

# Chapter 10

## General Polynomial Method for Controller Design



It was shown in Sect. 7.1 that the YOULA-parameterization can well be used for the design of optimal controllers in the case of stable processes. The only disadvantage of the general method is that it cannot be applied to unstable processes, so different kind of parameterization is required. Let us find the controller  $C(s)$  in the form of rational function.

$$C(s) = \frac{\mathcal{Y}(s)}{\mathcal{X}(s)} = \frac{\mathcal{Y}}{\mathcal{X}}. \tag{10.1}$$

Let the prescribed stable characteristic polynomial of the closed-loop be denoted by  $\mathcal{R}(s)$ , i.e., the characteristic equation is given by  $\mathcal{R}(s) = 0$ . Similarly to state feedback, here the design of the stability and performance is also carried out via prescribed poles (pole-placement). Let the transfer characteristics of the delay free process be

$$P(s) = \frac{\mathcal{B}(s)}{\mathcal{A}(s)} = \frac{\mathcal{B}}{\mathcal{A}}. \tag{10.2}$$

The characteristic equation expressing the design goal is

$$\mathcal{A}(s)\mathcal{X}(s) + \mathcal{B}(s)\mathcal{Y}(s) = \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{Y} = \mathcal{R} = \mathcal{R}(s) \tag{10.3}$$

where  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{R}$  are known polynomials, the unknown parameters to be determined are in polynomials  $\mathcal{X}$  and  $\mathcal{Y}$ . Equation (10.3) is called a *DIOPHANTINE equation (DE)*. Since it is not assumed that the process is stable, the resulting controller is therefore also called a *stabilizing controller*.

This *DE* has solution if, and only if, all common factors of  $\mathcal{A}$  and  $\mathcal{B}$  are also the common factors of  $\mathcal{R}$ . If  $\mathcal{A}$  and  $\mathcal{B}$  are relative prime (i.e., they have no polynomial common factor), this *DE* always has a solution for any  $\mathcal{R}$ , and the number of the solutions is infinity. If a pair  $\mathcal{X}_0, \mathcal{Y}_0$  fulfills the equation, then the pair

$$\mathcal{X} = \mathcal{X}_0 + \mathcal{D}\mathcal{B}; \quad \mathcal{Y} = \mathcal{Y}_0 - \mathcal{D}\mathcal{A} \quad (10.4)$$

is also a solution of this *DE*, where  $\mathcal{D}$  is an arbitrary polynomial. If the process polynomials are relative prime (if  $\mathcal{A} \neq 0$ ) then there is always a solution  $\mathcal{X}_0, \mathcal{Y}_0$  of this *DE* such that either  $\mathcal{Y}_0 = 0$  or  $\deg\{\mathcal{Y}_0\} < \deg\{\mathcal{A}\}$ . This latter solution  $\mathcal{X}_0, \mathcal{Y}_0$  is called the minimal one, because there is no other solution  $\mathcal{X}, \mathcal{Y}$  whose polynomials have degree less than the degree of  $\mathcal{Y}_0$ .

Since there are an infinite number of solutions of this *DE*, there exists a special one satisfying the assumption

$$\deg\{\mathcal{X}\} < \deg\{\mathcal{B}\}. \quad (10.5)$$

Similarly, there exists a solution for which

$$\deg\{\mathcal{Y}\} < \deg\{\mathcal{A}\}. \quad (10.6)$$

Both assumptions are fulfilled at the same time (simultaneously), if

$$\deg\{\mathcal{A}\} + \deg\{\mathcal{B}\} \geq \deg\{\mathcal{R}\}. \quad (10.7)$$

In the case of (10.6), the *DE* has a special minimum order solution if

$$\deg\{\mathcal{X}\} = \deg\{\mathcal{Y}\}. \quad (10.8)$$

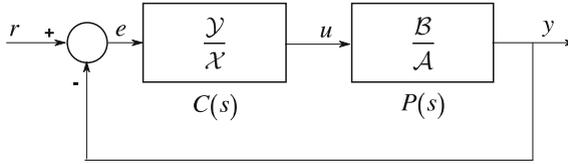
If (10.6) is not valid, then there exists a solution when  $\mathcal{X}$  or  $\mathcal{Y}$  is minimal.

In practice, two basic cases can be distinguished:

- (a) Let  $\mathcal{R}(s)$  be an arbitrary polynomial of order  $\deg\{\mathcal{R}\} = 2\deg\{\mathcal{A}\} - 1$ . In this case the solution of the *DE* can be sought by controller polynomials of order  $\deg\{\mathcal{X}\} = \deg\{\mathcal{A}\} - 1$  and  $\deg\{\mathcal{Y}\} = \deg\{\mathcal{A}\} - 1$ . Consequently the controller will be proper.
- (b) Let  $\mathcal{R}(s)$  be an arbitrary polynomial of order  $\deg\{\mathcal{R}\} = 2\deg\{\mathcal{A}\}$ . In this case the solution of the *DE* can be sought by controller polynomials of order  $\deg\{\mathcal{X}\} = \deg\{\mathcal{A}\}$  and  $\deg\{\mathcal{Y}\} = \deg\{\mathcal{A}\} - 1$ . Consequently the controller will be strictly proper.

Therefore for a process of degree  $n$  usually a stabilizing regulator of degree  $(n - 1)$  is searched, because in this case *DE* always has solution. It can be seen from (10.4), that the order of  $\mathcal{Y}$  can be less than the order of  $\mathcal{A}$ . Theoretically  $\mathcal{X}$  could be of lower order than  $\mathcal{B}$ , but in this case the obtained controller cannot be realized. That is why the stabilizing controller is sought as a transfer function of order  $(n - 1)$ .

It seems to be a reasonable choice if the order of  $\mathcal{R}$  is equal to the order of  $\mathcal{A}$ . In a fortunate case it is possible to find a stabilizing controller of corresponding order and pole excess to a process having a pole excess bigger than one. This procedure,



**Fig. 10.1** One-degree-of-freedom (ODOF) stabilized closed-loop controller

however, cannot be performed in a systematic way and according to (10.7) the solution is not surely minimal.

Equation (10.4) is also valid for the rational function  $D = \mathcal{G}/\mathcal{D}$ . In this case besides  $\mathcal{R}$ ,  $\mathcal{D}$  also appears in the denominator of the overall transfer function of the closed-loop. The form (10.4) parameterizes all stabilizing controllers by  $D$ . The parameter  $D$  is called the YOUNG-KUČERA parameter.

It can be seen easily that the transfer function of the one-degree-of-freedom (ODOF) stabilized closed-loop shown in Fig. 10.1 is

$$T = \frac{\mathcal{B}\mathcal{Y}}{\mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{Y}} = \frac{\mathcal{Y}}{\mathcal{R}}\mathcal{B} = R'_n\mathcal{B}. \tag{10.9}$$

Equation (10.9) shows that stabilization is achieved, but the numerator of the process and the polynomial  $\mathcal{Y}$  resulting from the solution of the DE appears in the numerator of the overall transfer function. Note that none of them can be directly influenced, so the numerator of the transfer function of the closed-loop cannot be designed. (See the similarities with the results obtained for state-feedback.)

In spite of the not completely preferable design possibilities, a TDOF control loop can be constructed where the reference signal tracking, at least, can be designed. This system is shown in Fig. 10.2a. An equivalent block scheme is presented in Fig. 10.2b which can be directly compared to the generic TDOF (GTDOF) control loop obtained by a YOUNG parameterized controller for stable processes according to Fig. 7.10a. The controller is obviously different now.

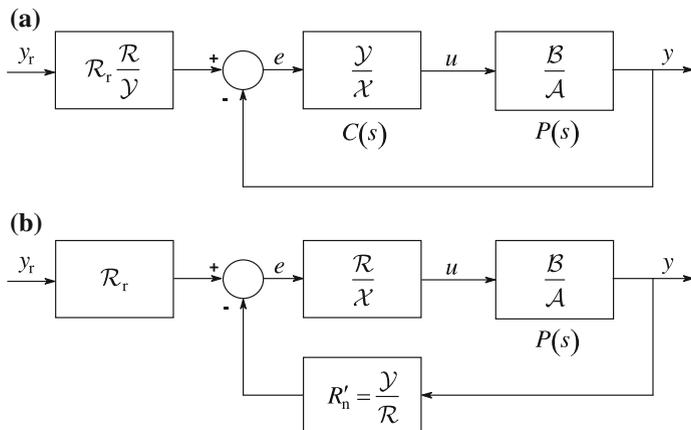
The transfer function of the control loop shown in Fig. 10.2 is

$$T_r = R_r\mathcal{B}. \tag{10.10}$$

Here  $\mathcal{Y}$  already does not appear, only the numerator of the process, and  $R_r$  is independent of  $R'_n$ , thus it is really a TDOF control. The noise-rejection behavior can be computed from  $T$

$$S = 1 - T = 1 - R'_n\mathcal{B} = 1 - \frac{\mathcal{Y}}{\mathcal{R}}\mathcal{B}. \tag{10.11}$$

It has been already seen in the discussion of the YOUNG-parameterized controller, that in the numerator of the transfer function of the process only the stable zeros can



**Fig. 10.2** TDOF stabilized closed-loop

be cancelled. This method can be extended to the stable poles of the denominator in the design method using the *DE*. Assume that the transfer function of the process is

$$P(s) = P_+(s)P_-(s) \text{ or, for short, } P = P_+P_-, \tag{10.12}$$

where  $P_+$  is stable, its inverse is also stable (*SIS: Stable Inverse Stable*).  $P_-$  is unstable, and its inverse is also unstable (*UIU: Unstable Inverse Unstable*). Thus a practical factorization is

$$P = \frac{B}{A} = \frac{B_+B_-}{A_+A_-} = \left(\frac{B_+}{A_+}\right)\left(\frac{B_-}{A_-}\right) = P_+P_-. \tag{10.13}$$

Here  $A_+$  contains the stable poles of the process and  $A_-$  does contains the unstable ones. Similarly  $B_+$  contains the stable zeros and  $B_-$  the unstable zeros. The *DE* has to be constructed to make possible the cancellation of the stable roots  $B_+$  and  $A_+$ . In order to define the design procedure in a completely general way, predefined polynomials  $\mathcal{Y}_d$  and  $\mathcal{X}_d$  are introduced in the numerator and denominator of the controller. The following design *DE* can be written for this most general case:

$$\begin{matrix} (A_+A_-)(B_+\mathcal{X}_d\mathcal{X}) + (B_+B_-)(A_+\mathcal{Y}_d\mathcal{Y}) = \mathcal{R} = A_+B_+\mathcal{R} \\ \mathcal{A} \quad \quad \mathcal{X} \quad + \quad \mathcal{B} \quad \quad \mathcal{Y} \quad = \mathcal{R}' \end{matrix} \tag{10.14}$$

A lower order  $DE$  can be obtained by simplifying with the reducing factors

$$\begin{aligned} (\mathcal{A}_- \mathcal{X}_d) \mathcal{X}' + (\mathcal{B}_- \mathcal{Y}_d) \mathcal{Y}' &= \mathcal{R} \\ \mathcal{A}' \mathcal{X}' + \mathcal{B}' \mathcal{Y}' &= \mathcal{R} \end{aligned} \tag{10.15}$$

where  $\mathcal{A}' = \mathcal{A}_- \mathcal{X}_d$  and  $\mathcal{B}' = \mathcal{B}_- \mathcal{Y}_d$  are known, and the controller is obtained as

$$C = \frac{\mathcal{Y}}{\mathcal{X}} = \frac{\mathcal{A}_+ \mathcal{Y}_d \mathcal{Y}'}{\mathcal{B}_+ \mathcal{X}_d \mathcal{X}'}. \tag{10.16}$$

It is evident, that the stabilizing controller cancelled only the stable zeros and poles, and introduced the desired polynomials  $\mathcal{Y}_d$  and  $\mathcal{X}_d$  into the numerator and denominator. The YOULA regulator is an integrating one, if a unit gain concerning the reference model is ensured:  $R_n(\omega = 0) = R_n(s = 0) = 1$ . This cannot be automatically ensured for the stabilizing controller resulting from a  $DE$ . It can be guaranteed only if  $\mathcal{X}_d$  brings a pole  $s = 0$  into the denominator.

Since now  $\mathcal{Y}_d$  can be considered as the numerator of the reference model, and  $\mathcal{R}$ , however, as the denominator, it follows that in the general case, the corrected reference model is

$$R'_n = \frac{\mathcal{Y}_d}{\mathcal{R}}, \tag{10.17}$$

which depends only on us, so it can completely be designed.

Equivalent block schemes of the general stabilized control loop are shown in Fig. 10.3.

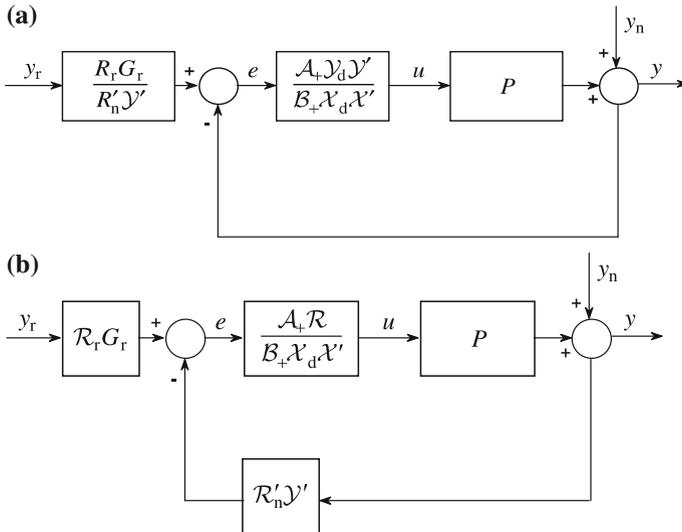


Fig. 10.3 TDOF general stabilized closed-loop

It can be easily checked that the transfer function of the whole loop is

$$T_r = P_r G_r \mathcal{B}_- \quad (10.18)$$

and the sensitivity function of the closed-loop is

$$S = 1 - P'_w \mathcal{Y}' \mathcal{B}_-. \quad (10.19)$$

So the transfer characteristics of the whole closed-loop control is

$$y = T_r y_r + S y_n = R_r G_r \mathcal{B}_- y_r + (1 - R'_n \mathcal{Y}' \mathcal{B}_-) y_n. \quad (10.20)$$

It is evident that the filter  $G_r$  can be freely chosen, and can be optimized to attenuate the effect of  $\mathcal{B}_-$ . Unfortunately, the same is not valid for the optimal design concerning the disturbance rejection, because there,  $\mathcal{Y}'$  results from the modified *DE* (10.15), so it cannot be freely chosen, therefore the attenuation of the effect of  $\mathcal{Y}'$  cannot be easily solved, as has been seen in the *YOULA*-parameterization for the tracking problem (10.20).

The form of the resulting stabilizing controller shown in (10.16) can be further simplified:

$$C = \frac{\mathcal{A}_+ \mathcal{Y}_d \mathcal{Y}'}{\mathcal{B}_+ \mathcal{X}_d \mathcal{X}'} = \frac{\left(\frac{\mathcal{Y}_d}{\mathcal{R}}\right) \mathcal{Y}' \mathcal{A}}{\mathcal{B}_+ \left(1 - \frac{\mathcal{Y}_d}{\mathcal{R}} \mathcal{Y}' \mathcal{B}_-\right)} = \frac{P'_w \mathcal{Y}'}{1 - P'_w \mathcal{Y}' \mathcal{B}_-} \frac{\mathcal{A}}{\mathcal{B}_+}, \quad (10.21)$$

which is very similar to the form of the optimal *YOULA* regulator (7.14). Observe that though only the stable factors  $\mathcal{A}_+$  and  $\mathcal{B}_+$  are cancelled, formally the controller cancels the whole denominator of the process.

If the feature obtained for the noise-rejection in (10.20) cannot be accepted, an outer cascade control loop has to be applied, which can already be designed by the *YOULA*-parameterization, since the system has already been stabilized by the inner loop. This two-step method was discussed in detail in the chapter on the control loops applying state-feedback [see Sect. 9.4].

The stabilizing controller obtained by the *DE* can be applied only to delay free processes. If the process has significant dead-time, then there is the possibility of approximating the delay by a rational function [see Sect. 2.5]. The other possibility is to use sampled data control system [see Chap. 14].

*Example 10.1* Let the controlled system be a first order ( $n = 1$ ) unstable process

$$P = \frac{\mathcal{B}}{\mathcal{A}} = \frac{0.5}{1 - 0.5s} = \frac{-1}{s - 2}, \quad (10.22)$$

whose pole  $p = 2$  is on the right half of the complex plane. Find the controller  $C = \mathcal{Y}'/\mathcal{X}'$  that stabilizes the process by prescribing the characteristic polynomial

$\mathcal{R}(s) = s + 2 = 0$ . The controller is sought in the form of order  $n - 1 = 0$ , which can be ensured by the structure

$$C = \frac{\mathcal{Y}}{\mathcal{X}} = \frac{K}{1} = K, \tag{10.23}$$

i.e., by a proportional controller. Based on (10.3) one can write

$$\begin{aligned} \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{Y} &= \mathcal{R} \\ (s - 2) - K &= s + 2 \end{aligned} \tag{10.24}$$

where  $C = K = -4$  is obtained for the controller. It can be checked by simple computation, that the transfer function of the closed-loop is

$$T = \frac{4}{s + 2} = \frac{2}{1 + 0.5s}, \tag{10.25}$$

thus the unstable pole can be mirrored about the imaginary axis, and by this means, the system is stabilized. The static gain of the closed-loop system is not unity, because the controller is proportional and not an integrating one. To reach better quality in performance, it is reasonable to use a further outer cascade control loop, as was seen with the state feedback controllers. Based on (10.4), the resulting stabilizing controllers  $C(s)$  and  $T(s)$  are given for different parameters  $D(s) = \mathcal{G}/\mathcal{D}$  in Table 10.1. ■

The first row of the Table 10.1 contains the first solution obtained in (10.23) and (10.25). It is well seen, that only the first controller can be realized, so the other solutions have only theoretical importance. For higher order processes the expressions are more complicated, but even for these cases it is reasonable to summarize the different order solutions in tables and choose the lowest order realizable controller. In the same way it is also reasonable to give the solutions being lower order than the  $(n - 1)$  order controller.

*Example 10.2* Let the controlled system be a first order ( $n = 1$ ) stable process

$$P = \frac{\mathcal{B}}{\mathcal{A}} = \frac{1}{1 + 10s} = \frac{0.1}{s + 0.1}, \tag{10.26}$$

**Table 10.1** .

$D(s) = \mathcal{G}/\mathcal{D}$	$C(s)$	$T(s)$
0	-4	$\frac{4}{s + 2}$
1	$\frac{s - 6}{2}$	$-\frac{s - 6}{s + 2}$
$1 + s$	$\frac{s^2 - s - 6}{s + 2}$	$-\frac{s^2 - s - 6}{s + 2}$
$\frac{s + 2}{s + 1}$	$\frac{s^2 - 4s - 8}{2s + 3}$	$-\frac{s^2 - 4s - 8}{(s + 1)(s + 2)}$

which we would like to speed up. Assuming an *ODOF* system, the design goal is expressed by the reference model

$$R_r = R_n = \frac{1}{1+2s} = \frac{0.5}{s+0.5}. \quad (10.27)$$

The *YOULA*-regulator

$$C_{\text{opt}} = C_{\text{id}} = \frac{R_n P_+^{-1}}{1 - R_n} = \left(1 - \frac{1}{1+2s}\right)^{-1} \frac{1+10s}{1+2s} = \frac{1+10s}{2s} \quad (10.28)$$

is now an integrating one, so the transfer function of the closed-loop is

$$T(s) = \frac{1}{1+2s}. \quad (10.29)$$

For the *DE* design, based on (10.27), the characteristic equation is  $\mathcal{R}(s) = s + 0.5 = 0$ . As in the previous example, the controller is again sought in a form of  $n - 1 = 0$  degree, thus the proportional controller (10.23) is employed. The Eq. (10.3) now becomes

$$\begin{aligned} \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{Y} &= \mathcal{R} \\ (s + 0.1) + 0.1K &= s + 0.5 \end{aligned} \quad (10.30)$$

where  $C = K = 4$  is obtained for the regulator. It can be easily checked that the transfer function of the closed system is

$$T = \frac{0.4}{s+0.5} = \frac{0.8}{1+2s}. \quad (10.31)$$

The prescribed pole  $-0.5$  is successfully placed, but the control loop has zero-type, therefore for the gain of  $T$  the value 0.8 is obtained. The above two examples well represent the practice of how the *YOULA*-parameterization can be reasonably applied to stable processes, while for stabilizing unstable processes the application of *DE* or the state-feedback discussed in Chap. 9 can provide a solution. ■