Chapter 2
Description of Continuous Systems in the Time-, Operator- and Frequency Domains

The behaviour of linear systems can be described in the time-, in the LAPLACE operator-, and in the frequency domain. The most straightforward information about the operation of practical systems is obtained by analysis in the time domain. The analysis in the frequency domain gives deeper insight into important properties of the systems. The design of control systems is frequently executed based on considerations in the frequency domain. In the LAPLACE operator domain the calculations related to the performance of the system become simpler than in the time domain. These domains can be converted to each other (Fig. 2.1).

2.1 Relationship Between the Time- and the Frequency Domain

A signal can be investigated in the frequency and in the time domain. Investigation in the frequency domain means that the signal is considered as a sum of sinusoidal components. Let us approximate a periodical rectangular signal by the sum of 4 sinusoidal signals. The odd coefficients of the frequency spectrum (FOURIER expansion) of the signal are: $4/\pi, 4/3\pi, 4/5\pi, 4/7\pi$. The approximation can be calculated by the MATLAB™ commands

```
w0=1; Ts=0.2;
t=0:Ts:51;
y=4/pi*(sin(w0*t)+ sin(3*w0*t)/3+ sin(5*w0*t)/5+
sin(7*w0*t)/7);
figure(1),plot(t,y);
```


L. Keviczky et al., *Control Engineering: MATLAB Exercises*,
Advanced Textbooks in Control and Signal Processing,
https://doi.org/10.1007/978-981-10-8321-1_2

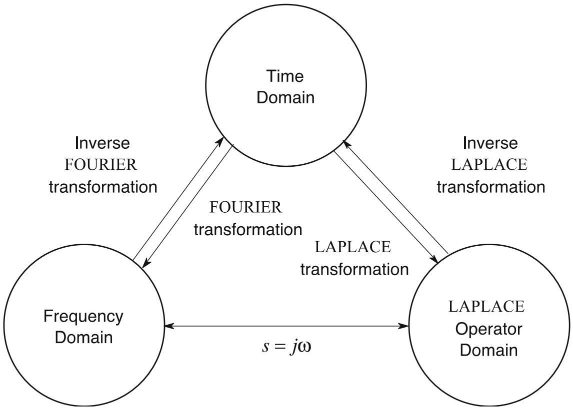


Fig. 2.1 Domains of calculations

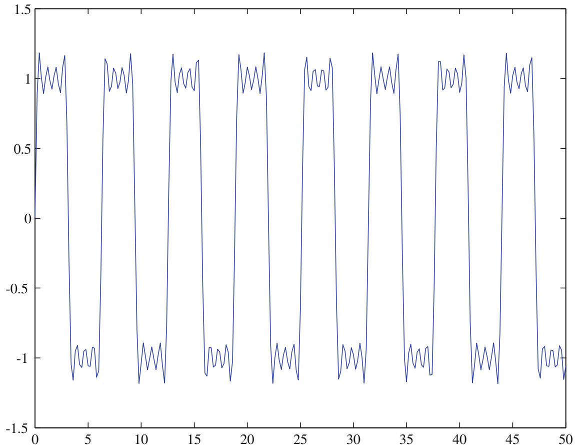


Fig. 2.2 A periodic signal approximated by 4 Fourier components

It can be seen that the approximation is already good with only 4 components (Fig. 2.2).

Let us plot the absolute value of the spectrum of the signal.

The command fft determines the Fourier transform of the signal.

```
Yf=fft(y);
n=length(t);nh=floor(n/2);
Yf=Yf(1:nh+1);
w=2*pi*(1:nh+1)/(n*Ts);
figure(2),plot(w,abs(Yf))
```

It can be seen that the spectrum (Fig. 2.3) contains values only at odd frequencies.

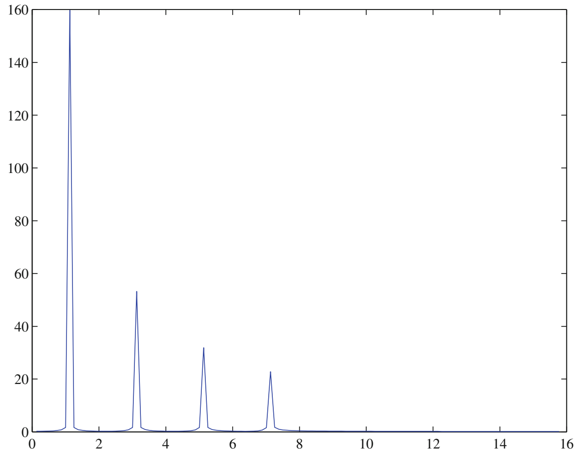


Fig. 2.3 Frequency spectrum

2.2 Laplace and Inverse Laplace Transformations

Analysing the behaviour of linear systems in the Laplace operator domain using Laplace transformation and the inverse Laplace transformation is easier than analysis in the time domain. Laplace transforms of the most frequently applied input signals:

$$\delta(t) \leftrightarrow 1 \quad ; \quad 1(t) \leftrightarrow 1/s \quad \text{and} \quad t \leftrightarrow 1/s^2.$$

Determine the step response of a system (Fig. 2.4).

In the Laplace operator domain the output signal is obtained by multiplication, $Y(s) = H(s)U(s)$, where $Y(s) = \mathcal{L}\{y(t)\}$ is the Laplace transform of the output signal $y(t)$ and $H(s)$ is the transfer function of the system, which is defined as the ratio of the Laplace transforms of the output and the input signal: $H(s) = \frac{Y(s)}{U(s)}$. The output signal is obtained by applying the inverse Laplace transformation: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Let us calculate the step response of the system given by the transfer function

$$H(s) = \frac{-2s^3 - 9s^2 - 5s + 18}{(s+2)(s+3)^2}.$$

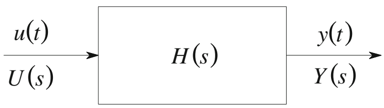


Fig. 2.4 System described by its transfer function

The input signal is a unit step, whose Laplace transform is $U(s) = \mathcal{L}\{1(t)\} = \frac{1}{s}$. The Laplace transform of the output signal is

$$Y(s) = U(s)H(s) = \frac{1}{s}\frac{-2s^3 - 9s^2 - 5s + 18}{(s+2)(s+3)^2}$$

The output signal $y(t)$ in the time domain can be obtained by inverse Laplace transformation. The Laplace transform of the signal is expanded to a sum of components whose Laplace transforms are known. The most common elements are

$$k \xrightarrow{\mathcal{L}^{-1}} k\,1(t) \quad ; \quad t \geq 0$$

$$\frac{r}{s+p} \xrightarrow{\mathcal{L}^{-1}} re^{-pt}$$

$$\frac{r}{(s+p)^2} \xrightarrow{\mathcal{L}^{-1}} rte^{-pt}$$

This form can be obtained by the partial fractional expansion of the Laplace transform of the output signal. In MATLAB™ this is executed by the command residue.

First give the Laplace transform of the output signal by the polynomials of its numerator and denominator.

```
s=zpk('s')
Y=(-2*s^3-9*s^2+-5*s+18)/(s*(s+2)*(s+3)*(s+3))
[num,den]=tfdata(Y,'v')
```

The polynomials can be given directly, as well.

```
num=[-2 -9 -5 18]
den=poly([-3 -3 -2 0])
```

Expansion in terms of partial fractions:

```
[r,p,k]=residue(num,den)
   r =  1.0000
        2.0000
       -4.0000
        1.0000
   p = -3.0000
       -3.0000
```

```
   -2.0000
     0
k = []
```

That means the result in the LAPLACE operator domain is

$$Y(s) = \frac{r(1)}{s-p(1)} + \frac{r(2)}{[s-p(2)]^2} + \frac{r(3)}{s-p(3)} + k = \frac{1}{s+3} + \frac{2}{(s+3)^2} - \frac{4}{s+2} + \frac{1}{s}$$

and in the time domain,

$$y(t) = e^{-3t} + 2te^{-3t} - 4e^{-2t} + 1(t), \quad t \geq 0$$

Let us observe the structure corresponding to the double pole in the vectors r and p in the partial fractional representation of the LAPLACE transform of the output signal and in the expression of the output signal in the time domain. The number of partial fractions belonging to a multiple pole is equal to the multiplicity of the pole.

Based on the analytical expression above the time function can be given in numerical form as

```
t=0:0.05:6;
y=r(1)*exp(p(1)*t)+r(2)*t.*exp(p(2)*t)+r(3)*exp(p(3)*t) +r(4)*exp(p(4)
*t);
```

In the second term on the right side the point besides t means that the operation is to be executed on the elements of the vector.

The values of $y(t)$ can be determined numerically even more simply.

```
yi=impulse(Y,t);
plot(t,y,t,yi),grid;
```

In the figure only one curve is seen, as the two curves coincide exactly.

Exercise:

Determine the inverse LAPLACE transform when there are conjugate complex poles.

$$Y(s) = \frac{2}{s^2 + 2s + 1.25}.$$

Find an analytical expression for the signal $y(t)$.

2.3 The Frequency Function

A basic property of a stable linear system is that for a sinusoidal input, it responds with a sinusoidal signal of the same frequency *in steady (quasi-stationary) state*. Applying the input signal

$$u(t) = A_u \sin(\omega t + \varphi_u) \quad t \geq 0,$$

the output signal is obtained as the sum of a quasi-stationary and a transient component.

$$y(t) = y_{steady}(t) + y_{transient}(t)$$

The output signal in quasi-stationary state (Fig. 2.5) is

$$y_{steady}(t) = A_y \sin(\omega t + \varphi_y)$$

The frequency function defines the *amplitude ratio* A_y/A_u and the *phase shift* $\varphi_y - \varphi_u$ as a function of frequency. Using the amplitude ratio and the phase shift within one single function the frequency function is derived as a complex function. It can be proven that formally the frequency function can be obtained from the transfer function by substituting $s = j\omega$.

$$H(j\omega) = H(s)|_{s=j\omega} = M(\omega)\, e^{j\varphi(\omega)}$$

$M(\omega)$ is the *amplitude function* (the absolute value of the frequency function) and $\varphi(\omega)$ is the *phase function*.

$$M(\omega) = |H(j\omega)| = \frac{A_y(\omega)}{A_u(\omega)} \quad ; \quad \varphi(\omega) = \arg\{H(j\omega)\} = \varphi_y(\omega) - \varphi_u(\omega)$$

The frequency function can be depicted in a given frequency range by plotting $M(\omega)$ and $\varphi(\omega)$ versus the frequency. The frequency scale is logarithmic. This technique gives the Bode diagram. A second possibility is to plot the points corresponding to pairs of $M(\omega)$ and $\varphi(\omega)$ of the frequency function calculated for various values of ω in the complex plane, while ω varies from zero to infinity. Connecting these points results in the contour of the so-called Nyquist diagram.

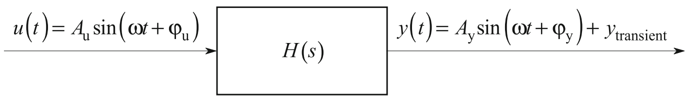


Fig. 2.5 System response to a sinusoidal input

2.3.1 Calculation and Visualization of the Frequency Function

Suppose the transfer function of a system is

$$H(s) = \frac{10}{s^2 + 2s + 10}.$$

Determine its output signal if the input signal is $u(t) = A_u \sin(\omega t)$, $A_u = 1, \ \omega = 3$.

```
num=10
den=[1, 2, 10]
H=tf(num,den)
t=0:0.05:10;
u=sin(3*t);
y=lsim(H,u,t);
```

Plot both the input (red) and output (blue) in the same diagram:

```
plot(t,u,'r',t,y,'b'), grid;
```

In steady, quasi-stationary state, after the decrease of the transient, the output signal is sinusoidal, its frequency is the same as that of the input signal, but its amplitude and phase angle differ from those of the input signal. Their values depend on the frequency. From the figure one sees ($M(3) = 1.64, \ \varphi = -80°$). The gain and the phase angle can be calculated from the frequency function $H(s = j\omega)$. In MATLAB™, the command bode can be employed to calculate these values at a given frequency or over a given frequency range. E.g. at $\omega = 3$,

```
[M,fi]=bode(H,3);
```

The values of the gain and the phase angle can be obtained from the complex frequency function as well.

$$H(j\omega) = \frac{10}{(j\omega)^2 + 2j\omega + 10} = \frac{10}{10 - \omega^2 + 2j\omega}$$

$$H(j3) = \frac{10}{10 - 3^2 + 2j3} = \frac{10}{1 + 6j}$$

```
H3=10/(1+6j)
M=abs(H3)
fi=angle(H3)*180/pi
```

Let us repeat the calculations if the frequency of the input signal is changed to $\omega = 10$. It can be seen that the values of the gain and the phase angle have changed.

The command bode plots the amplitude and the phase angle versus the frequency.

```
bode(H);
```

Check on the curve if at frequency $\omega = 3$ the gain and the phase angle are equal to the previously calculated values. The gain is given in decibels. The value of the gain $M(3) = 1.64$ in decibels is

```
20*log10(1.64)
```

The scale on the amplitude curve can be set from decibels to absolute values. Let us right click with the mouse on the white background of the amplitude diagram of the Bode diagram, then set on the appearing menu window *Properties, Units, magnitude in –> absolute*. Then the gain corresponding to the given frequency can be read directly from the amplitude diagram.

2.3.2 Plotting the Bode and the Nyquist Diagrams

The bode command shows the amplitudes of the Bode diagram in decibels and the phase angles in degrees. The frequency scale is logarithmic.

```
bode(H)
```

Let us calculate the amplitude and the phase values in vector format and then plot the diagram.

```
[gain,phase,w]=bode(H)
```

The bode command determines automatically the points of the frequency vector based on the poles and zeros of the system. These values are provided in the vector w on the left side of the command. If we would like to calculate the frequency function over a different frequency range, the frequency vector can be given by the command logspace which determines a row vector with logarithmically equidistant frequency points.

```
w=logspace(-1,2,200)
```

The first two parameters of logspace give the lower and the upper points of the frequency range in powers of 10. The above command calculates 200 logarithmically equidistant points between the lower point $10^{-1} = 0.1$ and the upper point $10^2 = 100$ (without giving the third variable, the command employees 50 points). If, e.g. the upper point of the frequency range is 300, the command is called in the following form:

```
w=logspace(-1,log10(300),200)
```

Let us repeat the calculation of the BODE diagram with this frequency vector.

```
[gain,phase]=bode(H,w)
```

(Remark: the variables on the right side of a MATLAB™ command are the input variables, while the variables on the left side are the output variables.)

The *LTI sys* structure generates three dimensional matrices (because of the possible *MIMO* systems). With the (:) operator the results can be transformed to vector form.

```
gain=gain(:),phase=phase(:)
```

The amplitude and the phase diagrams can be plotted in different windows of the screen by the command subplot. (E.g. subplot(211) generates 2×1 windows on the screen and refers to the first one.)

Plot the amplitude and the phase angle in linear scale, and the frequency with logarithmic scale by calling the command semilogx. This command is used similarly to plot.

```
subplot(211),semilogx(w,gain)
subplot(212),semilogx(w,phase)
```

Generally the amplitude is plotted in logarithmic scale, using the command loglog.

```
subplot(211),loglog(w,gain)
subplot(212),semilogx(w,phase)
```

On the amplitude scale, the powers of 10 do appear. To convert the values to decibels use the following commands.

```
subplot(211),semilogx(w,20*log10(gain)),grid
subplot(212),semilogx(w,phase),grid
```

The BODE diagram is advantageous when multiplying two transfer functions (calculating the resulting transfer functions of serially connected elements). Because of the logarithmic scale the BODE diagrams are just added. In most cases, approximate BODE diagrams, given by the asymptotes of the magnitude curve, provide a good approximation of the frequency characteristics. By sketching these approximate curves, a quick evaluation of the system's behaviour can be made.

The NYQUIST diagram plots the points of the frequency function in the complex plane. Its shape characterizes the system. The important properties of the system can be determined by analysing it.

```
nyquist(H)
```

Calling the command without variables on the left side plots the NYQUIST diagram extending the curve with points calculated for negative frequencies. This is the so called entire or total NYQUIST diagram.

The real and the imaginary components can be calculated from the values of the amplitudes and the phase angles.

```
re=real(gain.*exp(j*phase*pi/180));
im=imag(gain.*exp(j*phase*pi/180));
```

The real and imaginary values belonging to the different frequency values can also be calculated directly, with the command nyquist.

```
[re,im]=nyquist(H,w);
re=re(:);im=im(:);
```

Then the NYQUIST diagram for positive frequencies can be plotted.

```
plot(re,im)
```

To supplement the curve with the part belonging to negative frequencies, it has to be throwing back to the real axis.

```
re2=[re;flipud(re)]
im2=[im;flipud(-im)]
plot(re2,im2)
```

2.4 Operations with Basic Elements

In a control system, the basic connections of elements are the series connection, parallel connection, and feedback. With block diagram algebra a complex system can be built with these basic connections.

Given the following two systems with their transfer functions:

$$H_1(s) = \frac{10s+1}{s+1} \quad \text{and} \quad H_2(s) = \frac{s}{(10s+1)(5s+1)}$$

Determine the resulting transfer functions

- of the serially connected systems,
- of the parallel connected systems,
- when H_1 is fed back through a unity gain element (negative feedback)
- when H_1 is fed back through H_2 (negative feedback)

Let us define two systems in MATLAB™.

```
s=zpk('s')
H1=(10*s+1)/(s+1)
H2=s/((10*s+1)*(5*s+1))
```

Serially connected systems (Fig. 2.6):

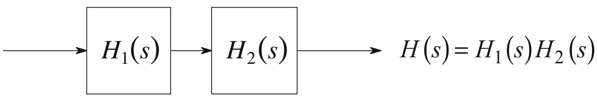


Fig. 2.6 Serially connected systems

The resulting transfer function:

```
H=H1*H2
```

The series command also calculates the resulting transfer function of a series connection:

```
H=series(H1,H2)
        0.2 s (s+0.1)
     ---------------------
     (s+1) (s+0.2) (s+0.1)
```

It can be seen that the numerator and the denominator have common roots (zeros, poles) which can be cancelled using the command minreal.

```
H=minreal(H)
          0.2 s
     -------------
     (s+1) (s+0.2)
```

Parallel connected systems (Fig. 2.7):
The resulting transfer function:

```
H=H1+H2
     10 (s+0.06876) (s^2 + 0.3332 s + 0.02909)
     -----------------------------------------
             (s+1) (s+0.2) (s+0.1)
```

Negative feedback through unit gain (Fig. 2.8):

Fig. 2.7 Parallel connection

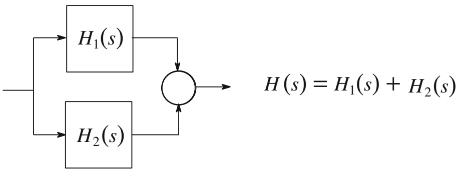
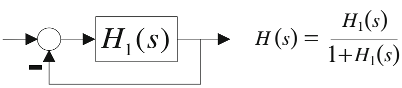


Fig. 2.8 Negative feedback through unit gain



The resulting transfer function:

```
H=H1/(1+H1)
```

The command feedback can also be applied to calculate the resulting transfer function. The second parameter is the transfer function in the feedback path, the third parameter shows that the feedback is negative.

```
H=feedback(H1,1,-1)
```
(or **H=feedback(H1,1)**, the basic definition is negative feedback.)

```
H=minreal(H)
      0.90909 (s+0.1)
      ---------------
        (s+0.1818)
```

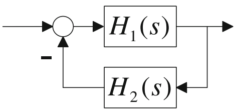
Negative feedback (Fig. 2.9):
The resulting transfer function:

```
H=H1/(1+H1*H2)
```

With the command feedback:

```
H=feedback(H1,H2,-1)
H=minreal(H)

    10 (s+0.1) (s+0.2)
    ---------------------
    (s+1.239) (s+0.1615)
```



$$H(s) = \frac{H_1(s)}{1+H_1(s)H_2(s)}$$

Fig. 2.9 Negative feedback

2.5 Basic Elements of a Linear System

A linear system generally can be given in the following time constant form:

$$H(s) = \frac{K}{s^i} \frac{\prod_1^c (1 + s\tau_j) \prod_1^d \left(1 + 2\zeta_j \tau_{0j} s + s^2 \tau_{0j}^2\right)}{\prod_1^e (1 + sT_j) \prod_1^f \left(1 + 2\xi_j T_{0j} s + s^2 T_{0j}^2\right)} e^{-sT_d}$$

where K is the gain, i is the number of the integrators, T_d is the dead-time, τ_o and T_o are time constants, and ζ and ξ are the damping factors.

In the sequel the time- and frequency characteristics of the most important elements will be analysed. These are the proportional, integrating, differentiating, dead-time, and lag elements, and more complex elements obtained by series connections of these basic elements.

2.5.1 *Proportional (P) Element*

$$H(s) = H_P(s) = K$$

The gain is K, the phase angle is zero at all frequencies.

2.5.2 *Integrating (I) Element*

$$H(s) = H_I(s) = \frac{K}{s}$$

Here $i = 1$: the system contains an integrator. The integrator has a “memory”: its output value depends on the values of the past inputs. Its output can be constant only if the input value is zero. Let us investigate the properties of a pure integrator given by a transfer function $H_I(s)$ for gains $K = 1$ and $K = 5$.

$$H_1(s) = \frac{1}{s} \quad \text{and} \quad H_2(s) = \frac{5}{s}.$$

```
clear             % clear all the previously defined variables.
s=zpk('s')        % define the symbolic s LAPLACE variable in zpk form.
H1=1/s
H2=5/s
```

The step response:

```
figure(1), step(H1,'r',H2,'g'), grid
```

Bode diagram:

```
figure(2), bode(H1,'r',H2,'g'), grid
```

Nyquist diagram:

```
figure(3), nyquist (H1,'r',H2,'g'),grid
```

It can be seen that the step response increases linearly. The amplitude of the frequency function at low frequencies is infinity. Its phase angle is −90° for all frequencies.

2.5.3 First-Order Lag Element (PT1)

$$H(s) = H_T(s) = \frac{K}{1+Ts}$$

Determine the step response and the Bode and Nyquist diagrams of the *PT1* element given by the transfer function

$$H(s) = \frac{2}{1+10s}.$$

Define the system by

```
H=2/(1+10*s)
```

or

```
H=tf(2,[10, 1])
```

The step response:

```
t=0:0.1:50;
y=step(H,t);
plot(t,y),grid
```

or simply

```
step(H)
```

Let us investigate the effect of the parameters K and T on the system response.

The steady value of the output signal, $y(t \to \infty)$, can be calculated by the final value theorem of the LAPLACE transformation:

$$y(t \to \infty) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sH(s)U(s),$$

where $U(s)$ is the LAPLACE transform of the input signal. For unit step input,

$$y(t \to \infty) = \lim_{s \to 0} s\,H(s)U(s) = \lim_{s \to 0} s\,H(s)\frac{1}{s} = \lim_{s \to 0} H(s)$$

In the case of the considered system,

$$y(t \to \infty) = \left.\frac{2}{1 + 10s}\right|_{s=0} = 2.$$

With MATLAB™:

```
yinf=dcgain(H)
```

Investigate the effect of the time constant parameter T on the transient behaviour. Repeat the command

```
y=step(2,[T 1],t);
```

for several different values of T.

The transfer function of the system can be given in *zero-pole form* as well:

$$H_1(s) = \frac{k_\mathrm{p}}{s - p} = \frac{2}{10(0.1 + s)} = \frac{0.2}{s + 0.1},$$

where

$$p = -\frac{1}{T} \quad \text{and} \quad k_{\mathrm{p}} = \frac{k}{T}$$

The absolute value of the pole gives the break-point frequency of the approximating BODE amplitude-frequency diagram.

The steady-state value of the step response

$$y(t \to \infty) = \lim_{s \to 0} H(s)$$

coincides with the low frequency value of the BODE amplitude-frequency function.

The BODE and NYQUIST diagrams of the system are obtained by the following commands:

```
bode(H);
nyquist(H);
```

The characteristic functions of the following first-order, second-order and third-order (*PT1*, *PT2*, *PT3*) elements can be calculated similarly.

$$H_1 = \frac{2}{1+10s}; \quad H_2 = \frac{2}{(1+10s)(1+2s)}; \quad H_3 = \frac{2}{(1+10s)(1+2s)(1+s)}$$

(1—is in red, 2—is in green, 3—is in blue)

```
H1=2/(1+10*s)
H2=2/((1+10*s)*(1+2*s))
H3=2/((1+10*s)*(1+2*s)*(1+s))
```

Step responses:

```
figure(1), step(H1,'r',H2,'g',H3,'b'),grid
```

BODE diagrams:

```
figure(2), bode(H1,'r',H2,'g',H3,'b'), grid
```

NYQUIST diagrams:

```
figure(3), nyquist(H1,'r',H2,'g',H3,'b'),grid
```

With more lags the step response is slower.

Remark: in the figure window the marked part of the plots can be enlarged by command *zoom*.

As it was shown previously, the characteristic functions of several elements can be investigated simultaneously also by the LTI Viewer.

2.5.4 Second-Order Oscillating (ξ) Element

$$H(s) = H_\xi(s) = \frac{1}{s^2 T_o^2 + 2\xi T_o s + 1}$$

Let us investigate the system:

$$H(s) = \frac{1}{9s^2 + 2s + 1} = \frac{1}{s^2 T_o^2 + 2\xi T_o s + 1}$$

where $\omega_o = \frac{1}{T_o}$ is the natural frequency and ξ is the damping factor ($T_o = \frac{1}{\omega_o} = 3$, $\xi = 1/3$).

```
num=1;
den=[9, 2, 1]
H=tf(num,den)
```

The poles of the system are calculated by the command roots,

```
roots(den)
```

or by the command damp:

```
damp(H)
```

The conjugate complex poles can be given in the following form: $p_1 = a + jb$, $p_2 = a - jb$.

The overshoot v_t of the step response is calculated by

$$\omega_o^2 = a^2 + b^2 \quad ; \quad \xi = -\frac{a}{\omega_o} \quad \text{and} \quad v_t = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} = e^{-\frac{\pi a}{b}}.$$

The oscillation frequency is

$$\omega_p = b = \omega_o\sqrt{1-\xi^2}$$

The time of the first maximum of the step response (the peak time) is $T_p = \pi/\omega_p = \pi/b$.

```
kszi=1/3
vt=exp(-kszi*pi/sqrt(1-kszi*kszi))
```

The step response can be obtained as follows:

```
[y,t]=step(H);
plot(t,y), grid
```

The maximum value of the step response:

```
ym=max(y)
```

Its steady state value:

```
ys=dcgain(H)
```

The overshoot can be calculated also as

```
yo=(ym-ys)/ys
```

Let us analyse the step responses, the Bode diagram and the Nyquist diagram, for several values of the damping factor: $\xi = 0.3, 0.7, 1, 2$.

```
kszi1=0.3, kszi2=0.7, kszi3=1, kszi4=2
T0=3
H1=1/(s*s*T0*T0+2*kszi1*T0*s+1)
H2=1/(s*s*T0*T0+2*kszi2*T0*s+1)
H3=1/(s*s*T0*T0+2*kszi3*T0*s+1)
H4=1/(s*s*T0*T0+2*kszi4*T0*s+1)
```

The step responses:

```
figure(1), step(H1,'r',H2,'g',H3,'b',H4,'m'),grid
```

The Bode diagrams:

```
figure(2), bode(H1,'r',H2,'g',H3,'b',H4,'m'),grid
```

The Nyquist diagrams:

```
figure(3), nyquist(H1,'r',H2,'g',H3,'b',H4,'m'),grid
```

The pole-zero configurations:

```
figure(4), pzmap (H1,'r',H2,'g',H3,'b',H4,'m')
```

The poles can be obtained also with command damp:

```
damp(H1)
damp(H2)
damp(H3)
damp(H4)
```

It can be seen that for damping factor $\xi = 0.3$ the step response is the most oscillating, the maximum amplification in the Bode amplitude diagram is the highest, and the Nyquist diagram crossing the imaginary axis gives the biggest magnitude for this case. The poles are complex conjugates. The imaginary value of the complex conjugate poles providing the frequency of oscillation in the time response is also the highest. High amplification in the Bode amplitude diagram indicates a high overshoot in the step response. If this should be avoided, no high amplification is allowed in the Bode amplitude diagram. The damping factor $\xi = 0.7$ provides a slight overshoot. Control systems can be designed for similar behaviour. For $\xi = 1$ the system has two coinciding real poles. In the case of $\xi > 1$ there are two different real poles, and the step response is aperiodic. There is no overshoot in the step response and no amplification in the Bode amplitude diagram.

2.5.5 Differentiating (D and DT) Elements

The transfer function of the ideal differentiating element is $H(s) = sT_{\mathrm{d}}$.

```
H=s
bode(H)
step(H)
   ??? Error using ==> rfinputs
   Not supported for non-proper models.
```

MATLAB™ can not evaluate the system responses as the element is non realizable, its transfer function is non-proper, the degree of its numerator is higher than that of its denominator. Its step response is the DIRAC delta.

The differentiating effect can be realized only together with lag elements.

$$H_1(s) = \frac{2s}{1+10s}; \quad H_2(s) = \frac{2s}{(1+10s)(1+2s)}$$

Give the step responses, the BODE and the NYQUIST diagrams of these elements.

```
H1=(2*s)/(1+10*s)
H2=(2*s)/((1+10*s)*(1+2*s))
```

Step responses:

```
figure(1); step(H1,'r',H2,'b'),grid
```

BODE diagrams:

```
figure(2), bode(H1,'r',H2, 'b'),grid
```

NYQUIST diagrams:

```
figure(3), nyquist(H1,'r',H2, 'b'),grid
```

It can be seen that a differentiating element behaves as a high pass filter. It supresses the DC (low frequency) component of a signal and amplifies the high frequency components.

2.5.6 The Effect of Zeros

Suppose the transfer function is given in the following form. The roots of the numerator are the zeros of the transfer function.

$$H(s) = \frac{k(s - z_1)(s - z_2)\ldots(s - z_m)}{\mathrm{D}(s)}$$

Let us analyse how the zeros affect the step response and the frequency response in case of the following transfer function:

$$H(s) = \frac{1 + \tau s}{(1 + s)(1 + 10s)}$$

The time constant τ in the numerator (the zero is $-1/\tau$) changes between -12 and 12. In the case of a positive zero (which is located in the right half-plane of the complex plane) the system is called a non-minimum phase system.

```
s=tf('s')
D=(1+s)*(1+10*s)
tau=[-12 -4 0 4 12]
for i=1:5,H(i)=(s*tau(i)+1)/D,end
figure(1),step(H(1),'r',H(2),'g',H(3),'k',H(4),'m',H(5),'b')
```

The step responses are seen in Fig. 2.10. Note that in the case of a non-minimum phase system, they behave unexpectedly. E.g. in the case of one right-side zero the step response starts in the opposite direction related to the steady state value, then it

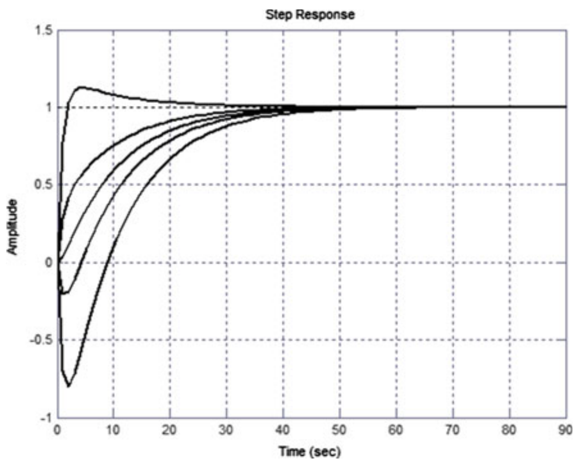


Fig. 2.10 Zeros affect the step response

changes direction and reaches the steady state value. Inserting a zero in the system results in an accelerated time response.

BODE and NYQUIST diagrams:

```
figure(2),bode(H(1),'r',H(2),'g',H(3),'k',H(4),'m',H(5),'b)
figure(3),nyquist(H(1),'r',H(2),'g',H(3),'k',H(4),'m',H(5),'b)
```

Let us evaluate the effect of a zero in the frequency domain, how it influences the BODE and the NYQUIST diagrams.

2.5.7 Dead-Time Element

Its transfer function is

$$H_{\mathrm{H}}(s) = H(s)e^{-sT_{\mathrm{d}}}$$

Its description in the time and in the LAPLACE operator domain is

$$y(t) \to y(t - T_{\mathrm{d}}); \quad Y(s) \to Y(s)e^{-sT_{\mathrm{d}}}$$

In the frequency domain, this is

$$|e^{-j\omega T_{\mathrm{d}}}| = 1, \quad \arg\{e^{-j\omega T_{\mathrm{d}}}\} = -\omega T_{\mathrm{d}} \text{ (in radians)}.$$

Its gain is calculated as $|H_{\mathrm{H}}| = |H|$, and its phase angle is $\arg(H_{\mathrm{H}}) = \arg(H) - \omega T_{\mathrm{d}}$.

Let us analyse the frequency functions of the elements

$$H_1(s) = \frac{1}{1+10s} \quad \text{and} \quad H_2(s) = \frac{1}{(1+10s)}e^{-2s}.$$

The amplitude and the phase angle of the dead-time element are

$$|H_1| = |H_2| \quad , \quad \mathsf{gain2} = \mathsf{gain1}$$

$$\arg(H_2) = \arg(H_1) - \omega T_{\mathrm{d}} \quad , \quad \mathsf{phase2} = \mathsf{phase1} - \mathsf{w}^*\mathsf{Td}$$

```
Td=2
H1=1/(1+10*s)
num1=1
den1=[10, 1]
```

Now when calculating the BODE diagram, use the polynomial form given by the num and den numerator and denominator polynomials.

First let us generate the frequency vector.

```
w=logspace(-2,2,500);
[gain1,phase1]=bode(H1,w);
```

Change the gain1, phase1 values to vector form.

```
gain1=gain1(:);phase1=phase1(:);
phasedelay=180/pi*Td*w';
```

The bode command calculates the phase angle in degrees. The phase delay $\varphi = -\omega T_\mathrm{d}$ of the dead-time element is obtained in radians. Therefore it has to be converted to degrees.

The amplitude and the phase angle considering the dead-time can be obtained as follows:

```
gain2=gain1;
phase2=phase1-phasedelay;
subplot(211),loglog(w, gain1,'r',w, gain2,'b'),grid;
subplot(212),semilogx(w, phase1,'r',w, phase2,'b'),grid
```

The linearity of the course of the phase angle can be seen better if drawing it command plot is used instead of semilogx.

```
figure(2),subplot(111),plot(w,phase1,'r',w,phase2,'b'),grid
```

Now let us plot the NYQUIST diagram. First calculate the real and imaginary values of the frequency function.

```
h1= gain1.*exp(j*phase1*pi/180);
h2= gain2.*exp(j*phase2*pi/180);
figure(2),plot(real(h1),imag(h1),'r',real(h2),imag(h2),'b')
```

The behaviour in the high frequency domain could be seen better if a bigger frequency range is given with the command logspace.

The behaviour of a dead-time element in the time domain can be investigated better in SIMULINK™, as the time-delay is offered as a single building block.

The transfer function of the dead-time element is not a rational function. Nevertheless it can be approximated by a non-minimum phase rational fraction where the first elements of its TAYLOR expansion are the same as those of the exponential transfer function characterizing the dead-time. These rational functions are called PADE functions. The higher the degree of the PADE function, the better is the approximation. It has to be mentioned that with this approximation the step response starts with +1 or –1 instead of zero. In MATLAB™, the command pade calculates the approximation.

Demonstrating the use of PADE approximation, use a 5-th order approximation.

$$H_{\mathrm{H}}(s) = e^{-sT_{\mathrm{d}}} \cong H_{\mathrm{PADE}}(s)$$

```
pade(Td,5);
```

As there is no output parameter, now this command shows graphically the step response.

```
[numd,dend]=pade(Td,5)
Hd=tf(numd,dend)
Hd=zpk(Hd)
H2=H1*Hd
```

The step responses:

```
figure(1), step(H1,'r',H2,'g'),grid
```

The BODE diagram:

```
figure(2), bode(H1,'r',H2,'g'),grid
```

The NYQUIST diagram:

```
figure(3), nyquist(H1,'r',H2,'g')
```

It should be emphasized that in the frequency domain it is better to consider the phase modifying effect of the dead-time than to employ the PADE approximation. In the time domain the analysis can be executed better by running the simulation in SIMULINK™.

Problem: Let us investigate how good is the PADE approximation in the above case of a first-order lag element with dead-time. Build a SIMULINK™ program using the "*transport delay*" block, and using the fifth-order PADE rational function. Compare the step responses.

2.5.8 *Evaluation of the Characteristics of the Elements, the Effects of Poles and Zeros*

The values of the step responses in steady state ($t \to \infty$) and the values of the amplitude response of the frequency function for $\omega \to 0$ are the same.

NYQUIST diagrams of proportional elements at $\omega = 0$ start from a point of the positive real axis, which characterizes the gain of the element. The NYQUIST diagram of an integrating element at $\omega = 0$ starts from infinity in the direction of the negative imaginary axis. The NYQUIST diagram of a double integrating element starts from infinity in the direction of the negative real axis. NYQUIST diagrams of derivative elements start from the zero point of the complex plane in the direction of the positive imaginary axis. In case of a transfer function containing only lags (no zeros), the NYQUIST diagram covers as many quarters in the complex plane as there are time lags. The zeros deteriorate the monotonic change of the phase angle. The BODE amplitude diagram of a proportional element starts parallel to the frequency axis with zero phase angle, the BODE amplitude diagram of a system containing one integrator starts with a slope of –20 dB/decade with a -90° phase angle, whereas the BODE diagram of a system with two integrators starts with a slope of −40 dB/decade and -180° phase angle. Time lags break down the slope of the BODE amplitude diagram by −20 dB/decade, while zeros make the slope go up by +20 dB/decade.

A time constant is the reciprocal of the pole. If the pole is located far away from the origin at the left side of the real axis, this means a fast transient behaviour. In the frequency domain it affects the frequency function at higher frequencies.