{\text{d}g(\boldsymbol{x}_\text{o}, u_\text{o})}{\text{d}\boldsymbol{x}^\text{T}}; \quad d = \frac{\text{d}g(\boldsymbol{x}_\text{o}, u_\text{o})}{\text{d}u} \end{aligned} \tag{3.9}$$

The obtained model is a linear time-invariant (*LTI*) system, i.e. it does not change in time. It is a widely used practice that the original variables $\boldsymbol{x}, u, y$ are used instead of the small changes $(\Delta\boldsymbol{x}, \Delta u, \Delta y)$ for simplicity, but they are considered as the changes around the operating point. In this way we arrive at the linear, constant parameter (*LTI*) state-space equation of the system generally applied in the theory of systems and control,

$$\begin{aligned} \frac{\text{d}x(t)}{\text{d}t} &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) \qquad & & \frac{\text{d}x}{\text{d}t} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u \\ & & \text{or simply} & \\ y(t) &= \boldsymbol{c}^\text{T}\boldsymbol{x}(t) + du(t) & & y = \boldsymbol{c}^\text{T}\boldsymbol{x} + du \end{aligned} \tag{3.10}$$

Here the parameter matrices of the system are $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}^\text{T}, d$. Since in this book single-input single-output (*SISO*) systems are considered, in the n-order case, $\boldsymbol{A}$ is an $(n \times n)$ square matrix, which is the so called state matrix, $\boldsymbol{b}$ is an $(n \times 1)$ column vector, $\boldsymbol{c}^\text{T}$ is a row vector of dimensions $(1 \times n)$, and d is a scalar. The block diagram of the state-Eq. (3.10) can be seen in Fig. 3.1.

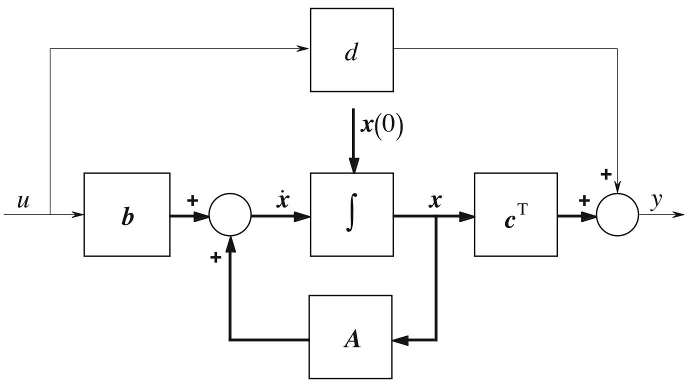

Fig. 3.1 Block diagram of the linear time invariant system

3.1 Solution of the State-Equations in the Complex Frequency Domain

The state-equations can be transferred to the complex frequency domain by the Laplace transformation of (3.10). Let us denote the transformed time functions $\boldsymbol{x}, u, y$ by $\boldsymbol{X}(s), U(s), Y(s)$. Taking the rules of transformation of derivatives into account we get

$$\begin{aligned} s\boldsymbol{X}(s) &= \boldsymbol{AX}(s) + \boldsymbol{b}U(s) + \boldsymbol{x}(0) = \boldsymbol{AX}(s) + \boldsymbol{b}U(s) + \boldsymbol{Ix}(0) \\ Y(s) &= \boldsymbol{c}^{\mathrm{T}}\boldsymbol{X}(s) + dU(s) \end{aligned} \tag{3.11}$$

In the first equation, the vector of the initial conditions $\boldsymbol{x}(0)$ can be considered as an input which has an effect on the system via the identity matrix $\boldsymbol{I}$. From the first equation we get

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{b}U(s) + \boldsymbol{x}(0)] = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}U(s) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0). \tag{3.12}$$

According to the rules of matrix inversion

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{\mathbf{adj}(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} = \frac{\mathbf{adj}(s\boldsymbol{I} - \boldsymbol{A})}{\mathcal{A}(s)} = \frac{\boldsymbol{\Psi}(s)}{\mathcal{A}(s)} = \boldsymbol{\Phi}(s). \tag{3.13}$$

Here $\boldsymbol{\Psi}(s) = \mathbf{adj}(s\boldsymbol{I} - \boldsymbol{A})$ is the transpose of a matrix whose elements are the signed sub-determinants belonging to the corresponding elements of the matrix $(s\boldsymbol{I} - \boldsymbol{A})$. The determinant of that matrix, $\det(s\boldsymbol{I} - \boldsymbol{A})$ is the denominator of the transfer function, and is an n-degree polynomial in s:

$$\mathcal{A}(s) = s^n + k_1 s^{n-1} + \cdots + k_{n-1}s + k_n = \prod_{i-1}^{n}(s - \lambda_i) = \det(s\boldsymbol{I} - \boldsymbol{A}). \tag{3.14}$$

$\mathcal{A}(s)$ is the so-called characteristic polynomial of the matrix $\boldsymbol{A}$. The roots $\lambda_1, \ldots \lambda_n$ of the characteristic equation $\mathcal{A}(s) = 0$ are the eigenvalues of $\boldsymbol{A}$, called the poles of the system.

The elements of the matrix in the numerator of (3.13) are also polynomials in s, but since they come from an $(n-1)$ order sub-determinant, they can have at most order $(n-1)$, consequently the quotients of each element and $\mathcal{A}(s)$ represent strictly proper transfer functions.

According to (3.12) the motion of the state vector is determined by the initial condition $\boldsymbol{x}(0)$ and the input signal $U(s)$. Since the characteristic polynomial is in the denominators of all the elements depending on $\boldsymbol{x}(0)$, as a consequence of the expansion theorem, their time functions are exclusively determined by the poles of the system. This part of the solution describes the motion of a un-excited system from any initial position to its equilibrium point and it exclusively depends on one of the parameters, i.e., on the state matrix $\boldsymbol{A}$ both in the frequency and time domains.

In the case of excitation, in each element of the solution depending on $U(s)$, the denominator contains not only $\mathcal{A}(s)$ but also the denominator polynomial of $U(s)$, so the time functions depend not only on the poles of the system but also on the poles of the input. This part of the solution gives the motion of the excited system.

From Eqs. (3.11) and (3.12), the output is

$$Y(s) = \boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{b}U(s) + \boldsymbol{x}(0)] + dU(s). \tag{3.15}$$

The output of the excited motion, when $\boldsymbol{x}(0) = \boldsymbol{0}$ is

$$Y(s) = \left[\boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + d\right]U(s). \tag{3.16}$$

Thus the transfer function of the system is

$$P(s) = \frac{Y(s)}{U(s)} = \boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + d\big|_{d=0} = \boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} = \frac{\mathcal{B}(s)}{\mathcal{A}(s)}. \tag{3.17}$$

The first term of $P(s)$ is strictly proper since it consists of a linear combination of only proper elements (see (3.13), i.e., the order of the adjoint is always lower than that of the determinant). Thus if $d = 0$, then $P(s)$ is strictly proper, the order of the numerator being lower by at least one than that of the denominator. If $d \neq 0$, then $P(s)$ is proper, i.e. the order of the numerator is equal to that of the denominator. The physical meaning of d is how the input directly influences the output without any dynamics. Note, that this effect does not disappear even for very high frequencies, thus $P(j\omega \to \infty) = d$. This means, at the same time, that the jump of the transfer function at time $t = 0$ is $v(t = 0) = d$. In practice the case $d \neq 0$ is usually traced back to the case $d = 0$ by introducing a new output $\tilde{y} = y - du$. The case $d \neq 0$ can also be considered as an imperfect linearization which needs a certain correction.

Example 3.1 Let the parameter matrices of the system be

$$\boldsymbol{A} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}; \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \boldsymbol{c}^{\mathrm{T}} = [2 \quad 2],$$

and compute the transfer function using (3.17).

$$P(s) = \frac{B(s)}{A(s)} = \boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} = \frac{1}{s^2 + 3s + 2}[2 \quad 2]\begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \frac{1}{s^2 + 3s + 2}[2s + 2 \quad -4 + 2s + 6]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2s + 2}{s^2 + 3s + 2} = \frac{s + 1}{0.5s^2 + 1.5s + 1}$$

■

3.2 Solution of the State-Equations in the Time Domain

The solution of the state-Eq. (3.10) in time domain can also be given in closed form

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)\mathrm{d}\tau = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \left[\int_0^t e^{\boldsymbol{A}(t-\tau)}u(\tau)\mathrm{d}\tau\right]\boldsymbol{b}. \quad (3.18)$$

The first term represents the motion of the un-excited system starting from the initial point $\boldsymbol{x}(0)$, the second term is the convolution integral, i.e., the excited motion starting from the initial point $\boldsymbol{x}(0) = \boldsymbol{0}$.

To check (3.18), let us differentiate the above equation with respect to time:

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{A}e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t \boldsymbol{A}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)\mathrm{d}\tau + \boldsymbol{b}u(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad (3.19)$$

which proves the correctness of (3.18). (See the detailed derivation in Chap. A.3.1 of Appendix A.5.) Here, $e^{\boldsymbol{A}t}$ is the fundamental matrix of the system, which is defined by its Taylor-series, convergent for all t, as is valid for matrix functions in general.

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{1}{2}(\boldsymbol{A}t)^2 + \cdots + \frac{1}{n!}(\boldsymbol{A}t)^n + \cdots. \quad (3.20)$$

By differentiating the equation, a very interesting and important feature of the fundamental matrix can be obtained.

$$\begin{aligned}\frac{\mathrm{d}e^{\boldsymbol{A}t}}{\mathrm{d}t} &= \boldsymbol{A} + \boldsymbol{A}^2 t + \frac{1}{2}\boldsymbol{A}^3 t^2 + \cdots + \frac{1}{(n-1)!}\boldsymbol{A}^n t^{n-1} + \cdots \\ &= \boldsymbol{A}\left(\boldsymbol{I} + \boldsymbol{A}t + \frac{1}{2}(\boldsymbol{A}t)^2 + \cdots + \frac{1}{n!}(\boldsymbol{A}t)^n + \cdots\right) = \boldsymbol{A}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}\end{aligned} \quad (3.21)$$

Comparing (3.12) and (3.18), the Laplace-transform of the fundamental matrix for $U(s) = 0$ is

$$\mathcal{L}\{e^{\boldsymbol{A}t}\} = (s\boldsymbol{I} - \boldsymbol{A})^{-1} = \boldsymbol{\Phi}(s), \quad (3.22)$$

which provides a new relationship for the computation of the fundamental matrix:

$$e^{\boldsymbol{A}t} = \mathcal{L}^{-1}\left\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}\right\} = \mathcal{L}^{-1}\{\boldsymbol{\Phi}(s)\} \quad (3.23)$$

Combining (3.10) and (3.18) shows that the output of the system is

$$y(t) = \boldsymbol{c}^{\mathrm{T}} e^{\boldsymbol{A}t} \boldsymbol{x}(0) + \boldsymbol{c}^{\mathrm{T}} \left[\int_0^t e^{\boldsymbol{A}(t-\tau)} u(\tau) \mathrm{d}\tau \right] \boldsymbol{b} + du(t) \tag{3.24}$$

In the case of zero initial conditions ($\boldsymbol{x}(0) = \boldsymbol{0}$) and $d = 0$, the weighting function of the system for the excitation $u(t) = \delta(t)$ can be easily obtained from the last equation

$$\begin{aligned} w(t) &= \boldsymbol{c}^{\mathrm{T}} e^{\boldsymbol{A}t} \boldsymbol{b} = \boldsymbol{c}^{\mathrm{T}} \mathcal{L}^{-1} \left\{ (s\boldsymbol{I} - \boldsymbol{A})^{-1} \right\} \boldsymbol{b} = \mathcal{L}^{-1} \left\{ \boldsymbol{c}^{\mathrm{T}} (s\boldsymbol{I} - \boldsymbol{A})^{-1} \boldsymbol{b} \right\} \\ &= \mathcal{L}^{-1} \left\{ \boldsymbol{c}^{\mathrm{T}} \boldsymbol{\Phi}(s) \boldsymbol{b} \right\} = \mathcal{L}^{-1} \left\{ P(s)|_{d=0} \right\} \end{aligned} \tag{3.25}$$

See the details for the weighting function computation in Chap. A.3.2 of Appendix A.5.

As a consequence of matrix function operations and the Cayley-Hamilton theorem the fundamental matrix can also be computed in the form of finite sum:

$$e^{\boldsymbol{A}\tau} = \alpha_{\mathrm{o}}(\tau)\boldsymbol{I} + \alpha_1(\tau)\boldsymbol{A} + \cdots + \alpha_{\mathrm{n}-1}(\tau)\boldsymbol{A}^{n-1} \tag{3.26}$$

since the state matrix $\boldsymbol{A}$ satisfies its characteristic equation, i.e.

$$\mathcal{A}(\boldsymbol{A}) = \boldsymbol{0}. \tag{3.27}$$

(See the proofs in A.3.3 of Appendix A.5.)

3.3 Transformation of the State-Equations, Canonical Forms

The input and output signals of a system are usually certain physical variables. The state variables, however, depend on the chosen coordinate system. The parameter matrices $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}^{\mathrm{T}}$ also depend on the coordinate system. Introduce the new state vector $\boldsymbol{z}$, which can be obtained from $\boldsymbol{x}$ by a linear transformation $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$ where $\boldsymbol{T}$ is regular. Using (3.10) the new state-equations are

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} &= \boldsymbol{T}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u) = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}\boldsymbol{z} + \boldsymbol{T}\boldsymbol{b}u = \tilde{\boldsymbol{A}}\boldsymbol{z} + \tilde{\boldsymbol{b}}u \\ y &= \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + du = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{T}^{-1}\boldsymbol{z} + du = \tilde{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{z} + \tilde{d}u \end{aligned} \tag{3.28}$$

where

$$\tilde{\boldsymbol{A}} = \boldsymbol{TAT}^{-1}, \quad \tilde{\boldsymbol{b}} = \boldsymbol{Tb}, \quad \tilde{\boldsymbol{c}}^{\mathrm{T}} = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{T}^{-1}, \quad \tilde{d} = d. \tag{3.29}$$

It is easy to check that the weighting function and the transfer function of the system are invariant under linear transformations:

$$w(t) = \tilde{\boldsymbol{c}}^{\mathrm{T}} e^{\tilde{\boldsymbol{A}}t}\tilde{\boldsymbol{b}} = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{T}^{-1}e^{\boldsymbol{TAT}^{-1}t}\boldsymbol{Tb} = \boldsymbol{c}^{\mathrm{T}}e^{\boldsymbol{A}t}\boldsymbol{b} \tag{3.30}$$

$$H(s) = \tilde{\boldsymbol{c}}^{\mathrm{T}}\left(s\boldsymbol{I} - \tilde{\boldsymbol{A}}\right)^{-1}\tilde{\boldsymbol{b}} = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{T}^{-1}\left(s\boldsymbol{I} - \boldsymbol{TAT}^{-1}\right)^{-1}\boldsymbol{Tb} = \boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} \tag{3.31}$$

In (3.30) the following simple identity (obtained by Taylor series of e^x) was employed

$$e^{\left(\boldsymbol{TAT}^{-1}\right)t} = \boldsymbol{T}e^{\boldsymbol{A}t}\boldsymbol{T}^{-1}. \tag{3.32}$$

It is well known, that a linear transformation has certain special directions in which the vectors keep their directions, only their lengths change by a factor of λ_i, i.e.

$$\boldsymbol{A}\boldsymbol{v}_i = \lambda_i\boldsymbol{v}_i, \quad i = 1, \ldots, n. \tag{3.33}$$

Here $\boldsymbol{v}_i$ is the eigenvector of $\boldsymbol{A}$, and λ_i is its eigenvalue. The eigenvalue problem can also be formulated in a different way, i.e. as a homogeneous system equations in n unknown variables

$$(\lambda_i\boldsymbol{I} - \boldsymbol{A})\boldsymbol{v} = \boldsymbol{0}, \tag{3.34}$$

where the variables are the components of $\boldsymbol{v}$. This system of equations has a solution different from the trivial one ($\boldsymbol{v} = \boldsymbol{0}$) if the condition

$$\det(\lambda_i\boldsymbol{I} - \boldsymbol{A}) = \mathcal{A}(\lambda_i) = 0 \tag{3.35}$$

is satisfied, i.e. the eigenvalues λ_i are the roots of the characteristic polynomial. If the roots are single, then the total number of the eigenvalues is n, and each has only one eigenvector of unit length.

3.3.1 Diagonal Canonical Form

In the case of single eigenvalues, by choosing a special transformation matrix $\boldsymbol{T}_{\mathrm{d}}$, one can make $\boldsymbol{T}_{\mathrm{d}}\boldsymbol{A}(\boldsymbol{T}_{\mathrm{d}})^{-1}$ diagonal:

$$\tilde{\boldsymbol{A}}_{\mathrm{d}} = \boldsymbol{T}_{\mathrm{d}}\boldsymbol{A}(\boldsymbol{T}_{\mathrm{d}})^{-1} = \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = \boldsymbol{A}_{\mathrm{d}} = \mathbf{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]. \tag{3.36}$$

The necessary transformation matrix $\boldsymbol{T}_{\mathrm{d}}$ is the inverse of the matrix of the eigenvectors

$$\boldsymbol{T}_{\mathrm{d}} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n]^{-1}. \tag{3.37}$$

The canonical state-equation (canonical form) obtained by the diagonal transformation is

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} z + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} u = \boldsymbol{\Lambda} z + \boldsymbol{\beta} u$$

$$y = [\gamma_1 \quad \gamma_2 \quad \dots \quad \gamma_n] z + du = \boldsymbol{\gamma}^{\mathrm{T}} z + du \tag{3.38}$$

The transfer function of the transformed system is

$$P(s) = \sum_{i=1}^{n} \frac{\beta_i \gamma_i}{s - \lambda_i} + d. \tag{3.39}$$

Thus the transfer function can be obtained in partial fraction form from the canonical one. Note that the eigenvalues of $\boldsymbol{A}$ appear in the denominator. The transfer function remains unchanged if the product of β_i and γ_i remains constant. Thus there are a great many canonical forms which are different in the matrices $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}^{\mathrm{T}}$ but have the same transfer function.

In the case of single poles, the state-equation system in the canonical coordinates consists of n independent first order differential equations. Each individual state variable can be assigned to an individual pole of the system.

If the characteristic equation has multiple roots, the matrix $\boldsymbol{A}_{\mathrm{d}}$ can be diagonalized only in exceptional cases, but its JORDAN-form, in general, can be given by

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{J}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{J}_m \end{bmatrix}. \tag{3.40}$$

Here, each $\boldsymbol{J}_i$ is a square matrix (a Jordan block) of dimension equal to the multiplicity of the eigenvalue λ_i, whose main diagonal contains the eigenvalues and there are ones in the first off-diagonal right from the main diagonal, all the other elements are zeros. If, e.g., λ_1 is a triple eigenvalue, the sub-matrix $\boldsymbol{J}_1$ has the form

$$\boldsymbol{J}_1 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}. \tag{3.41}$$

(The number of ones depends on how many linearly independent eigenvectors can be found for the multiple eigenvalue λ_1. If only one such an eigenvector exists —which is the normal case corresponding to (3.38)—then all elements of the off-diagonal are ones. If the number of the independent eigenvectors has increased by one compared to the previous case, then the number of ones decreases by one. If there exists the same number of independent eigenvectors as the multiplicity, the Jordan block is diagonal. In other cases, finding the transformation matrix needs special considerations, which are not discussed here.)

3.3.2 Controllable Canonical Form

It is the most common practice in modeling to directly derive the state-equations from the differential equations formulated for the physical variables. In many cases, however, the initial information is a transfer function, or a linear differential equation of order n. This procedure is often called the description or construction (reconstruction) of the state-equations. Suppose that the operation of the system is described by the differential equation

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_1 \frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_n y = b_1 \frac{\mathrm{d}^{n-1} u}{\mathrm{d}t^{n-1}} + \cdots + b_n u. \tag{3.42}$$

The equation valid for the Laplace-transforms is

$$Y(s) = \frac{b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} U(s) = \frac{\mathcal{B}(s)}{\mathcal{A}(s)} U(s) = P(s) U(s). \tag{3.43}$$

Introduce the following state variables with their Laplace-transforms

$$\begin{aligned} X_1(s) &= \frac{s^{n-1}}{\mathcal{A}(s)} U(s) && \\ X_2(s) &= \frac{s^{n-2}}{\mathcal{A}(s)} U(s) = \frac{1}{s} X_1(s) && \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 \\ &\vdots && \vdots \\ X_n(s) &= \frac{1}{\mathcal{A}(s)} U(s) = \frac{1}{s} X_{n-1}(s) && \frac{\mathrm{d}x_n}{\mathrm{d}t} = x_{n-1} \end{aligned} \tag{3.44}$$

On this basis,

$$\begin{aligned} sX_1(s) &= -a_1X_1(s) - \cdots - a_nX_n(s) + U(s) & \frac{\mathrm{d}x_1}{\mathrm{d}t} &= -a_1x_1 - \cdots - a_nx_n + u \\ Y(s) &= b_1X_1(s) + \cdots + b_nX_n(s) & y &= b_1x_1 + \cdots + b_nx_n \end{aligned} \tag{3.45}$$

Thus the resulting state-equations are

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \tag{3.46}$$

$$y = [\, b_1 \quad b_2 \quad \dots \quad b_{n-1} \quad b_n \,]\boldsymbol{x}$$

This form with its special system matrices is called the *controllable canonical form* or *phase-variable form*

$$\boldsymbol{A}_\mathrm{c} = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}; \boldsymbol{b}_\mathrm{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \boldsymbol{c}_\mathrm{c}^\mathrm{T} = [\, b_1 \quad b_2 \quad \dots \quad b_{n-1} \quad b_n \,]. \tag{3.47}$$

The special feature of this form is that every state variable, except the last one, is the derivative of the next state variable in the action direction, and all state variables are fed back to the first one. The feedback factors are the negative coefficients of the characteristic equation which appear in the first row of matrix $\boldsymbol{A}$. The input has effect only on x_1. The feedforward factors representing the output are the coefficients of the numerator of the transfer function.

If $P(s)$ is not strictly proper, i.e., $\mathcal{B}(s) = b'_\mathrm{o}s^n + b'_1s^{n-1} + \cdots + b'_{n-1}s + b'_n$, the $d = b'_\mathrm{o}$ also occurs in the state-equation. In this case new coefficients b_i must be computed from the original coefficients b'_i by the following decomposition

$$\begin{aligned} P(s) &= \frac{\mathcal{B}(s)}{\mathcal{A}(s)} = \frac{b'_\mathrm{o}s^n + b'_1s^{n-1} + \cdots + b'_{n-1}s + b'_n}{s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n} \\ &= b_\mathrm{o} + \frac{b_1s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n}. \end{aligned} \tag{3.48}$$

The second term is already strictly proper and the coefficients of the numerator can be computed by the relationships $b_i = b'_i - b'_o a_i = b'_i - b_o a_i \; \left(b_o = b'_o\right)$.

The characteristic polynomial of the controllable canonical form is

$$\mathcal{A}(s) = \det \begin{bmatrix} s + a_1 & a_2 & \dots & a_{n-1} & a_n \\ -1 & s & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & s \end{bmatrix} = \mathcal{A}_n(s) = s\mathcal{A}_{n-1}(s) + a_n, \qquad (3.49)$$

where a recursive relationship is obtained by decomposing the last row. It is obvious that

$$\mathcal{A}_n(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n = \mathcal{A}(s) \qquad (3.50)$$

i.e. the characteristic polynomial is the denominator of the transfer function. Therefore the special matrix $\boldsymbol{A}_c$ is called the accompanying (complementary) matrix of $\mathcal{A}(s)$.

Note that the parameter matrix selection

$$\bar{\boldsymbol{A}}_c = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & \dots & -a_2 & -a_1 \end{bmatrix}; \; \bar{\boldsymbol{b}}_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}; \; \bar{\boldsymbol{c}}_c^T = \begin{bmatrix} b_n & b_{n-1} & \dots & b_2 & b_1 \end{bmatrix} \qquad (3.51)$$

also provides the controllable canonical form, where the serial number of the state variables is the opposite of what appeared in the form (3.47).

3.3.3 Observable Canonical Form

To create this form, let us introduce the state variables with their LAPLACE-transforms according to the recursions

$$\begin{aligned} X_1(s) &= Y(s) & & \\ sX_1(s) &= -a_1 X_1(s) + X_2(s) + b_1 U(s) & \frac{dx_1}{dt} &= -a_1 x_1 + x_2 + b_1 u \\ sX_2(s) &= -a_2 X_1(s) + X_3(s) + b_2 U(s) & \frac{dx_2}{dt} &= -a_2 x_1 + x_3 + b_2 u \\ &\vdots & &\vdots \\ sX_n(s) &= -a_n X_1(s) + b_n U(s) & \frac{dx_n}{dt} &= -a_n x_1 + b_n u \end{aligned} \qquad (3.52)$$

where $Y(s)$ corresponds to (3.43). Based on the relationships above the following state-equations can be written

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u \tag{3.53}$$

$$y = [1 \quad 0 \quad \dots \quad 0 \quad 0]\boldsymbol{x}$$

This form with its special system matrices

$$\boldsymbol{A}_\mathrm{o} = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix}, \quad \boldsymbol{b}_\mathrm{o} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}, \quad \boldsymbol{c}_\mathrm{o}^\mathrm{T} = [1 \quad 0 \quad \dots \quad 0 \quad 0] \tag{3.54}$$

is called observable canonical form. The special feature of this form is that its output is the state variable x_1 itself, which is fed back to the inputs of all the state variables. The feedback factors are the negative coefficients of the characteristic equation, and thus they appear in the first column of $\boldsymbol{A}_\mathrm{o}$. Note that the parameter matrix selection

$$\bar{\boldsymbol{A}}_\mathrm{o} = \begin{bmatrix} 0 & \dots & 0 & 0 & -a_n \\ 1 & \dots & 0 & 0 & -a_{n-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & -a_2 \\ 0 & \dots & 0 & 1 & -a_1 \end{bmatrix}, \quad \bar{\boldsymbol{b}}_\mathrm{o} = \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_2 \\ b_1 \end{bmatrix}, \quad \bar{\boldsymbol{c}}_\mathrm{o}^\mathrm{T} = [0 \quad 0 \quad \dots \quad 0 \quad 1] \tag{3.55}$$

also provides the observable canonical form, where the serial number of the state variables is the opposite to what appeared in (3.53).

If $P(s)$ is not strictly proper, $d = b_\mathrm{o}$ also appears in the state-equation and the statements made in connection with (3.48) are also valid.

(If the poles and the partial fractional forms of the transfer function are known, then further canonical forms can be constructed.)

3.4 The Concepts of Controllability and Observability

A very important question of control is how to influence arbitrarily all state variables by the input. This question can be answered by the controllability theorem introduced by KALMAN.

A system is *state controllable* if its state vector can be driven from an initial state $\boldsymbol{x}(t_o)$ to an arbitrary final state $\boldsymbol{x}(t_v)$ in a finite time $(t_v - t_o)$ by a control signal u. If this definition is fulfilled only for the output, then the system is *output controllable*. In the case of linear time invariant systems, the starting time is chosen $(t_o = 0)$, and the initial state can be given as $\boldsymbol{x}(0)$. By this definition, the controllability is connected to the system. If the controllability exists for a certain initial state, then it remains for any initial state, since from any $\boldsymbol{x}(0)$ the system can be driven to $\boldsymbol{x}(t_v)$ by an appropriate control signal.

Controllability can best be explained in canonical coordinates. If in the canonical form (3.38) β_i is zero for a state variable, then this state can not be controlled. This means, that there is no parallel component, but only a perpendicular component, of any control to the eigenvector belonging to the eigenvalue λ_i, thus the effect of the control always remains in the plane perpendicular to the eigenvector. (As a consequence of the canonical form the system can only be controlled if the poles of the canonical coordinates are different.)

If the system is not state controllable, the output, however, may be controllable if at least one state variable is controllable and the γ_i belonging to it is not zero [see (3.39)].

In coordinates different from the canonical ones, the above conditions can not be directly recognized because of the relationships between the state variables, therefore they have to be replaced by more general criteria.

For simplicity, choose the initial condition $\boldsymbol{x}(0) = \boldsymbol{0}$. Then the solution of the state-Eq. (3.18) has the form

$$\boldsymbol{x}(t) = \left[\int_0^t e^{\boldsymbol{A}(t-\tau)} u(\tau)\mathrm{d}\tau\right]\boldsymbol{b} = \left[\int_0^t e^{\boldsymbol{A}\tau} u(t-\tau)\mathrm{d}\tau\right]\boldsymbol{b} \tag{3.56}$$

and using the finite sum form of the fundamental matrix [see (3.26)], it can be written as

$$e^{\boldsymbol{A}\tau} = \alpha_o(\tau)\boldsymbol{I} + \alpha_1(\tau)\boldsymbol{A} + \cdots + \alpha_{n-1}(\tau)\boldsymbol{A}^{n-1} \tag{3.57}$$

The solution of the state-equation is obtained in closed form as

$$\boldsymbol{x}(t) = \boldsymbol{b}\int_0^t \alpha_o(\tau)u(\tau)\mathrm{d}\tau + \boldsymbol{A}\boldsymbol{b}\int_0^t \alpha_1(\tau)u(\tau)\mathrm{d}\tau + \cdots + \boldsymbol{A}^{n-1}\boldsymbol{b}\int_0^t \alpha_{\mathrm{n}-1}(\tau)u(\tau)\mathrm{d}\tau. \tag{3.58}$$

Thus the right-hand side of the equation is a linear combination of the columns of the *controllability matrix*

$$\boldsymbol{M}_{\mathrm{c}} = \left[\boldsymbol{b} \quad \boldsymbol{A}\boldsymbol{b} \quad \ldots \quad \boldsymbol{A}^{n-1}\boldsymbol{b}\right] \tag{3.59}$$

Thus the condition of that each point of the state-space be reachable means that $\boldsymbol{M}_{\mathrm{c}}$ must have n linearly independent columns, i.e. $\boldsymbol{M}_{\mathrm{c}}$ must be invertible and regular. Since $\boldsymbol{M}_{\mathrm{c}}$ depends on $\boldsymbol{A}$ and $\boldsymbol{b}$, the controllability of the pair $\boldsymbol{A};\boldsymbol{b}$ is a quite accepted convention.

If the above statements are referred to the output, then the condition of *output controllability* is that at least one element of

$$\boldsymbol{m}_{\mathrm{c}}^{\mathrm{T}} = \left[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b} \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b} \ldots \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}^{n-1}\boldsymbol{b}\right] \tag{3.60}$$

must be non zero.

The controllability matrix of the controllable form (3.46) has the special form

$$\boldsymbol{M}_{\mathrm{c}}^{\mathrm{c}} = \begin{bmatrix} 1 & a_1 & a_2 & \ldots & a_{n-1} \\ 0 & 1 & a_1 & \ldots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_1 \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}^{-1}, \tag{3.61}$$

which can be seen very easily by taking the product $\boldsymbol{M}_{\mathrm{c}}^{\mathrm{c}}\left(\boldsymbol{M}_{\mathrm{c}}^{\mathrm{c}}\right)^{-1}$

$$\begin{aligned} &\left[\boldsymbol{b}_{\mathrm{c}} \quad \boldsymbol{A}_{\mathrm{c}}\boldsymbol{b}_{\mathrm{c}} \quad \ldots \quad (\boldsymbol{A}_{\mathrm{c}})^{n-1}\boldsymbol{b}_{\mathrm{c}}\right] \begin{bmatrix} 1 & a_1 & a_2 & \ldots & a_{n-1} \\ 0 & 1 & a_1 & \ldots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_1 \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \\ &= \left[\boldsymbol{w}_{\mathrm{o}} \quad \boldsymbol{w}_1 \quad \boldsymbol{w}_2 \quad \ldots \quad \boldsymbol{w}_{n-1}\right] \end{aligned} \tag{3.62}$$

Based on the special construction of the state matrices $\boldsymbol{A}_{\mathrm{c}}$ and $\boldsymbol{b}_{\mathrm{c}}$ [see (3.47)], it can be seen that

$$\begin{aligned} \boldsymbol{w}_{\mathrm{o}} &= \boldsymbol{b}_{\mathrm{c}} \\ \boldsymbol{w}_1 &= a_1\boldsymbol{b}_{\mathrm{c}} + \boldsymbol{A}_{\mathrm{c}}\boldsymbol{b}_{\mathrm{c}} \\ &\vdots \\ \boldsymbol{w}_{n-1} &= a_{n-1}\boldsymbol{b}_{\mathrm{c}} + a_{n-2}\boldsymbol{A}_{\mathrm{c}}\boldsymbol{b}_{\mathrm{c}} + \cdots + (\boldsymbol{A}_{\mathrm{c}})^{n-1}\boldsymbol{b}_{\mathrm{c}} \end{aligned} \tag{3.63}$$

where the following recursive relationship holds:

$$\boldsymbol{w}_k = a_k \boldsymbol{b}_\mathrm{c} + \boldsymbol{A}_\mathrm{c} \boldsymbol{w}_{k-1}. \tag{3.64}$$

The use of the recursive relationship

$$\begin{bmatrix} \boldsymbol{w}_\mathrm{o} & \boldsymbol{w}_1 & \boldsymbol{w}_2 & \dots & \boldsymbol{w}_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = \boldsymbol{I} \tag{3.65}$$

proves the validity of (3.61).

The special matrix $\boldsymbol{M}_\mathrm{c}^\mathrm{c}$ obtained by the controllable canonical form—which is always derived from the transfer function—is regular, since it is the inverse of a regular matrix. (The determinant of a triangular matrix is the product of its diagonal elements, which is now equal to one.) The name of this canonical form comes from the above features, where only the observability (see later) can be investigated by the pair $\boldsymbol{A}_\mathrm{c}; \boldsymbol{c}_\mathrm{c}^\mathrm{T}$.

It is an interesting question how the linear transformation $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$ influences the controllability matrix. Based on (3.29), one can write

$$\begin{aligned} \tilde{\boldsymbol{b}} &= \boldsymbol{T}\boldsymbol{b} \\ \tilde{\boldsymbol{A}}\tilde{\boldsymbol{b}} &= \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}\boldsymbol{T}\boldsymbol{b} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{b} \\ &\vdots \\ \tilde{\boldsymbol{A}}^{n-1}\tilde{\boldsymbol{b}} &= \boldsymbol{T}\boldsymbol{A}^{n-1}\boldsymbol{b} \end{aligned} \tag{3.66}$$

on the basis of which it follows that

$$\tilde{\boldsymbol{M}}_\mathrm{c} = \begin{bmatrix} \tilde{\boldsymbol{b}} & \tilde{\boldsymbol{A}}\tilde{\boldsymbol{b}} & \dots & \tilde{\boldsymbol{A}}^{n-1}\tilde{\boldsymbol{b}} \end{bmatrix} = \boldsymbol{T}\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \dots & \boldsymbol{A}^{n-1}\boldsymbol{b} \end{bmatrix} = \boldsymbol{T}\boldsymbol{M}_\mathrm{c} \tag{3.67}$$

Based on the above form of the controllability matrix, any controllable systems can be rewritten into controllable canonical form by using the transformation matrix $\boldsymbol{T}_\mathrm{c} = \boldsymbol{M}_\mathrm{c}^\mathrm{c}(\boldsymbol{M}_\mathrm{c})^{-1}$.

The controllability matrix, however, is not always derived from the transfer function. In this case, of course, the direct investigation of the controllability matrix $\boldsymbol{M}_\mathrm{c}$ is required.

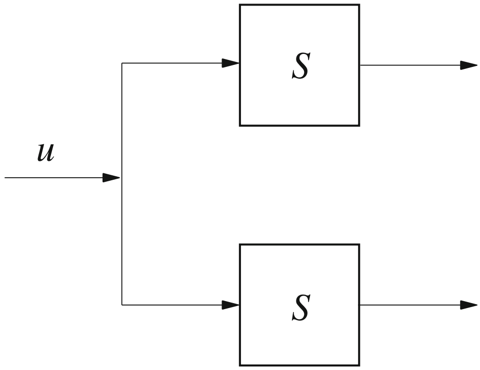

Fig. 3.2 A non-controllable system

Example 3.2 The complete state-equation of a system (see the block diagram in Fig. 3.2) consisting of identical first order sub-systems is

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u = \boldsymbol{Ax} + \boldsymbol{b}u. \tag{3.68}$$

The controllability matrix is

$$\boldsymbol{M}_\mathrm{c} = [\boldsymbol{b} \quad \boldsymbol{Ab}] = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \tag{3.69}$$

which is singular, so the system is not controllable. ■

An other essential question of control is, whether each state variable can be observed by measuring the output. This question can be answered by the observability theorem introduced by KALMAN.

Observability—being related to the controllability—gives an answer to the question, whether the initial state at the starting point of the measurements can be reconstructed by measuring the input and output signals of a system of unknown state during a certain time. The system is observable if $\boldsymbol{x}(t_\mathrm{o})$ can be determined from the signals $y(t)$ and $u(t)$ observed in the interval $t_\mathrm{o} < t < t_\mathrm{v}$.

It is enough to perform the investigation only for $u(t) \equiv 0$, i.e., for the motion generated by the initial values. Observability can be diagnosed in the most easiest way in canonical coordinates. Two criteria have to be fulfilled: the signal y must depend on all canonical state variables; and the poles of the systems must be different. Thus if any γ_i in (3.38) is zero, the output does not have any information concerning the given canonical state variable, so it cannot be reconstructed from the measurements. This means that there is no observation which would have parallel component, to the eigenvector belonging to the eigenvalue λ_i, only perpendicular component, so the effect of the observation always remains in the plane perpendicular to the eigenvector.

In coordinates different from the canonical ones, the above conditions can not directly be recognized due to the interrelationships between the state variables, therefore they have to be replaced by more general criteria.

In the discussion of controllability, the controllability of the state variables was investigated and the output was disregarded. Here in the discussion of observability, the input is disregarded, as was mentioned earlier. Consider the following system.

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \boldsymbol{A}\boldsymbol{x} \\ y &= \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}\end{aligned} \tag{3.70}$$

By consecutive differentiations of the output the equation

$$\left[y, \frac{\mathrm{d}y}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}y}{\mathrm{d}t^{n-1}}\right]^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{c}^{\mathrm{T}} \\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A} \\ \vdots \\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}^{n-1} \end{bmatrix} \boldsymbol{x} \tag{3.71}$$

is obtained and the state vector can be unambiguously determined from the output and its derivatives, if the *observability matrix*

$$\boldsymbol{M}_{\mathrm{o}} = \begin{bmatrix} \boldsymbol{c}^{\mathrm{T}} \\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A} \\ \vdots \\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}^{n-1} \end{bmatrix} \tag{3.72}$$

has n linearly independent rows. Thus $\boldsymbol{M}_{\mathrm{o}}$ must be invertible and regular. Since $\boldsymbol{M}_{\mathrm{o}}$ depends on $\boldsymbol{A}$ and $\boldsymbol{c}^{\mathrm{T}}$, this problem is used to be cited as the observability of the pair $\boldsymbol{A};\boldsymbol{c}^{\mathrm{T}}$. (In (3.71)—due to the CAYLEY-HAMILTON theorem—there is no need to compute derivates of higher order than $(n-1)$, see A.3.3 of Appendix 5.).

The observability matrix of the observable canonical form is very special

$$\boldsymbol{M}_{\mathrm{o}}^{\mathrm{o}} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ a_1 & 1 & \ldots & 0 & 0 \\ a_2 & a_1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ldots & 1 & 0 \\ a_{n-1} & a_{n-2} & \ldots & a_1 & 1 \end{bmatrix}^{-1} \tag{3.73}$$

which can be seen very easily, if the product $\left(\boldsymbol{M}_{\mathrm{o}}^{\mathrm{o}}\right)^{-1}\boldsymbol{M}_{\mathrm{o}}$ is computed, i.e.

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ a_1 & 1 & \dots & 0 & 0 \\ a_2 & a_1 & \ddots & 0 & 0 \\ \vdots & \vdots & \dots & 1 & 0 \\ a_{n-1} & a_{n-2} & \dots & a_1 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_o^T \\ \boldsymbol{c}_o^T \boldsymbol{A}_o \\ \vdots \\ \boldsymbol{c}_o^T (\boldsymbol{A}_o)^{n-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_o^T \\ \boldsymbol{w}_1^T \\ \vdots \\ \boldsymbol{w}_{n-1}^T \end{bmatrix} \tag{3.74}$$

Based on the special construction [see (3.54)] of the system matrices, one has

$$\begin{aligned} \boldsymbol{w}_o^T &= \boldsymbol{c}_o^T \\ \boldsymbol{w}_1^T &= a_1 \boldsymbol{c}_o^T + \boldsymbol{c}_o^T \boldsymbol{A}_o \\ &\vdots \\ \boldsymbol{w}_{n-1}^T &= a_{n-1} \boldsymbol{c}_o^T + a_{n-2} \boldsymbol{c}_o^T \boldsymbol{A}_o + \dots + \boldsymbol{c}_o^T (\boldsymbol{A}_o)^{n-1} \end{aligned} \tag{3.75}$$

where there exists the following recursive relationship

$$\boldsymbol{w}_k^T = a_k \boldsymbol{c}_o^T + \boldsymbol{w}_{k-1}^T \boldsymbol{A}_o \tag{3.76}$$

Using this recursive relationship

$$\begin{bmatrix} \boldsymbol{w}_o^T \\ \boldsymbol{w}_1^T \\ \vdots \\ \boldsymbol{w}_{n-1}^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = \boldsymbol{I} \tag{3.77}$$

which proves the validity of (3.73).

The special $\boldsymbol{M}_o^o$ obtained by the observable canonical form—which is always derived from the transfer function—is always regular, since it is the inverse of a regular matrix. (The determinant of a triangle matrix is the product of the diagonal elements, which is now equal to one.) The name of this canonical form comes from the above features, where only the controllability can be investigated by the pair $\boldsymbol{A}_o; \boldsymbol{b}_o$.

It is an interesting question how the linear transformation $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$ influences the observability matrix. Based on (3.29), one has that

$$\begin{aligned} \tilde{\boldsymbol{c}}^T &= \boldsymbol{c}^T \boldsymbol{T}^{-1} \\ \tilde{\boldsymbol{c}}^T \tilde{\boldsymbol{A}} &= \boldsymbol{c}^T \boldsymbol{T}^{-1} \boldsymbol{T} \boldsymbol{A} \boldsymbol{T}^{-1} = \boldsymbol{c}^T \boldsymbol{A} \boldsymbol{T}^{-1} \\ &\vdots \\ \tilde{\boldsymbol{c}}^T \tilde{\boldsymbol{A}}^{n-1} &= \boldsymbol{c}^T \boldsymbol{A}^{n-1} \boldsymbol{T}^{-1} \end{aligned} \tag{3.78}$$

In matrix form, this is

$$\tilde{\boldsymbol{M}}_{\mathrm{o}} = \begin{bmatrix} \tilde{\boldsymbol{c}}^{\mathrm{T}} \\ \tilde{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{A}} \\ \vdots \\ \tilde{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{A}}^{n-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}^{\mathrm{T}} \\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A} \\ \vdots \\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}^{n-1} \end{bmatrix} \boldsymbol{T}^{-1} = \boldsymbol{M}_{\mathrm{o}}\boldsymbol{T}^{-1} \tag{3.79}$$

Based on the above form of the observability matrix, any observable system can be rewritten in observable canonical form by using the transformation matrix $\boldsymbol{T}_{\mathrm{o}}^{-1} = (\boldsymbol{M}_{\mathrm{o}})^{-1}\boldsymbol{M}_{\mathrm{o}}^{\mathrm{o}}$ (i.e., $\boldsymbol{T}_{\mathrm{o}} = \left(\boldsymbol{M}_{\mathrm{o}}^{\mathrm{o}}\right)^{-1}\boldsymbol{M}_{\mathrm{o}}$).

The observability matrix, however, is not always derived from the transfer function. In this case, of course, the direct investigation of the observability matrix $\boldsymbol{M}_{\mathrm{o}}$ is required.

Example 3.3 The complete state-equation of a system consisting of identical first order subsystems (see the block diagram in Fig. 3.3) is

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{A}\boldsymbol{x}$$
$$y = [1 \quad 1] = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \tag{3.80}$$

The observability matrix is

$$\boldsymbol{M}_{\mathrm{o}} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \tag{3.81}$$

which is singular, thus the system is not observable. ■

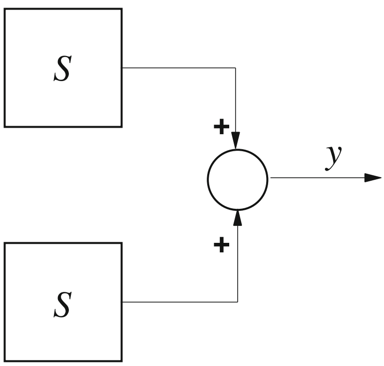

Fig. 3.3 A non-observable system

3.4.1 The Kalman Decomposition

The concepts of controllability and observability make it possible to understand the structure of a linear system. Remember that the space of controllable states is the sub-space defined by the columns of the controllability matrix. If its dimension is n, then the whole space is controllable. Let us introduce the notation $\boldsymbol{x}_{\mathrm{c}}$ for the controllable states, and $\boldsymbol{x}_{\bar{\mathrm{c}}}$ for the non-controllable states. In this case the state-equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{x}_{\mathrm{c}}\\ \boldsymbol{x}_{\bar{\mathrm{c}}}\end{bmatrix}=\begin{bmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12}\\ \mathbf{0} & \boldsymbol{A}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_{\mathrm{c}}\\ \boldsymbol{x}_{\bar{\mathrm{c}}}\end{bmatrix}+\begin{bmatrix}\boldsymbol{b}_{1}\\ \mathbf{0}\end{bmatrix}u \tag{3.82}$$

where it can be clearly seen from the structure that the states $\boldsymbol{x}_{\bar{\mathrm{c}}}$ cannot be influenced by u. Similarly, let us introduce the notation $\boldsymbol{x}_{\mathrm{o}}$ for the observable states and $\boldsymbol{x}_{\bar{\mathrm{o}}}$ for the non-observable states. Then the state-equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{x}_{\mathrm{o}}\\ \boldsymbol{x}_{\bar{\mathrm{o}}}\end{bmatrix}&=\begin{bmatrix}\boldsymbol{A}_{11} & \mathbf{0}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_{\mathrm{o}}\\ \boldsymbol{x}_{\bar{\mathrm{o}}}\end{bmatrix}\\ y&=\begin{bmatrix}\boldsymbol{c}_1^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_{\mathrm{o}}\\ \boldsymbol{x}_{\bar{\mathrm{o}}}\end{bmatrix}\end{aligned} \tag{3.83}$$

is obtained, where it can be well seen, that there is no component in the output for the states $\boldsymbol{x}_{\bar{\mathrm{o}}}$.

A linear system can be decomposed into four sub-systems:

- S_{co} controllable and observable $\boldsymbol{x}_{\mathrm{co}}$
- $S_{\mathrm{c\bar{o}}}$ controllable and non-observable $\boldsymbol{x}_{\mathrm{c\bar{o}}}$
- $S_{\mathrm{\bar{c}o}}$ non-controllable and observable $\boldsymbol{x}_{\mathrm{\bar{c}o}}$
- $S_{\overline{\mathrm{co}}}$ non-controllable and non-observable $\boldsymbol{x}_{\overline{\mathrm{co}}}$

where the corresponding state variables are also presented (the entering arrows mean the effect of the input and the regarding state sub-system). The complete Kalman decomposition of the linear system is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{x}_{\mathrm{co}}\\ \boldsymbol{x}_{\mathrm{c\bar{o}}}\\ \boldsymbol{x}_{\mathrm{\bar{c}o}}\\ \boldsymbol{x}_{\overline{\mathrm{co}}}\end{bmatrix}&=\begin{bmatrix}\boldsymbol{A}_{11} & \mathbf{0} & \boldsymbol{A}_{13} & \mathbf{0}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} & \boldsymbol{A}_{24}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{A}_{33} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{A}_{43} & \boldsymbol{A}_{44}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_{\mathrm{co}}\\ \boldsymbol{x}_{\mathrm{c\bar{o}}}\\ \boldsymbol{x}_{\mathrm{\bar{c}o}}\\ \boldsymbol{x}_{\overline{\mathrm{co}}}\end{bmatrix}+\begin{bmatrix}\boldsymbol{b}_{1}\\ \boldsymbol{b}_{2}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}u=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}u\\ y&=\begin{bmatrix}\boldsymbol{c}_1^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \boldsymbol{c}_2^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}}\end{bmatrix}\boldsymbol{x}\end{aligned} \tag{3.84}$$

The block diagram representing each sub-system is shown in Fig. 3.4. Following the arrows of the block diagram it can be seen that the input influences the

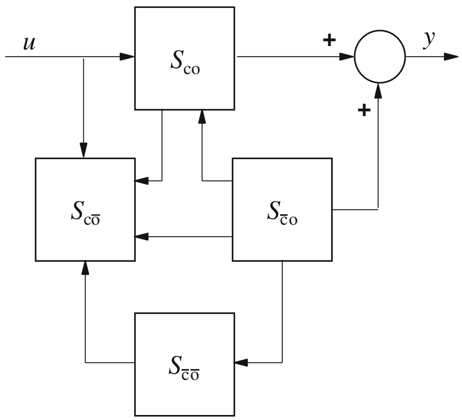

Fig. 3.4 KALMAN decomposition of the linear system

sub-systems S_{co} and $S_{c\bar{o}}$, but the output depends only on the sub-systems S_{co} and $S_{\bar{c}o}$. The sub-system $S_{\bar{c}\bar{o}}$ does not belong either to the input or to the output.

The transfer function of the entire system can be obtained by simple computation

$$P(s) = \boldsymbol{c}_1^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A}_{11})^{-1}\boldsymbol{b}_1, \tag{3.85}$$

i.e., it is completely determined by the sub-system S_{co}. In contrast it can be stated that only the controllable and observable sub-system of the whole system can be determined from the transfer function.

3.4.2 The Effect of Common Poles and Zeros

A very old problem of control, namely the canceling of poles and zeros, can be explained by the KALMAN decomposition. To illustrate it let us consider the following example.

Example 3.4 Let the transfer function of the process be

$$P(s) = \frac{Y(s)}{U(s)} = \frac{s-1}{s-1} = 1, \tag{3.86}$$

i.e., the numerator and the denominator have common roots, thus one zero and one pole are equal. In this case a common root, at the same time, means also an unstable pole. It can be seen easily that the following differential equation corresponds formally to the transfer function (3.86):

$$\frac{dy}{dt} - y = \frac{du}{dt} - u. \tag{3.87}$$

The solution obtained by integration of the differential equation is

$$y(t) = u(t) + ce^t \tag{3.88}$$

where c is a constant. In the method of canceling, it must never be forgotten that the complete solution of the state-equation is performed according to (3.18), which contains also the initial condition, whose dynamics (an un-excited system) depends on the poles of the whole system, even also on the possibly cancelled pole. If this pole is unstable, then its non-disappearing effect occurs unpleasantly in the solution.

The trivial system $y(t) = u(t)$ obtained from (3.86) after the pole cancellation is obviously not equal to (3.88). The Eq. (3.86) can be brought to the following form

$$P(s) = b_o + \frac{b_1}{s-1} = d + \frac{b_1}{s-1} = 1 + \frac{0}{s-1} \tag{3.89}$$

Based on this the controllable canonical form can be easily written as

$$\frac{dx_1}{dt} = x_1 + u; \quad y = u \tag{3.90}$$

which is not observable, and an observable canonical form can also be defined as

$$\frac{dx_2}{dt} = x_2; \quad y = x_2 + u \tag{3.91}$$

which is not controllable. ■

The KALMAN-form of the whole system corresponding to Eqs. (3.90) and (3.91) is shown in Fig. 3.5, which consists of the sub-systems $S_{c\bar{o}}$, $S_{\bar{c}o}$ and S_{co}. The S_{co} is a static system with transfer function $P(s) = 1$. The $S_{c\bar{o}}$ is a non-observable but controllable subsystem, while $S_{\bar{c}o}$ is not controllable, but is an observable sub-system.

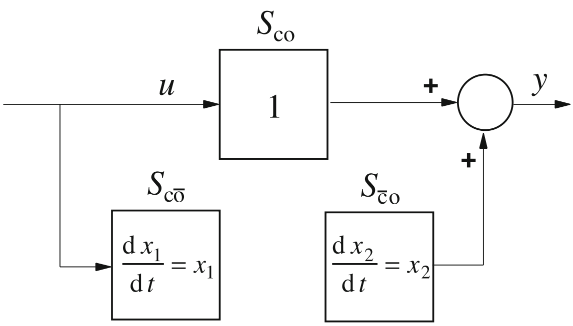

Fig. 3.5 The complete KALMAN-form of the system of transfer function (3.86)

Note that if the transfer function of the system is given, then first the common divisors of the numerator and denominator have to be investigated. The common factor can only be a common root. It is reasonable to continue the simplification until there are no more common divisors. Such polynomials are called relatively prime. A transfer function $P(s) = \mathcal{B}(s)/\mathcal{A}(s)$ is called irreducible (i.e., can not be simplified) if the polynomials $\mathcal{A}(s)$ and $\mathcal{B}(s)$ are relatively prime, which is an algebraic condition for the special Diophantine (or Bezout) equation

$$\mathcal{A}(s)\mathcal{X}(s) + \mathcal{B}(s)\mathcal{Y}(s) = 1 \tag{3.92}$$

to have a solution, i.e., the corresponding Silvester matrix must be regular (see more details in Chap. 9).

If a transfer function is not reducible, the related state-equation corresponds to the controllable and observable sub-system S_{co} of the Kalman-form and the other sub-systems do not exist.

Equations (3.90) and (3.91) can be generalized to the case when the controllable and observable system $S_{co}\{\boldsymbol{A}; \boldsymbol{b}; \boldsymbol{c}^T; d\}$ is irreducible, and the numerator and also the denominator of the transfer function $P(s)$ are extended by a common factor $(s-p)$ referring to a real pole. For this general case the state-equation of the redundant non-controllable and non-observable system can be given as

$$\begin{aligned} \dot{\boldsymbol{x}}_r &= \begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0}^T & p & 0 \\ \boldsymbol{0}^T & 0 & p \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{b} \\ 1 \\ 0 \end{bmatrix} u = \boldsymbol{A}_r \boldsymbol{x}_r + \boldsymbol{b}_r u \\ y &= \begin{bmatrix} \boldsymbol{c}^T & 0 & 1 \end{bmatrix} \boldsymbol{x}_r + du = \boldsymbol{c}_r^T \boldsymbol{x}_r + du \end{aligned} . \tag{3.93}$$

Example 3.5 Assume that the transfer function of the process is

$$P(s) = \frac{2(s+1)}{(s+1)(s+2)} = \frac{2s+2}{s^2+3s+2} = \frac{b_1 s + 2}{s^2+3s+2}.$$

It is easy to form the parameter matrices of the controllable canonical form, which are

$$\boldsymbol{A}_c = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}; \quad \boldsymbol{b}_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{c}_c^T = [2 \quad 2].$$

The controllability matrix of this canonical form is

$$\boldsymbol{M}_c^c = [\boldsymbol{b}_c \quad \boldsymbol{A}_c \boldsymbol{b}_c] = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}.$$

It is easy to check that the determinant of this matrix is $\det\left(\boldsymbol{M}_{\mathrm{c}}^{\mathrm{c}}\right) = 1$, as a consequence the process is controllable. The observability matrix of this canonical form is

$$\boldsymbol{M}_{\mathrm{o}}^{\mathrm{c}} = \begin{bmatrix} \boldsymbol{c}_{\mathrm{c}}^{\mathrm{T}} \\ \boldsymbol{c}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{A}_{\mathrm{c}} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & -4 \end{bmatrix}.$$

Note that the determinant of this matrix is $\det\left(\boldsymbol{M}_{\mathrm{o}}^{\mathrm{c}}\right) = 0$; as a consequence the process is not observable.

Now form the parameter matrices of the observable canonical form, which are

$$\boldsymbol{A}_{\mathrm{o}} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}; \quad \boldsymbol{b}_{\mathrm{o}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{c}_{\mathrm{o}}^{\mathrm{T}} = [\,1 \quad 0\,].$$

The controllability matrix of this canonical form is

$$\boldsymbol{M}_{\mathrm{c}}^{\mathrm{o}} = [\,\boldsymbol{b}_{\mathrm{o}} \quad \boldsymbol{A}_{\mathrm{o}}\boldsymbol{b}_{\mathrm{o}}\,] = \begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix}.$$

It is easy to check that the determinant of this matrix is $\det\left(\boldsymbol{M}_{\mathrm{c}}^{\mathrm{o}}\right) = 0$; as a consequence the process is not controllable. The observability matrix of this canonical form is

$$\boldsymbol{M}_{\mathrm{o}}^{\mathrm{o}} = \begin{bmatrix} \boldsymbol{c}_{\mathrm{o}}^{\mathrm{T}} \\ \boldsymbol{c}_{\mathrm{o}}^{\mathrm{T}}\boldsymbol{A}_{\mathrm{o}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}.$$

Note that the determinant of this matrix is $\det\left(\boldsymbol{M}_{\mathrm{o}}^{\mathrm{o}}\right) = 1$; as a consequence the process is observable. This example shows and explains very nicely the meaning of the above matrices.

Observe and check that the irreducible equivalent transfer function

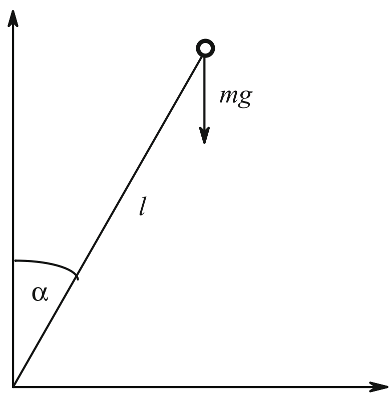

Fig. 3.6 Scheme of the inverted pendulum

$$P(s) = \frac{2(s+1)}{(s+1)(s+2)} = \frac{2}{s+2}$$

is already controllable and observable. ■

3.4.3 The Inverted Pendulum

Next the simplest case of the moving inverted pendulum shown in Chap. 2.6 is investigated, i.e., when the suspension of the pendulum is fixed. Via this example almost all of the methods of this Chapter from the linear modeling to the investigation of the controllability and observability issues can be demonstrated.

In order to determine the state-equation of the inverted pendulum, given in a simple schematic form in Fig. 3.6, introduce the following state variables: $x_2 = \mathrm{d}\alpha/\mathrm{d}t$ and $x_1 = \alpha$ (the angular velocity and the angular position). From the equality of the moments calculated for the center of the angular position it follows that

$$J\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = mgl\sin(\alpha) + mugl\cos(\alpha), \tag{3.94}$$

where it is assumed that the mass m is concentrated at the end of an ideal, weightless pendulum of length l, the inertia relating to the center of the rotation is denoted by J. The actuating signal is the horizontal acceleration of the value ug (measured in g), the output is the angular position α. The non-linear state-equation is obtained as

$$\begin{aligned} &\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u) = \begin{bmatrix} \mathrm{d}\alpha/\mathrm{d}t \\ \dfrac{mgl}{J}\sin(\mathrm{d}\alpha/\mathrm{d}t) + \dfrac{mglu}{J}\cos(\alpha) \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) + u\cos(x_1) \end{bmatrix} \\ &y = \alpha = x_1 \end{aligned} \tag{3.95}$$

where choosing $\sqrt{J/mgl}$ as a time unit, the last, normalized form in Eq. (3.95) results. Thus the state-equation is a nonlinear, time-invariant, second order vector differential equation.

Let us linearize the equation in the case of zero actuating signal. The equilibrium point is

$$\dot{\boldsymbol{x}} = \boldsymbol{0} = f(u = 0) = \begin{bmatrix} x_2 \\ \sin(x_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \begin{matrix} x_2 = \mathrm{d}\alpha/\mathrm{d}t = 0 \\ \sin(x_1) = \sin(\alpha) = 0 \end{matrix} \tag{3.96}$$

where $\alpha = 0$ and $\alpha = \pi$. At the first equilibrium point the pendulum is in the position upside, but in the second it is in down-side. Determining the derivatives with respect to $\boldsymbol{x}$ and u of the function $\boldsymbol{f}(\boldsymbol{x}, u)$ yields

$$\frac{\mathrm{d}\boldsymbol{f}(\boldsymbol{x},u)}{\mathrm{d}\,\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ \cos(x_1) - u\sin(x_1) & 0 \end{bmatrix} \quad \text{and} \quad \frac{\mathrm{d}\boldsymbol{f}(\boldsymbol{x},u)}{\mathrm{d}\,u} = \begin{bmatrix} 0 \\ \cos(x_1) \end{bmatrix}. \tag{3.97}$$

Evaluating the derivates at the upper point ($u = 0$; $x_1 = 0$ and $x_2 = 0$) the parameter matrices

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{c}^{\mathrm{T}} = [\,1 \quad 0\,] \tag{3.98}$$

are obtained. Computing the transfer function (by using A.1.10) yields

$$\begin{aligned} G(s) &= \boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} = \frac{1}{\det\begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}}[\,1 \quad 0\,]\begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2 - 1}[\,s \quad 1\,]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2 - 1} = \frac{1}{(s+1)(s-1)} \end{aligned} \tag{3.99}$$

The root $s = 1$ shows that at this operating point the system is unstable. It can be easily checked that the controllability matrix

$$\boldsymbol{M}_{\mathrm{c}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.100}$$

is regular, thus the system is controllable. In this case the observability matrix, which is

$$\boldsymbol{M}_{\mathrm{o}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.101}$$

is also regular, thus the system is observable. Coming from the simplicity of the above task, the DT control of the inverted pendulum is a typical and spectacular laboratory example all over the world for controlling an unstable process. (The complexity of the task increases drastically by placing more pendulums on top of each other.)

Evaluating the derivates in the lower point ($u = 0$; $x_1 = \pi$ and $x_2 = 0$), the parameter matrices

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{c}^{\mathrm{T}} = [\,1 \quad 0\,] \tag{3.102}$$

are obtained. Calculate the transfer function [by using (A.1.10)] yields

$$G(s) = \boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} = \frac{1}{\det\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$= \frac{1}{s^2+1}\begin{bmatrix} s & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \frac{-1}{s^2+1} = \frac{-1}{(s+j)(s-j)} \tag{3.103}$$

The roots on the imaginary axis indicate that at this operating point the process is an oscillating system without any damping. Do not forget that no kind of damping (e.g., air or ordinary friction) is taken into consideration in the model. Simple computations similar to the above show that even at this operating point the system is controllable and observable.