

# Chapter 15

## General Polynomial Method for the Design of Discrete-Time Controllers



Unfortunately the application of the *DE* to CT processes cannot handle a time-delay, since the method can be used only for polynomials. Time-delay systems can be stabilized only in the discrete-time case. Assume that the pulse transfer function of the process is

$$G(z^{-1}) = G_+(z^{-1})\bar{G}_-(z^{-1}) = G_+(z^{-1})G_-(z^{-1})z^{-d}, \text{ or } G = G_+\bar{G}_- = G_+G_-z^{-d} \tag{15.1}$$

where  $G_+$  is stable and its inverse is also stable (*SIS*: Stable Inverse Stable).  $\bar{G}_-$  is unstable and its inverse is also unstable (*UIU*: Unstable Inverse Unstable).  $G_-$  is also *UIU*. Here, in general, the inverse of the time-delay part cannot be realized, because it would be an ideal predictor. Thus a reasonable factorization of the process is

$$G = \frac{\mathcal{B}}{\mathcal{A}}z^{-d} = \frac{\mathcal{B}_+\mathcal{B}_-}{\mathcal{A}_+\mathcal{A}_-}z^{-d} = \left(\frac{\mathcal{B}_+}{\mathcal{A}_+}\right)\left(\frac{\mathcal{B}_-}{\mathcal{A}_-}\right)z^{-d} = G_+G_-z^{-d} \tag{15.2}$$

Here  $\mathcal{A}_+$  contains the stable poles,  $\mathcal{A}_-$  the unstable ones. Similarly,  $\mathcal{B}_+$  includes the stable zeros,  $\mathcal{B}_-$  the unstable ones. The general design *DE* for discrete systems is simply obtained from (10.14) by formally changing  $\mathcal{B}_-$  to  $\mathcal{B}_-z^{-d}$ . The new form of (10.14) becomes

$$\begin{matrix} (\mathcal{A}_+\mathcal{A}_-) & (\mathcal{B}_+\mathcal{X}_d\mathcal{X}') & + & (\mathcal{B}_+\mathcal{B}_-z^{-d}) & (\mathcal{A}_+\mathcal{Y}_d\mathcal{Y}') & = & \mathcal{R}' = \mathcal{A}_+\mathcal{B}_+\mathcal{R} \\ \mathcal{A} & \mathcal{X} & + & \mathcal{B} & \mathcal{Y} & = & \mathcal{R}' \end{matrix} \tag{15.3}$$

The modified  $DE$  is

$$\begin{array}{rcl} (\mathcal{A}_- \mathcal{X}_d) & \mathcal{X}' & + (\mathcal{B}_- z^{-d} \mathcal{Y}_d) & \mathcal{Y}' & = & \mathcal{R}' \\ \mathcal{A}' & \mathcal{X}' & + & \mathcal{B}' & \mathcal{Y}' & = & \mathcal{R}' \end{array} \quad (15.4)$$

where  $\mathcal{A}' = \mathcal{A}_- \mathcal{X}_d$  and  $\mathcal{B}' = \mathcal{B}_- z^{-d} \mathcal{Y}_d$  are known and the controller is obtained again as

$$C = \frac{\mathcal{Y}}{\mathcal{X}} = \frac{\mathcal{A}_+ \mathcal{Y}_d \mathcal{Y}'}{\mathcal{B}_+ \mathcal{X}_d \mathcal{X}'} = \frac{\left(\frac{\mathcal{Y}_d}{\mathcal{R}}\right) \mathcal{Y}' \mathcal{A}}{\mathcal{B}_+ \left(1 - \frac{\mathcal{Y}_d}{\mathcal{R}} \mathcal{Y}' \mathcal{B}_- z^{-d}\right)} = \frac{P'_w \mathcal{Y}'}{1 - P'_w \mathcal{Y}' \mathcal{B}_- z^{-d}} \frac{\mathcal{A}}{\mathcal{B}_+} \quad (15.5)$$

The YOULA-regulator is integrating if a unit gain is ensured for the reference model:  $R_n(\omega = 0) = R_n(z = 1) = 1$ . This cannot be automatically guaranteed for the stabilizing controller coming from the  $DE$ . This solution is guaranteed if  $\mathcal{X}_d$  brings the pole  $z = 1$  into the denominator. To solve the  $DE$  Eq. (15.4), the equation has to be formed in powers of  $z$ .

What was discussed in detail in the Chap. 10 relating to the  $DE$  will not be repeated here. Note that the transfer characteristic of the whole control loop is

$$y = T_r y_r + S y_n = R_r G_r \mathcal{B}_- z^{-d} y_r + (1 - R'_n \mathcal{Y}' \mathcal{B}_- z^{-d}) y_n \quad (15.6)$$

It can be clearly seen that the filter  $G_r$  can be chosen arbitrarily and can be optimized to attenuate the effect of  $\mathcal{B}_-$ . Unfortunately the same statement cannot be made about the optimization of the disturbance rejection. Here  $\mathcal{Y}'$  comes from the modified  $DE$  (15.4), so it cannot be chosen arbitrarily, therefore the attenuation of the effect of  $\mathcal{B}_-$  cannot be solved as easily, as was seen with the YOULA-parameterization and tracking properties (15.6).

*Example 15.1* Let the controlled system be a first order ( $n = 1$ ), unstable DT process

$$G(z^{-1}) = \frac{\mathcal{B}(z^{-1})}{\mathcal{A}(z^{-1})} = \frac{-0.2z^{-1}}{1 - 1.2z^{-1}} = \frac{-0.2}{z - 1.2} \quad (15.7)$$

whose pole  $p = 1.2$  is outside the unit circle. Determine the controller  $C = \mathcal{Y}/\mathcal{X}$  which stabilizes the process by prescribing the characteristic polynomial  $\mathcal{R}(z) = z - 0.2 = 0$ . The controller is sought in the form of  $n - 1 = 0$  order, which can be reached by the structure

$$C = \frac{\mathcal{Y}}{\mathcal{X}} = \frac{K}{1} = K \quad (15.8)$$

i.e. by a proportional controller. Based on (15.4), we have

$$\begin{aligned} \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{Y} &= \mathcal{R} \\ (z - 1.2) - 0.2K &= z - 0.2 \end{aligned} \quad (15.9)$$

from which  $C = K = -5$  is obtained for the controller. It can be checked by simple computation that the pulse transfer function of the closed system is

$$T = \frac{1}{z - 0.2} = \frac{z^{-1}}{1 - 0.2z^{-1}}, \quad (15.10)$$

thus the unstable pole has been successfully allocated to the prescribed place inside the unit circle, by means of which the system is stabilized. The static transfer of the closed-loop is not unity, because the controller is proportional and not integrating. To get better control, it is reasonable to apply a further outer cascade loop, as was seen with state feedback control. ■

*Example 15.2* Let the controlled system be a first order ( $n = 1$ ), stable DT process

$$G(z^{-1}) = \frac{\mathcal{B}(z^{-1})}{\mathcal{A}(z^{-1})} = \frac{0.2z^{-1}}{1 - 0.8z^{-1}} = \frac{0.2}{z - 0.8} \quad (15.11)$$

and the goal is to make it faster. Assuming an *ODOF* system, our design goal is expressed by the reference model

$$R_r = R_n = \frac{0.8z^{-1}}{1 - 0.2z^{-1}} = \frac{0.8}{z - 0.2} \quad (15.12)$$

Now the YOUCLA-regulator is of integrating type, i.e.,

$$C_{\text{opt}} = C_{\text{id}} = \frac{R_n G_+^{-1}}{1 - R_n} = \frac{1}{1 - \frac{0.8z^{-1}}{1 - 0.2z^{-1}}} \frac{0.8z^{-1}}{1 - 0.2z^{-1}} \frac{1 - 0.8z^{-1}}{0.2z^{-1}} = 4 \frac{1 - 0.8z^{-1}}{1 - z^{-1}} \quad (15.13)$$

(because the zero of the denominator is  $z = 1$ ), and the transfer function of the closed system is

$$T = \frac{0.8z^{-1}}{1 - 0.2z^{-1}} = \frac{0.8}{z - 0.2} \quad (15.14)$$

whose static transfer is unity corresponding to a control of 1-type.

Based on (15.12), the characteristic equation for the design by *DE* is  $\mathcal{R}(z) = z - 0.2 = 0$ . Now the controller is also sought in the form of order  $n - 1 = 0$ , thus according to (15.8), proportional controller is applied. Equation (15.9) becomes

$$\begin{aligned} \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{Y} &= \mathcal{R} \\ (z - 0.8) + 0.2K &= z - 0.2 \end{aligned} \quad (15.15)$$

from which the controller is  $C = K = 3$ . It can be checked by simple computation that the overall transfer function of the closed-loop system is now

$$T = \frac{0.6}{z - 0.2} = \frac{0.6z^{-1}}{1 - 0.2z^{-1}} \quad (15.16)$$

The prescribed pole 0.2 is successfully allocated, but the loop is of 0-type, therefore the gain of  $T$  is 0.75. The above two examples represent well the practice, i.e., for stable systems the Y<sub>OU</sub>LA-parameterization has to be applied, while for stabilizing unstable systems, the application of  $DE$ , or the state feedback discussed in this chapter can provide the solution. ■

*Example 15.3* Let the controlled system be a first order ( $n = 1$ ), unstable, time-delay DT process

$$P(z^{-1}) = \frac{\mathcal{B}(z^{-1})}{\mathcal{A}(z^{-1})} = \frac{-0.2z^{-1}}{1 - 1.2z^{-1}} z^{-1} = \frac{-0.2z^{-2}}{1 - 1.2z^{-1}} = \frac{-0.2}{z(z - 1.2)} \quad (15.17)$$

whose pole  $p = 1.2$  is outside the unit circle. Observe that this formally corresponds to a second order process because of the time-delay. Therefore the stabilizing controller  $C = \mathcal{Y}/\mathcal{X}$  is sought in a first order form with three parameters

$$C = \frac{\mathcal{Y}}{\mathcal{X}} = \frac{y_0z + y_1}{z + x_1} = \frac{y_0 + y_1z^{-1}}{1 + x_1z^{-1}} \quad (15.18)$$

Because of realizability conditions it is reasonable to select a stable third degree characteristic polynomial  $\mathcal{R}(z) = z(z - 0.2)^2 = z(z^2 - 0.4z + 0.04)$  for the controller design. The number of unknown parameters is three and the relevant  $DE$  is

$$\begin{aligned} \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{Y} &= \mathcal{R} \\ (z^2 - 1.2z)(z + x_1) - 0.2(y_0z + y_1) &= z(z^2 - 0.4z + 0.04) \end{aligned} \quad (15.19)$$

and solving the equation for  $y_0$ ,  $y_1$  and  $x_1$  we get

$$C = \frac{-5z}{z + 0.8} = \frac{-5}{1 + 0.8z^{-1}} \quad (15.20)$$

It can be checked easily that the overall transfer function of the closed-loop system is

$$T = \frac{z}{z(z-0.2)^2} = \frac{z^{-1}}{(1-0.2z^{-1})^2} z^{-1} = \frac{z^{-1}}{1-0.4z^{-1}+0.04z^{-2}} z^{-1} \quad (15.21)$$

The prescribed double poles at 0.2 have been successfully allocated, but the control loop is of 0-type, thus the gain of  $T$  is 1.5625. Thus a controller having a relatively simple structure could solve a difficult problem, i.e., it can stabilize an unstable time-delay process. ■