Chapter 15
General Polynomial Method to Design Discrete Regulators

Polynomial design is an important method of regulator design. The main idea is that the transfer function of the closed loop control system is prescribed as the aim of the control, then the regulator is calculated using the knowledge of the process. The principle seems simple, nevertheless the calculation of the regulator necessitates the solution of a polynomial (Diophantine) equation, whose solvability sometimes requires the fulfilment of complicated conditions. Handling the unstable poles and inverse unstable zeros of the process requires further considerations. In Chap. 10, the design method was shown for continuous systems, and some examples demonstrated its application. For continuous systems the method can be applied only to systems without dead-time. For discrete (sampled) systems, polynomial design can be applied also to systems with dead-time.

Let the pulse transfer function of the process be

$$G\left(z^{-1}\right)=\frac{\mathcal{B}}{\mathcal{A}}\,z^{-d}=\frac{\mathcal{B}_{+}\mathcal{B}_{-}}{\mathcal{A}_{+}\mathcal{A}_{-}}\,z^{-d}=\left(\frac{\mathcal{B}_{+}}{\mathcal{A}_{+}}\right)\left(\frac{\mathcal{B}_{-}}{\mathcal{A}_{-}}\right)z^{-d}=G_{+}G_{-}\,z^{-d}.$$

Here $\mathcal{A}_{+}$ contains the stable, while $\mathcal{A}_{-}$ contains the unstable poles of the process. $\mathcal{B}_{+}$ contains the stable, compensable, while $\mathcal{B}_{-}$ contains the unstable, non compensable zeros.

The pulse transfer function of the regulator is sought in the following form:

$$C\left(z^{-1}\right)=\frac{\mathcal{Y}}{\mathcal{X}}=\frac{\mathcal{A}_{+}\,\mathcal{Y}_{\mathrm{d}}\,\mathcal{Y}'}{\mathcal{B}_{+}\,\mathcal{X}_{\mathrm{d}}\,\mathcal{X}'}$$

Here $\mathcal{Y}_{\mathrm{d}}$ and $\mathcal{X}_{\mathrm{d}}$ are given polynomials. In the design the polynomials $\mathcal{Y}'$ and $\mathcal{X}'$ have to be determined so as to ensure that the poles of the closed loop control system have the prescribed values.

L. Keviczky et al., *Control Engineering: MATLAB Exercises*,
Advanced Textbooks in Control and Signal Processing,
https://doi.org/10.1007/978-981-10-8321-1_15

The characteristic equation is

$$1 + CG = 1 + \frac{\mathcal{A}_+ \, \mathcal{Y}_\mathrm{d} \, \mathcal{Y}'}{\mathcal{B}_+ \, \mathcal{X}_\mathrm{d} \, \mathcal{X}'} \frac{\mathcal{B}_+ \mathcal{B}_-}{\mathcal{A}_+ \mathcal{A}_-} z^{-d} = 0$$

or

$$\mathcal{X}_\mathrm{d}\mathcal{X}'\mathcal{A}_- + \mathcal{Y}_\mathrm{d}\mathcal{Y}'\mathcal{B}_- z^{-d} = \mathcal{R} = 0.$$

The roots of the characteristic polynomial $\mathcal{R}$ are the prescribed values. For stable behaviour they have to be located inside the unit circle.

Choose the values $\mathcal{Y}_\mathrm{d} = 1$ and $\mathcal{X}_\mathrm{d} = 1$. Let us remark that if the polynomial $\mathcal{X}_\mathrm{d} = 1 - z^{-1}$ is chosen, then an integrator is introduced into the regulator.

Suppose that the degrees of the numerator and of the denominator of the regulator are the same. Let the degree of the denominator of the regulator be less by 1 than the degree of the denominator of the process. The numerator and denominator of the regulator are obtained by solving the DIOPHANTINE equation.

Let us consider the MATLAB™ simulation of some examples discussed in this chapter of the textbook [1].

Example 15.1 The pulse transfer function of the unstable process is: $G = \frac{\mathcal{B}}{\mathcal{A}} = \frac{-0.2}{z-1.2}$.

Find a regulator in the form $C = \mathcal{Y}/\mathcal{X}$ which stabilizes the process through prescribing the characteristic polynomial $\mathcal{R} = z - 0.2 = 0$. The regulator is supposed to be of order $n - 1 = 0$. $C = \mathcal{Y}/\mathcal{X} = K/1$. Solving the characteristic equation $\mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{Y} = \mathcal{R}$,

$$(z - 1.2) - 0.2K = z - 0.2.$$

The regulator is $C = K = -5$. (Let us remark that here $\mathcal{Y}_\mathrm{d} = 1$ and $\mathcal{X}_\mathrm{d} = 1$ were chosen.)

Simulate the behaviour of the closed loop with MATLAB™:

```
z=zpk('z');
Gz=-0.2/(z-1.2);
Cz=-5;
Lz=Gz*Cz;
Lz=minreal(Gz*Cz);
Tz=Lz/(1+Lz);Tz=minreal(Tz)
subplot(211),step(Tz,10); grid
subplot(212),
step(Cz/(1+Lz),10);grid
```

The step response of the controlled output signal and the regulator output are shown in Fig. 15.1. The closed loop is stable but there is a static error. This can be compensated by a prefilter, or by an extra integrator in the regulator design phase.

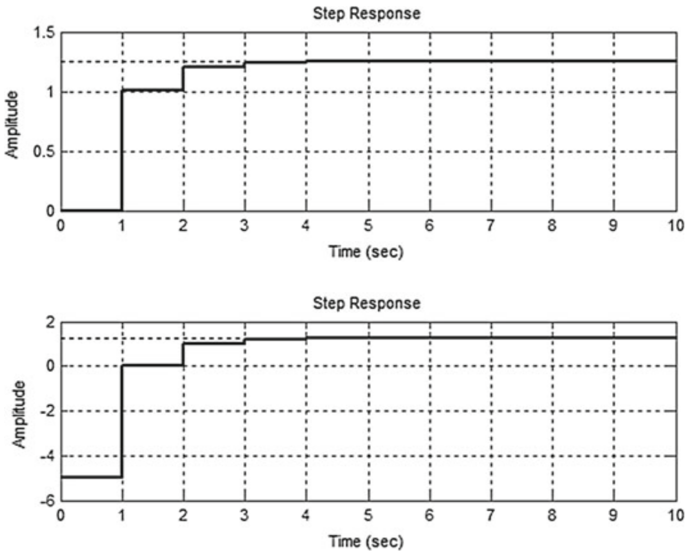

Fig. 15.1 Output and control signals for step reference input

In this system there are only discrete time pulse transfer functions, therefore the simulation can be executed without specifying the sampling time.

Example 15.2 (Example 15.3 in the textbook [1])

The pulse transfer function of the unstable process containing also dead-time is:

$$G(z^{-1}) = \frac{\mathcal{B}(z^{-1})}{\mathcal{A}(z^{-1})} = \frac{-0.2z^{-1}}{1 - 1.2z^{-1}} z^{-1} = \frac{-0.2}{z(z - 1.2)}$$

Find a stabilizing regulator $C = \mathcal{Y}/\mathcal{X}$ prescribing the characteristic polynomial $\mathcal{R}(z) = (1 - 0.2z^{-1})^2$. Formally the process is of second order, therefore a characteristic polynomial of second degree is chosen. The regulator is chosen to be of first degree.

$$C = \frac{\mathcal{Y}}{\mathcal{X}} = \frac{y_o}{1 + x_o z^{-1}} = \frac{y_o z}{z + x_o}.$$

The Diophantine equation is

$$\left(1 - 1.2z^{-1}\right)\left(1 + x_o z^{-1}\right) - 0.2 y_o z^{-2} = \left(1 - 0.2z^{-1}\right)^2.$$

Its solution yields $y_o = -5$ and $x_o = 0.8$. So the regulator is

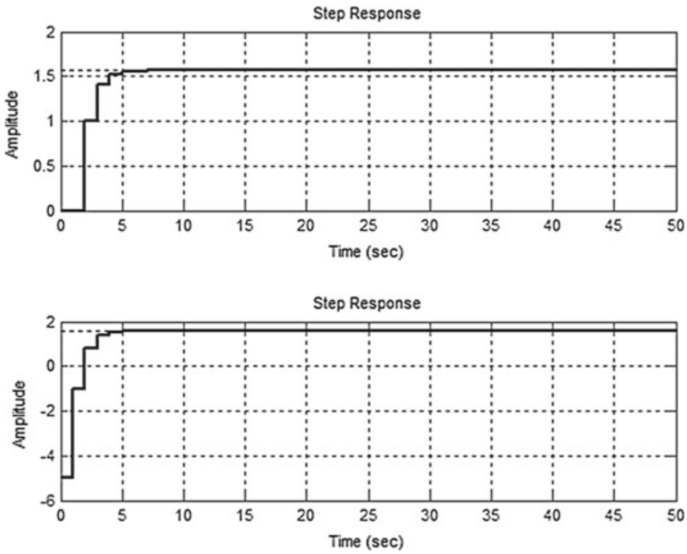

Fig. 15.2 Output and control signals for step reference input

$$C(z^{-1}) = \frac{-5}{1+0.8z^{-1}} = \frac{-5z}{z+0.8}.$$

The MATLAB™ simulation results from the following code:

```
z=zpk('z');
Gz=-0.2/z/(z-1.2);
Cz=-5*z/(z+0.8)
Lz=Gz*Cz; Lz=minreal(Lz);
Tz=Lz/(1+Lz);Tz=minreal(Tz);
subplot(211),step(Tz,50),grid
Uz=Cz/(1+Lz)
subplot(212),step(Uz,50),grid
```

Figure 15.2 shows the output and the control signals for a unit step reference signal. The control system is stable, but there is a static error.

Problem Introduce an integrator in the regulator. Write the Diophantine equation and determine the parameters and the pulse transfer function of the regulator.