Chapter 10
General Polynomial Method for Regulator Design

The theoretical considerations of Chap. 10 of the textbook [1] are summarized here to support the introduction of the variable names in the corresponding MATLAB™ programs.

Process: $P = \frac{\mathcal{B}}{\mathcal{A}}$

Regulator: $C = \frac{\mathcal{Y}}{\mathcal{X}}$

Resulting transfer function: $T = \frac{CP}{1+CP} = \frac{\frac{\mathcal{Y}\mathcal{B}}{\mathcal{X}\mathcal{A}}}{1+\frac{\mathcal{Y}\mathcal{B}}{\mathcal{X}\mathcal{A}}} = \frac{\mathcal{Y}\mathcal{B}}{\mathcal{Y}\mathcal{B}+\mathcal{X}\mathcal{A}} = \frac{\mathcal{Y}\mathcal{B}}{\mathcal{R}}$, where the characteristic polynomial of the closed loop is $\mathcal{R}(s) = \mathcal{X}(s)\mathcal{A}(s) + \mathcal{Y}(s)\mathcal{B}(s)$ and the characteristic equation is $\mathcal{R}(s) = \mathcal{X}(s)\mathcal{A}(s) + \mathcal{Y}(s)\mathcal{B}(s) = 0$.

Sensitivity function: $S = \frac{1}{1+CP} = \frac{1}{1+\frac{\mathcal{Y}\mathcal{B}}{\mathcal{X}\mathcal{A}}} = \frac{\mathcal{X}\mathcal{A}}{\mathcal{Y}\mathcal{B}+\mathcal{X}\mathcal{A}} = \frac{\mathcal{X}\mathcal{A}}{\mathcal{R}}$

Let the order of the system be n, i.e. $\deg\{\mathcal{A}\} = n$. A realizable regulator $C(s)$ is to be given, which

- yields the given characteristic polynomial $\mathcal{R}(s)$. The regulator is determined by solving the Diophantine equation $\mathcal{X}(s)\mathcal{A}(s) + \mathcal{Y}(s)\mathcal{B}(s) = \mathcal{R}(s)$
- at the initial time instant $t = 0$ for a unit step reference signal it provides a control signal value $u(0) \neq 0$, i.e. $\deg\{\mathcal{X}\} = \deg\{\mathcal{Y}\}$.

An important remark is that in the resulting transfer function of the closed-loop $T(s) = \frac{\mathcal{Y}(s)\mathcal{B}(s)}{\mathcal{R}(s)}$

- $\mathcal{R}(s)$ is a given polynomial determined by the designer.
- $\mathcal{Y}(s)$ is calculated by solving the Diophantine equation $\mathcal{X}(s)\mathcal{A}(s) + \mathcal{Y}(s)\mathcal{B}(s) = \mathcal{R}(s)$.
- $\mathcal{B}(s)$ is given: it is the numerator of the transfer function of the process.

Let us choose the degree of the characteristic polynomial $\mathcal{X}(s)\mathcal{A}(s) + \mathcal{Y}(s)\mathcal{B}(s) = \mathcal{R}(s)$ to be $\deg\{\mathcal{R}\} = 2n - 1$. Then the Diophantine equation always has a solution and the degree of the regulator is $(n-1)$.

L. Keviczky et al., *Control Engineering: MATLAB Exercises*,
Advanced Textbooks in Control and Signal Processing,
https://doi.org/10.1007/978-981-10-8321-1_10

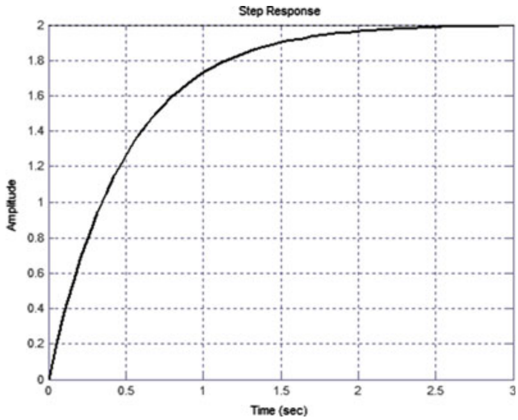

Fig. 10.1 Step response of a control system with unstable process

Example 10.1 Repeating Example 10.1 of the textbook [1] the transfer function of the unstable process is $P(s) = \frac{-1}{s-2}$. Then $n = 1$ and the degree of the regulator is 0, $C(s) = \frac{\mathcal{Y}}{\mathcal{X}} = \frac{K}{1} = K$. Let the first order characteristic polynomial be $\mathcal{R}(s) = s+2$ (The unstable pole of the process is reflected in the imaginary axis). Then from the solution of the DIOPHANTINE equation $\mathcal{X}(s)\mathcal{A}(s)+\mathcal{Y}(s)\mathcal{B}(s) = 1\,(s-2)+K(-1) = \mathcal{R}(s) = s+2$, the gain $K = -4$ is obtained for the regulator.

```
s=zpk('s')
P=-1/(s-2)
C=-4
T=C*P/(1+C*P), T=minreal(T)
           4
         -----
         s + 2
step(T),grid
```

In Fig. 10.1 it can be seen that with the designed regulator the unstable process has been stabilized, the characteristic equation of the closed loop is indeed $\mathcal{R}(s) = s+2$. But there is a significant static error: $\lim_{t\to\infty} y(t) = 2$. With a constant $F(s) = 0.5$ precompensator the static error becomes zero (Fig. 10.2).

```
step(0.5*T),grid
```

Let us remark that the resulting transfer function of the closed loop with the precompensator is $T_e(s) = F(s)\frac{\mathcal{Y}(s)\mathcal{B}(s)}{\mathcal{R}(s)}$. With the precompensator $F(s)$, the effect of the zeros in the numerator of $T(s)$ can also be compensated. If the transfer function

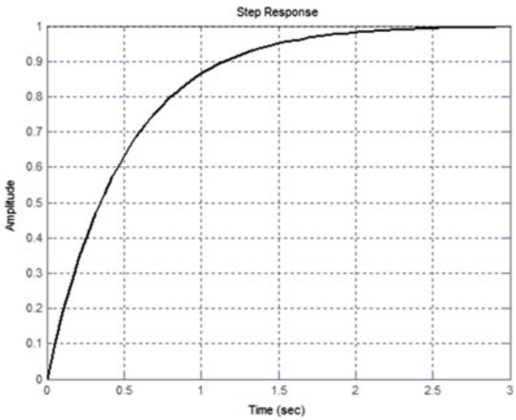

Fig. 10.2 The precompensator ensures accurate settling

$F(s)$ is not a constant, the dynamics of reference signal tracking and the dynamics of output disturbance rejection will be different, the control system will become of *two-degree-of-freedom* (*2DOF*).

Example 10.2 Take into consideration that the stable poles of the process (which are included in the polynomial $\mathcal{A}_+$) and its inverse stable zeros (included in the polynomial $\mathcal{B}_+$) can be cancelled with the regulator. ($\mathcal{A}_-$ and $\mathcal{B}_-$ denote the non-cancellable factors.) The transfer function of the process can be given by $P = \frac{\mathcal{B}_+}{\mathcal{A}_+}\frac{\mathcal{B}_-}{\mathcal{A}_-}$, and the regulator transfer function is expressed as $C = \frac{\mathcal{A}_+}{\mathcal{B}_+}\frac{\mathcal{Y}}{\mathcal{A}\mathcal{X}}$.

Following the notations of the textbook [1] the polynomials are factored as $P(s) = P_+(s)P_-(s)$ where the roots of $P_+(s)$ are located in the left half-plane.

The resulting transfer function of the closed loop is

$$T = \frac{CP}{1+CP} = \frac{\frac{\mathcal{A}_+}{\mathcal{B}_+}\frac{\mathcal{Y}}{\mathcal{X}}\frac{\mathcal{B}_+}{\mathcal{A}_+}\frac{\mathcal{B}_-}{\mathcal{A}_-}}{1+\frac{\mathcal{A}_+}{\mathcal{B}_+}\frac{\mathcal{Y}}{\mathcal{X}}\frac{\mathcal{B}_+}{\mathcal{A}_+}\frac{\mathcal{B}_-}{\mathcal{A}_-}} = \frac{\frac{\mathcal{Y}}{\mathcal{X}}\frac{\mathcal{B}_-}{\mathcal{A}_-}}{1+\frac{\mathcal{Y}}{\mathcal{X}}\frac{\mathcal{B}_-}{\mathcal{A}_-}} = \frac{\mathcal{Y}\mathcal{B}_-}{\mathcal{X}\mathcal{A}_-+\mathcal{Y}\mathcal{B}_-} = \frac{\mathcal{Y}\mathcal{B}_-}{\mathcal{R}},$$

where $\mathcal{X}\mathcal{A}_- + \mathcal{Y}\mathcal{B}_- = \mathcal{R}$. The sensitivity function is

$$S = \frac{1}{1+CP} = \frac{1}{1+\frac{\mathcal{A}_+}{\mathcal{B}_+}\frac{\mathcal{Y}}{\mathcal{X}}\frac{\mathcal{B}_+}{\mathcal{A}_+}\frac{\mathcal{B}_-}{\mathcal{A}_-}} = \frac{1}{1+\frac{\mathcal{Y}}{\mathcal{X}}\frac{\mathcal{B}_-}{\mathcal{A}_-}} = \frac{\mathcal{X}\mathcal{A}_-}{\mathcal{X}\mathcal{A}_-+\mathcal{Y}\mathcal{B}_-} = \frac{\mathcal{X}\mathcal{A}_-}{\mathcal{R}} = 1-T.$$

If the transfer function of the process is $P(s) = \frac{s+7}{(s-2)(s+10)}$, then $\mathcal{B}_+ = s+7$, $\mathcal{B}_- = 1$, $\mathcal{A}_+ = s+10$ and $\mathcal{A}_- = s-2$. The Diophantine equation $\mathcal{X}\mathcal{A}_- + \mathcal{Y}\mathcal{B}_- = \mathcal{R}$ with $\frac{\mathcal{Y}}{\mathcal{X}} = \frac{K}{1}$ can be of first degree. Let us choose $\mathcal{R}(s) = s+2$ as the characteristic polynomial. So $\mathcal{X}\mathcal{A}_- + \mathcal{Y}\mathcal{B}_- = \mathcal{R}$ and $K = 4$. The regulator is $C = \frac{\mathcal{A}_+}{\mathcal{B}_+}\frac{\mathcal{Y}}{\mathcal{X}} = \frac{s+10}{s+7}K = 4\frac{s+10}{s+7}$.

Steps of the MATLAB™ simulation:

```
C=4*(s+10)/(s+7)
P=(s+7)/(s-2)/(s+10)
T=C*P/(1+C*P)
T=minreal(T)
        4
      -----
      s + 2
step(T),grid
```

Figure 10.3 shows that the regulator stabilizes the unstable process. The static error can be eliminated by a precompensator.

Example 10.3 Consider the plant given by the transfer function $P(s) = \frac{(s-5)(s+7)}{(s-2)(s+10)}$. Here $\mathcal{B}_+ = s+7$, $\mathcal{B}_- = s-5$, $\mathcal{A}_+ = s+10$ and $\mathcal{A}_- = s-2$. Let one root of the characteristic polynomial be again $s=-2$, and the other $s=-6$. The characteristic polynomial is now $\mathcal{R}(s) = K(s+2)(s+6)$, so $\frac{\mathcal{Y}}{\mathcal{X}} = \frac{s-z}{s-p}$ and the characteristic equation can be written as $\mathcal{X}\mathcal{A}_- + \mathcal{Y}\mathcal{B}_- = \mathcal{R}$

$$(s-p)(s-2)+(s-z)(s-5) = \mathcal{R}(s) = K(s+2)(s+6)$$

Comparing the coefficients $K=2$, $z{=}70/3$, $p{=}-139/3$; and the regulator is $C = \frac{\mathcal{A}_+}{\mathcal{B}_+}\frac{\mathcal{Y}}{\mathcal{X}} = \frac{s+10}{s+7}\frac{s-70/3}{s+139/3}$.

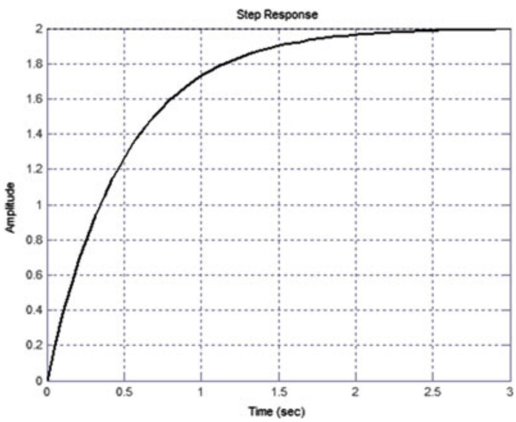

Fig. 10.3 Step response of a control system with unstable process

Steps of the simulation in MATLAB™:

```
s=zpk('s')
P=(s-5)*(s+7)/(s-2)/(s+10)
C=(s+10)*(s-70/3)/(s+7)/(s+139/3)
T=C*P/(1+C*P)
T=minreal(T)
    0.5(s-5)(s-23.33)
   --------------
       (s+6)(s+2)
figure(1)
step(T),grid
```

The step response of the control system is shown in Fig. 10.4. If we would like to eliminate the static error and decrease the under sweeping that resulted because of the non-minimum phase feature of the process, the precompensator can be extended by a filter allocating a pole e.g. to $s = -1$: $F(s) = \frac{1}{(s+1)T(0)}$, where $T(0)$ is the static gain of the system without the precompensator. The MATLAB™ code for this is

```
F= 1/(dcgain(T)*(s+1))
figure(2)
step(F*T,6),grid
```

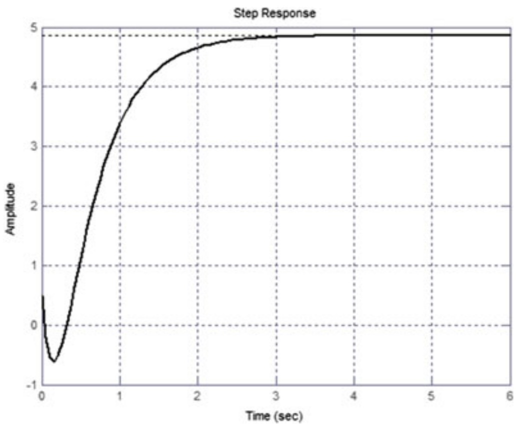

Fig. 10.4 Step response of a control system with unstable and non-minimumphase process

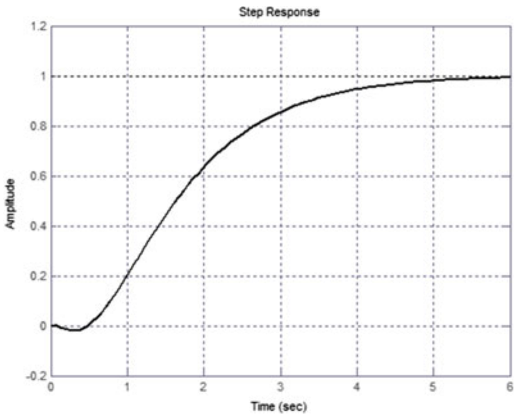

Fig. 10.5 Step response of a control system with unstable and non-minimumphase process with precompensator

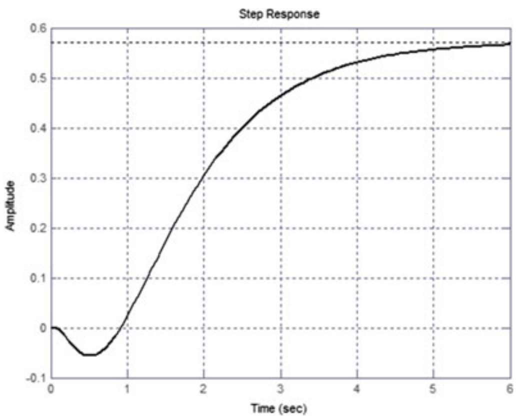

Fig. 10.6 The control signal

It can be seen that with the filter introducing the pole at $s = -1$ the under-sweeping has been decreased significantly, but the settling time has been increased (Fig. 10.5). The control signal is given in Fig. 10.6.

```
U=F*C/(1+C*P)
figure(3)
step(U,6),grid
```

Refining the design of the precompensator the behaviour of the control system can be improved further.

Example 10.4 As was seen in Example 10.3, the regulator contains fixed and calculated components. The fixed components result from cancellation of the stable process poles and the inverse stable zeros; the calculated components result from the solution of the DIOPHANTINE equation. There are practical cases when it is favourable to include a further given component in the regulator. As was seen in Example 4.2, Sect. 4.4 of the textbook [1], if the requirement is to follow an exponential reference signal then a zero in the regulator corresponding to the pole of the exponential signal would ensure tracking without error. Similarly, a pole at $s = 0$ in the regulator forces an integrating effect in the regulator. Let the structure of the regulator be $C = \frac{\mathcal{A}_+}{\mathcal{B}_+}\frac{\mathcal{Y}}{\mathcal{X}}\frac{\mathcal{Y}_\mathrm{d}}{\mathcal{X}_\mathrm{d}}$, where $\mathcal{Y}_\mathrm{d}$ and $\mathcal{X}_\mathrm{d}$ are polynomials representing the given zeros and poles, respectively. Now the characteristic polynomial is $\mathcal{X}\mathcal{X}_\mathrm{d}\mathcal{A}_- + \mathcal{Y}\mathcal{Y}_\mathrm{d}\mathcal{B}_- = \mathcal{R}$. To ensure the solvability of the DIOPHANTINE equation considering the degrees of polynomials $\mathcal{A}_-$ and $\mathcal{B}_-$, the degrees of $\mathcal{Y}_\mathrm{d}$, $\mathcal{X}_\mathrm{d}$ and $\mathcal{R}$ should be chosen according to quite complex conditions.

Let us consider the process $P(s) = \frac{(s-5)(s+7)}{(s-2)(s+10)}$ analysed in Example 10.3. Here $\mathcal{B}_+ = s+7$, $\mathcal{B}_- = s-5$, $\mathcal{A}_+ = s+10$ and $\mathcal{A}_- = s-2$. Apply an integrator in the regulator, $\mathcal{X}_\mathrm{d} = s$ and $\mathcal{Y}_\mathrm{d} = 1$. The characteristic polynomial is $\mathcal{X}\mathcal{X}_\mathrm{d}\mathcal{A}_- + \mathcal{Y}\mathcal{Y}_\mathrm{d}\mathcal{B}_- = \mathcal{R}$.

In this example the essence of polynomial design is summarized in three points:

- An integrator is introduced in the loop transfer function because of the required static accuracy.
- The performance of the closed loop as the aim of the design is specified by prescribing the poles of the closed loop transfer function, i.e. prescribing the characteristic equation of the closed loop.
- The degrees of the polynomials in the numerator and the denominator of the regulator should be the same, otherwise the regulator would be non-realizable, or at the instant when the error appears it would not produce a control signal which is proportional to the value of the error.

In our example for introducing an integrator let be $\mathcal{X}_\mathrm{d} = s$ and $\mathcal{Y}_\mathrm{d} = 1$.

Suppose the degrees of the numerator and the denominator of the regulator are the same. $\deg\{\mathcal{A}_+\} + \deg\{\mathcal{Y}\} = \deg\{\mathcal{B}_+\} + \deg\{\mathcal{X}\} + 1$; in our case $1 + \deg\{\mathcal{Y}\} = 1 + \deg\{\mathcal{X}\} + 1$.

If the degree of $\mathcal{X}$ is zero, then the degree of $\mathcal{Y}$ is 1.

Let the prescribed roots of the characteristic equation be -2 and -6.

The characteristic equation is then

$$s x_\mathrm{o}(s-2) + (y_\mathrm{o} + s y_1)(s-5) = \alpha(s+2)(s+6).$$

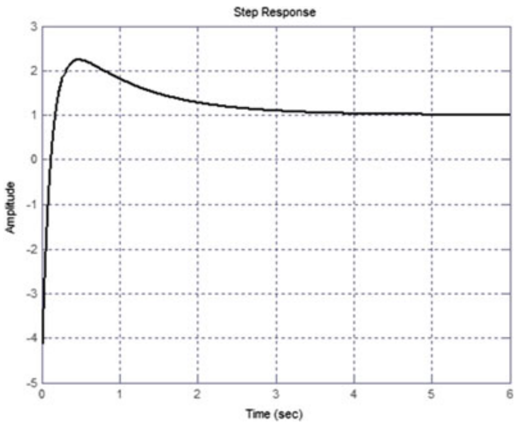

Fig. 10.7 Step response of the control system with a controller containing integrator

The solution of the DIOPHANTINE equation taking $\alpha = 1$ is $x_o = 231/31$, $y_o = -12/5$ and $y_1 = -6$. So the transfer function of the stabilizing regulator is

$$C(s) = \frac{-0.80519(s+0.4)(s+10)}{s(s+7)}$$

The step response of the control system is shown in Fig. 10.7. The dynamics and the overshoot can be modified by the polynomials $\mathcal{X}_d$ and $\mathcal{Y}_d$.