# Chapter 8
# *PID* Regulator Design

## 8.1 Characteristics of *PID* Elements

### *8.1.1 Characteristics of the PI Element*

The transfer function of an ideal *PI* element is $C(s) = A_P \frac{1+sT_I}{sT_I} = A_P\left(1 + \frac{1}{sT_I}\right)$. Plot its step response in case of $T_I = 10$ and $A_P = 2$.

```
s=tf('s')
C=2*(1+s*10)/(s*10);
step(C),grid
```

It can be seen that the initial jump of the curve is $A_P = 2$ (which can be calculated by limit values for $t \to 0$ or $s \to \infty$). The slope of the curve depends on the time constant $T_I = 10$ (Fig. 8.1).

Let us draw the Bode diagram of the element (Fig. 8.2).

```
bode(C),grid
```

In the low frequency range the Bode amplitude diagram can be approximated by a straight line of slope −20 dB/decade, then from frequency $1/T_I = 0.1$ with a horizontal straight line (Fig. 8.2).



L. Keviczky et al., *Control Engineering: MATLAB Exercises*,
Advanced Textbooks in Control and Signal Processing,
https://doi.org/10.1007/978-981-10-8321-1_8

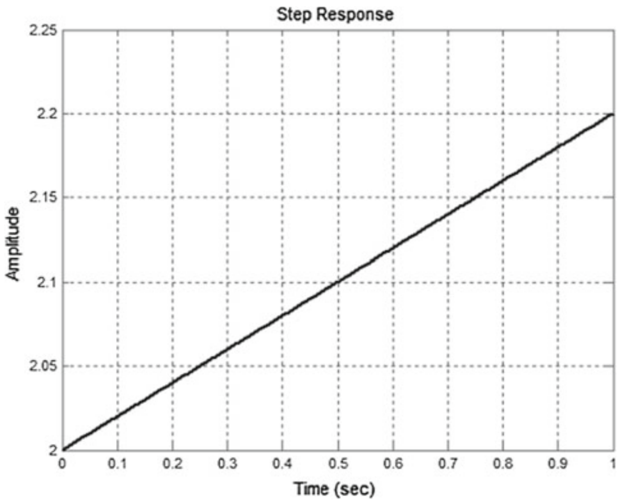



**Fig. 8.1** Step response of a *PI* element

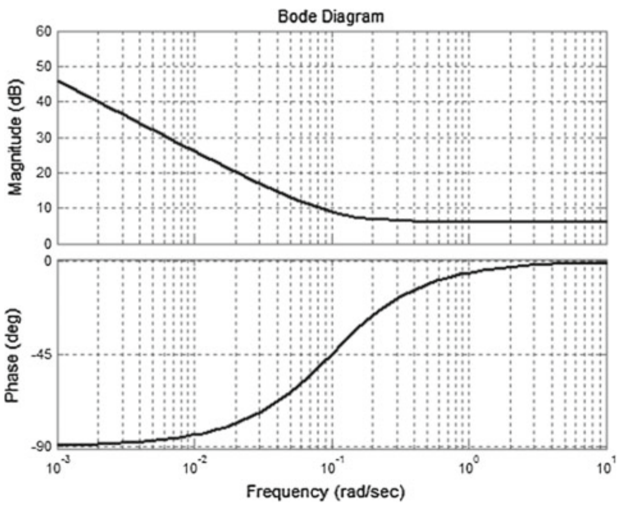



**Fig. 8.2** Bode diagram of a *PI* element

### 8.1.2 Characteristics of the PD Element

The transfer function of the ideal *PD* element is $C(s) = 1 + sT_D$. Plot its step response in case of $T_D = 10$;

```
s=tf('s')
C=1+s*10;
step(C)
```

MATLAB™ respond with an error message, as the regulator is non-realizable: the degree of its numerator is higher than the degree of its denominator. Its step response is Dirac delta, which can not be handled numerically.

```
??? Error using ==> rfinputs
Not supported for non-proper models.
```

The transfer function of a non ideal *PD* element is

$$C(s) = A_P\left(1 + \frac{s\tau}{1 + sT}\right) = A_P\left(\frac{1 + sT_D}{1 + sT}\right),\ T_D = T + \tau > T.$$

Plot its step response and Bode diagram.

```
C=(1+s*10)/(1+2*s)
figure(1),step(C)
figure(2),bode(C)
```

It can be seen that the step response in point $t = 0$ starts at 5 and in steady state approximates 1 (Fig. 8.3). The ratio of the initial and final values gives the overexcitation, which ensures acceleration in the control system. The Bode diagram is shown in Fig. 8.4. Between the frequencies 0.1 and 0.5 the amplitude diagram can be approximated by a straight line of slope +20 dB/decade. This regulator is also called phase lead regulator as its phase angle is positive over the whole frequency range.

### 8.1.3 Characteristics of the PID Element

A *PID* regulator can be built of a proportional element (*P*), an integrating element (*I*) and a differentiating element (*D*). For regulator design in the frequency domain

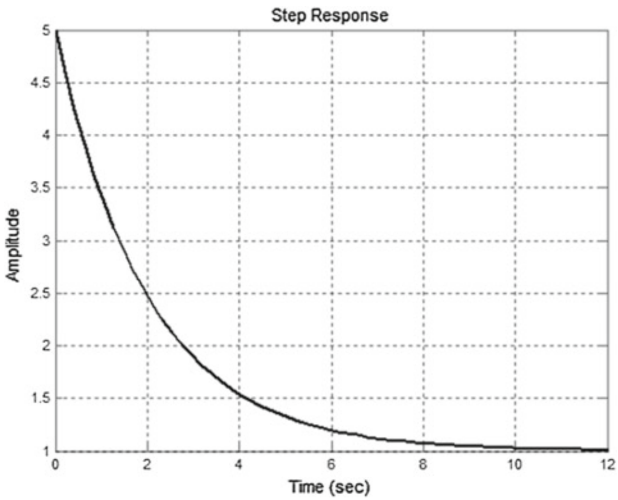



**Fig. 8.3** Step response of a *PD* element

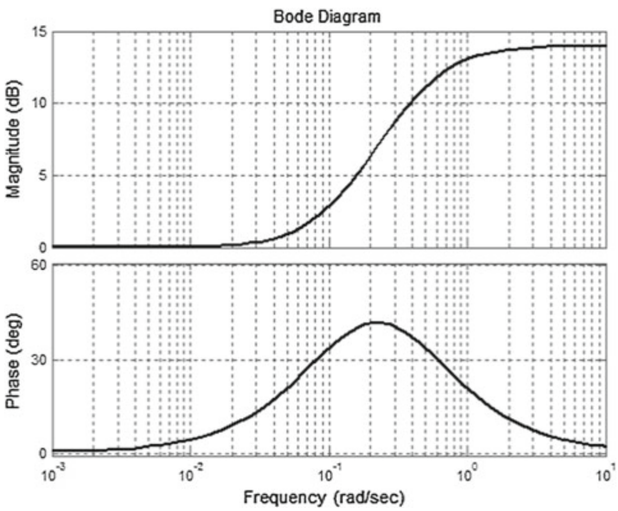



**Fig. 8.4** Bode diagram of a *PD* element

the series form is more advantageous: the transfer function of the regulator is approximated by serially connected *PI* and *PD* elements (see formula (8.8) in the textbook [1]).

$$C(s) = A_{\mathrm{P}} \frac{sT_{\mathrm{I}} + 1}{sT_{\mathrm{I}}} \frac{sT_{\mathrm{D}} + 1}{sT + 1}$$

Plot the step response and the Bode diagram of the element.

```
C=(10*s+1)/(10*s)*(s+1)/(0.5*s+1)
t=0:0.1:15;
step(C,t),grid
bode(C),grid
```

In the step response (Fig. 8.5), at the beginning the differentiating effect that is responsible for acceleration is dominant, whereas for later times the integrating effect is dominant. The initial low frequency part of the Bode amplitude diagram (Fig. 8.6) can be approximated by an asymptote of slope −20 dB/decade, and the middle frequency part by an asymptote of slope +20 dB/decade.

## 8.2 Design of a *PID* Regulator

Consider the closed loop control system shown in Fig. 8.7, with $P(s)$ being the transfer function of the process (plant) to be controlled, and $C(s)$ the transfer function of the regulator.

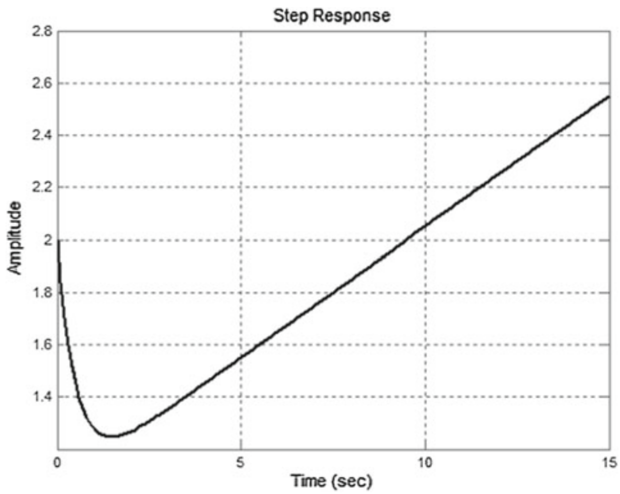



**Fig. 8.5** Step response of a *PID* element

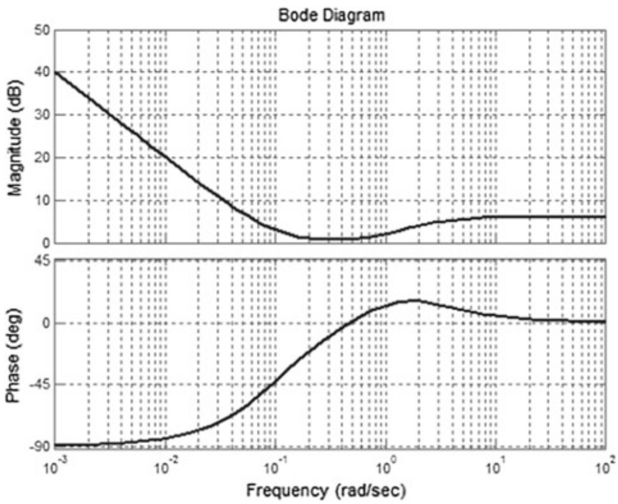
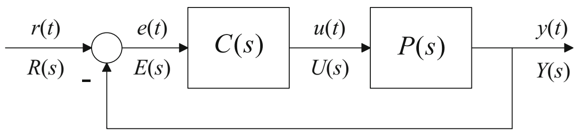



**Fig. 8.6** Bode diagram of a *PID* element

**Fig. 8.7** Control system




For the given process, a series regulator is to be designed which ensures the fulfilment of the quality specifications.

## *8.2.1 Design Considerations*

The following requirements are set for a closed loop control system:

- Stability;
- Static response (reference signal tracking, disturbance rejection);
- Transient response (overshoot in the output signal, settling time);
- Robustness;
- Limitation of the control signal.

Quality specifications (requirements) can be formulated in the frequency domain as well (see also Chap. 6). The overshoot of the step response of the closed loop is related to the maximum amplification of the amplitude-frequency function of the closed loop, and also to the phase margin calculated from the frequency function of the open loop. If the phase margin is about 60°, the overshoot will be about 10%. The settling time depends on the cut-off frequency, and can be approximated by $\frac{3}{\omega_c} \le t_s \le \frac{10}{\omega_c}$.

In practice the manipulated variable, the control signal, is restricted: its value should not exceed a given limit.

The structure and the parameters of the regulator have to be chosen considering the design specifications.

In more complex control problems several restrictions can be imposed on the output and control signals. For example, the integral of the quadratic error and also that of the control signal have to be minimized. The restrictions can be non-linear and also can be contradictory. In general, the parameters of the regulator can be determined by optimization procedures.

In the sequel a simple practical method will be presented for regulator design (also called *compensation*), when the phase margin is prescribed. The design is based on the frequency function of the open loop, and from the properties of the open loop consequences are drawn for the behaviour of the closed loop.

When the requirement is to track the step reference signal accurately, without steady state error, a *PI* regulator is employed. This requirement can be fulfilled by using an integrating effect in the open loop. With a *PD* regulator, the control system can be accelerated. If both the accuracy and the settling time should be improved, a *PID* regulator is employed. The *PD* element causes a significant increase of the initial value of the control signal, which is responsible for the acceleration.

The regulator is designed considering the model of the process and the quality specifications. A usual technique is pole cancellation, when the zeros of the transfer function of the regulator cancel the unfavourable poles of the process, and so a desired dynamics is ensured in the closed loop control circuit. For the pole cancellation technique, it is advantageous to give the transfer function of the regulator in product form.

In a *PID* regulator there are 4 parameters: $k_c = A_P$, $T_I$, $T_D$, $T$. When designing a regulator with the pole cancellation technique, these parameters are chosen as follows: $T_I$ should be equal to the biggest time constant (the pole with the lowest frequency), and $T_D$ should be equal to the second biggest time constant. Thus the zeros introduced by the regulator cancel the poles of the process. The parameter $T$ is given in the form $T = \frac{T_D}{n_p}$, where $n_p$ is the ratio of pole replacement, which indicates how far away the *PD* element pushes the compensated pole of the system. A good experimental rule is to choose the value of $n_p$ in the range 2–10. If it is higher, the control system will be faster, but at the cost of a higher maximum of the control signal. As the value of $k_c$ does not influence the phase of the open loop frequency function, this parameter can be used to set the value of the prescribed phase margin.

The steps of regulator design with *P*, *PI*, *PD* and *PID* regulators are shown for compensating the process given by the transfer function $P(s) = 1/[(1 + 10s)\,(1 + s)\,(1 + 0.2s)]$.

Design series *P*, *PI*, *PD* and *PID* regulators to ensure a phase margin of about 60°. Give the quality characteristics of the compensated system. Calculate and draw the output and the control signals for a unit step reference signal.

## 8.2.2 Design of a P Regulator

The regulator is given as $C(s) = k_c$. So only the value of $k_c$ has to be determined.

Give the transfer function of the process.

```
s=zpk('s')
P=1/((1+10*s)*(1+s)*(1+0.2*s))
```

First let $k_c = 1$.

```
kc=1
C=kc
L=C*P
```

Draw the Bode diagram of the open loop and determine its characteristic values (phase margin, gain margin, and cut-off frequency) using the command margin.

```
margin(L)
```

The system has significant phase and gain margins. The phase angle decreases monotonically from zero to $-270°$, so by changing the gain the required phase margin could be set. The gain of the regulator will be the reciprocal of the gain belonging to the phase value $\varphi = \varphi_t - 180°$. This can be read off the Bode diagram of the open loop, or calculated from a table containing the corresponding frequency, gain and phase values, or found by using command margin.

As the calculation of the gain margin $g_t$ is similar to the calculation of $k_c$ (e.g. if $\varphi_t = 60°$).

$$k_c = 1 \Big/ |L(j\omega)|_{\varphi=-120°} \quad \text{and} \quad g_t = 1 \Big/ |L(j\omega)|_{\varphi=-180°},$$

therefor $k_c$ can also be found using the command margin. The input parameter of the command margin can be a transfer function given in *LTI sys* structure, or the gain, phase and frequency vector calculated by the command bode. The command margin calculates the gain margin from the gain belonging to the phase value $-180°$. If the phase angles are decreased by the value of the required phase margin, then margin will calculate $g_t$ as the reciprocal of the gain belonging to the given phase margin.

```
[mag,phase,w]=bode(L);
gt=margin(mag,phase-60,w)
kc=gt
```

So the regulator is $C_P(s) = k_c = 7.51$. The parameter $k_c$ can also be calculated from the table below containing the corresponding data of the frequency function.

```
Table=[mag(:), phase(:), w]
      mag          phase          w
    0.2756       -95.0730       0.3290
    0.1960      -107.0164       0.4520
    0.1340      -119.7735       0.6210
    0.0873      -133.4679       0.8532
```

The parameter $k_c$ is calculated as the reciprocal of the gain corresponding to the phase angle $-120°$, and the corresponding frequency is the cut-off frequency: $k_c = 1/0.134 = 7.4627$, $\omega_c = 0.621$.

Refining the resolution of the frequency vector w, the two methods give the same result.

Similarly to the method using the table above, $k_c$ can be found also from the BODE diagram.

```
bode(L)
```

Change the scale of the amplitude curve from decibels to *absolute*: clicking on the white background of the BODE amplitude diagram, choose with the right mouse button *Properties –> Units –> Magnitude in—absolute*. Find phase angle $-120°$ in the phase diagram then read off the gain belonging to this frequency from the BODE amplitude diagram. The reciprocal of this gain will be $k_c$, i.e. $k_c = \frac{1}{|L(j\omega)|_{\varphi=-120°}} = \frac{1}{0.134} = 7.46$.

Check the behaviour of the system.

```
C=kc
L=kc*L
```

Check the parameters characterizing the stability margins (the gain margin and phase margin).

```
margin(L)
[gt,pm,wg,wc]=margin(L)
```

The phase margin is indeed 60°. The cut-off frequency is $\omega_c = 0.6245$.

Calculate the resulting transfer function of the closed loop system.

```
T=L/(L+1)
```

The calculations may result in coinciding zero-pole pairs, which can be cancelled using the command minreal.

```
T=minreal(T)
```

Or, in one step,

```
T=minreal(L/(1+L))
```

The same result is obtained using the command feedback.

```
T=feedback(L,1)
```

Plot the BODE diagrams of the open and the closed loop on one diagram.

```
bode(L,'r',T,'b')
```

It can be seen that in the low frequency range the amplitude diagram of the closed loop is approximately 1, and in the high frequency range the two curves are approximately the same.

Plot the step response of the closed loop.

```
step(T)
```

Calculate its values.

```
t=0:0.05:10;
y=step(T,t);
```

The maximum value of the step response:

```
ym=max(y)
```

The steady state value of the step response:

```
ys=dcgain(T)
```

From these values the overshoot can be calculated as

```
yt=(ym-ys)/ys
```

The error in steady state:

```
es=1-ys
```

Let us analyse the behaviour of the control signal $u(t)$. This is important because this is the input of the process, and it is not allowed to exceed the given limits. Let us calculate the resulting transfer function between the control signal and the reference signal.

```
U=minreal(C/(1+L))
```

or

```
U=feedback(C,P)
```

For a step reference signal:

```
ut=step(U,t);
plot(t,ut)
```

The constraint is generally imposed as the maximum value of the control signal.

```
um=max(ut)
```

### *8.2.3 Design of P, PI, PD and PID Regulators*

*PI*, *PD* and *PID* regulators can be designed similarly. Let us apply the pole cancellation technique. The table below summarizes the structure of the regulators and the characteristic values of the control systems with the different regulators. Let us generate a new *m*-file for regulator design. The program calculates the gain $k_c$ of that regulator which ensures a phase margin of 60°, then evaluates the characteristic parameters of the control system. In the table yt denotes the overshoot of the output signal, es is the value of the static error for unit step reference signal, um denotes the maximum value of the control signal, and ts gives the settling time.

To write a MATLAB™ program, let us generate a new *m*-file. This text file can be opened in the file menu of MATLAB™ with the extension ".*m*". Write the MATLAB™ commands into the empty file. Then save it: *Save As, C:/Matlab/work/myfile.m*.

To call the program simply write the name of the program, without its extension, in the command window of MATLAB™.

```
myfile
```

The following MATLAB™ program realizes the regulator design.

```
clear; s=zpk('s');
P=1/((1+10*s)*(1+s)*(1+0.2*s))
Cp=1; Cpi=(1+10*s)/(10*s)
Cpd=(1+s)/(1+0.2*s)
Cpid=Cpi*Cpd
[mag,phase,w]=bode(Cp*P);
kp=margin(mag,phase-60,w)
Cp=kp*Cp;

[mag,phase,w]=bode(Cpi*P);
kpi=margin(mag,phase-60,w);
Cpi=kpi*Cpi;

[mag,phase,w]=bode(Cpd*P);
kpd=margin(mag,phase-60,w);
Cpd=kpd*Cpd;

[mag,phase,w]=bode(Cpid*P);
kpid=margin(mag,phase-60,w);
Cpid=kpid*Cpid;

Lp=Cp*P;
Lpi=minreal(Cpi*P,0.0001)
Lpd=minreal(Cpd*P,0.0001)
Lpid=minreal(Cpid*P,0.0001)
%Resulting transfer functions:
Tp=Lp/(1+Lp);
% or Tp=feedback(Lp,1);
Tpi=Lpi/(1+Lpi);
Tpd=Lpd/(1+Lpd);
Tpid=Lpid/(1+Lpid);
%Transfer functions of U(s):
Up=Cp/(1+Lp);
Upi=Cpi/(1+Lpi);
Upd=Cpd/(1+Lpi);
Upid=Cpid/(1+Lpid);
t=0:0.05:10;

figure(1),step(Tp,'r',Tpi,'b'
,Tpd,'g',Tpid,'m',t)
figure(2),step(Up,'r',Upi,'b'
,t)
figure(3),step(Upd,'g',Upid,'m'
,t)

yp=step(Tp,t);
ypi=step(Tpi,t);
ypd=step(Tpd,t);
ypid=step(Tpid,t);

ysp=dcgain(Tp)
yspi=dcgain(Tpi)
yspd=dcgain(Tpd)
yspid=dcgain(Tpid)
ep=1-ysp
epi=1-yspi
epd=1-yspd
epid=1-yspid

ytp=(max(yp)-ysp)/ysp
ytpi=(max(ypi)-yspi)/yspi
ytpd=(max(ypd)-yspd)/yspd
ytpid=(max(ypid)-yspid)/yspid
up=step(Up,t);
upi=step(Upi,t);
upd=step(Upd,t);
upid=step(Upid,t);
upim=max(upi)
updm=max(upd)
upidm=max(upid)
```

The step responses are shown in Fig. 8.8. The control signals for the *P* and *PI* controls are given in Fig. 8.9. The control signals for the *PD* and *PID* controls are shown in Fig. 8.10.

The following table summarizes the structure and the parameters of different regulators given in the MATLAB™ example, and presents the characteristic values of the closed system.

| | $C(s)$ | $L(s) = C(s)P(s)$ | $k_c$ | $\omega_c$ | yt | es | um | ~ts |
|---|---|---|---|---|---|---|---|---|
| | No control case | $\frac{1}{(1+10s)(1+s)(1+0.2s)}$ | | 0 | 0 | 0.5 | 2 | 12 |
| *P* | $k_c$ | $\frac{k_c}{(1+10s)(1+s)(1+0.2s)}$ | 7.51 | 0.62 | 0.153 | 0.117 | 7.5 | 8 |
| *PI* | $k_c\frac{1+10s}{10s}$ | $\frac{k_c}{s(1+s)(1+0.2s)}$ | 5.04 | 0.46 | 0.078 | 0 | 5.16 | 9 |
| *PD* | $k_c\frac{1+s}{1+0.2s}$ | $\frac{k_c}{(1+10s)(1+0.2s)^2}$ | 16.5 | 1.51 | 0.103 | 0.057 | 82.7 | 2 |
| *PID* | $k_c\frac{(1+10s)(1+s)}{10s(1+0.2s)}$ | $\frac{k_c}{s(1+0.2s)^2}$ | 14.3 | 1.33 | 0.076 | 0 | 71.3 | 2 |

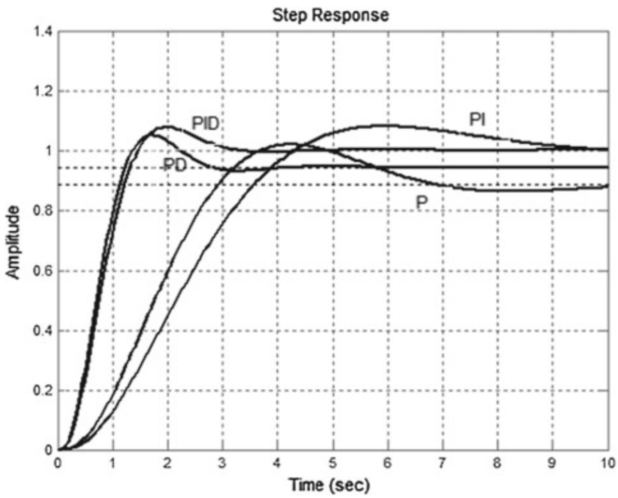



**Fig. 8.8** Step responses of the control system in case of *P*, *PI*, *PD* and *PID* controllers

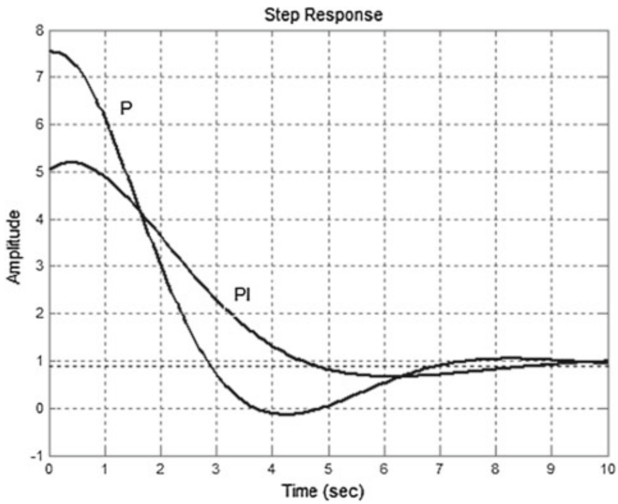



**Fig. 8.9** The control signals in case of *P* and *PI* controllers

The requirements imposed on the control system are a fast settling process and good reference signal tracking. In Fig. 8.8 it can be seen that *P* compensation does not fulfill these conditions. The settling is slow and the output signal does not reach

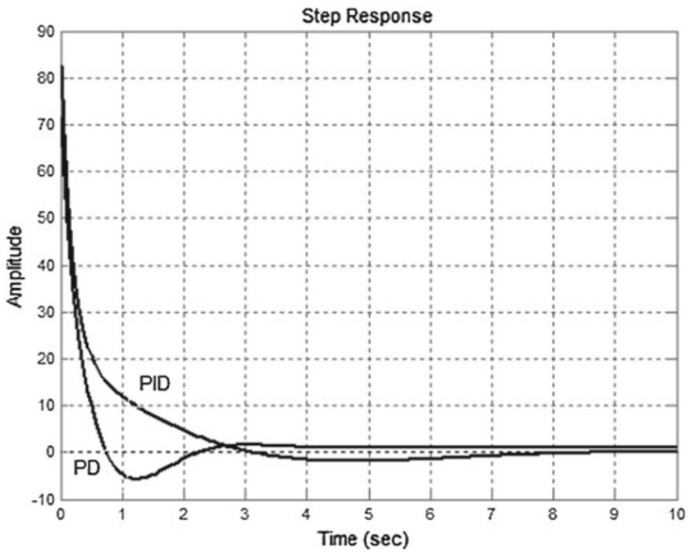



**Fig. 8.10** The control signals in case of *PD* and *PID* controllers

the required value $y(\infty) = 1$. With *PI* compensation the static error has been decreased to zero, but the settling process is slow. The *PD* compensation accelerates the control system, but there is a static error. The reason for this acceleration (Fig. 8.10) is the significant increase of the control signal $u(t)$. With the *PID* regulator the system became fast and the static error is zero.

The behaviour of the control system can also be analysed by building a SIMULINK™ block diagram and running it with the given process and with the designed regulators.

### *8.2.4 Regulator Design for a Second-Order Oscillating Element*

The process is given by the following transfer function:

$$P(s) = \frac{A'}{(s - p_1)(s - p_2)} = \frac{A}{s^2 T_o^2 + 2\xi T_o s + 1}, \text{ where } p_{1,2} = a \pm jb$$

The poles are complex conjugates. The breakpoint frequency of the Bode diagram is $\omega_o = 1/T_o$. Here the slope of the approximate Bode amplitude diagram changes from 0 to −40 dB/decade. In this case a possible *PID* pole cancellation technique can be, if the time constants of both the *PI* and the *PD* elements are chosen to be the reciprocals of the natural frequency, i.e. $T_I = T_o$ and $T_D = T_o$, so

$$C(s) = k_c \frac{1 + T_o s}{s} \frac{1 + T_o s}{1 + T_1 s}.$$

Another possibility is to employ a pure integrating element as a regulator, whose gain is set to ensure a phase margin of about 60°. Let us remark that for a small damping factor $\xi$, the prescribed phase margin alone will not always ensure the appropriate transient response. It is necessary to arrange that in the vicinity of the cut-off frequency the BODE amplitude diagram does not move close to the 0 dB axis.

### 8.2.5 Applying Experimental Tuning Rules

Besides the discussed regulator design methods, there are several practical regulator tuning methods. The most frequently used methods which give rules of thumb for the parameter tuning of the *PID* regulator based on the model of the process are:

- The ZIEGLER-NICHOLS rules
- The OPPELT method
- The CHIEN-HRONES-RESWICK method
- The STREJC method
- The ÅSTRÖM relay method
- The ÅSTRÖM-HÄGGLUND method.

The rules of thumb are given in the textbook [1].

## 8.3 *PID* Regulator Design for a Dead-Time System

Compensating a system containing dead-time is more complicated than for a system without dead-time, as the transfer function of the dead-time element can not be represented accurately by a rational function. The phase shift caused by the dead-time has to be taken into consideration in regulator design.

Let us consider the control system given in Fig. 8.11.

Here $P(s) = P_+(s)e^{-sT_d}$, where $P_+(s)$ is the transfer function of the process without the dead-time and $T_d$ denotes the dead-time, $C(s)$ is the transfer function of the regulator, and $L(s) = C(s)\,P(s)$ is the loop transfer function.

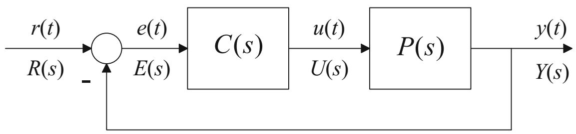



**Fig. 8.11** Control system

Consider the following example:

$$P(s) = P_+(s)\,\mathrm{e}^{-sT_\mathrm{d}} = \frac{\mathrm{e}^{-s}}{1+20s}.$$

The transfer function of the regulator $C(s)$ has to be chosen to ensure the fulfilment of the quality specifications.

Prescriptions: For a step reference signal, the static error for reference signal tracking should be zero, and the overshoot of the output signal should be below 10%. These requirements can be ensured using a *PI* regulator: $C(s) = k_\mathrm{c}(1+20s)/s$.

The loop transfer function is

$$L(s) = C(s)P(s) = k_\mathrm{c}\frac{1+20s}{s}\frac{\mathrm{e}^{-s}}{1+20s} = k_\mathrm{c}\frac{\mathrm{e}^{-s}}{s}.$$

The constant $k_\mathrm{c}$ is chosen to ensure a phase margin of about 60°.

The amplitude of the frequency function of the process is calculated from its part without the dead-time:

$|P(j\omega)| = |P_+(j\omega)\mathrm{e}^{-j\omega T_\mathrm{d}}| = |P_+(j\omega)|$, as $|e^{-j\omega T_\mathrm{d}}| = 1$

The phase angle is

$$\arg\{P(j\omega)\} = \arg\{P_+(j\omega)\} + \arg\left\{\mathrm{e}^{-j\omega T_\mathrm{d}}\right\} = \arg\{P_+(j\omega)\} - \omega T_\mathrm{d}$$

Regulator design in the MATLAB™ environment can be executed in two ways. In the first way, in the frequency domain the phase angle of the process without the dead-time is calculated, and then it is modified by $-\omega T_\mathrm{d}$, the phase angle of the dead-time element. The disadvantage of this method is that the simulation can not be done in the MATLAB™ environment alone, analysis in SIMULINK™ is also required. In the second way the dead-time is approximated by a rational function. For the approximate process the regulator design can be executed according to the method applied for rational functions (see Sect. 8.2 in textbook [1]). With this method, the behaviour of the system can be analysed in the MATLAB™ environment.

Let us emphasize that the first method is preferred.

### *8.3.1 Regulator Design for a Dead-Time System Considering the Phase Shift*

Let us design a *PI* regulator for the process given in Sect. 8.3. Write $P_+(\mathrm{s})$ = P.

```
s=zpk('s')
P=1/(1+20*s)
Td=1
```

```
kc=1
C=kc*(1+20*s)/s
```

The transfer function of the open loop is

```
L=C*P,L=minreal(L)
```

The amplitude-frequency function of the open loop coincides with the amplitude of the system without dead-time. Its phase angle is modified by a linear term. To execute the calculations a frequency vector w has to be defined, which contains the cut-off frequency. In many cases the command bode itself calculates the frequency vector.

```
[mag,phase,w]=bode(L);
```

If this frequency vector is not satisfactory, the user has to define it so that it ensures a frequency range which is wide enough. This can be done using the knowledge of the process or by trial and error.

```
w=logspace(-1,1,100)'
[mag,phase]=bode(L,w);
magd=mag(:);
phased=phase(:)-w*Td*180/pi;
```

The gain $k_c$ can be calculated in two ways.

1. *Method 1*: with the command margin.

   The gain margin for a phase angle −120° is found by

```
gm=margin(magd,phased-60,w)
```

This will be the value of gain a $k_c$:

```
kc=gm
      0.5235
```

2. *Method 2*: using a table.

```
Tabl=[phased, magd, w]
    -116.5943   2.1544   0.4642
    -117.8607   2.0565   0.4863
    -119.1873   1.9630   0.5094
    >> -120.5770   1.8738   0. 5337  <<
    -122.0330   1.7886   0.5591
    -123.5583   1.7074   0.5857
```

The value of magd at phased $= -120$ is 1.8738.
Hence $k_c = 1/1.8738$ and $\omega_c = 0.5337$.

```
kc=1/1.8738
        0.5337
```

Let us calculate the regulator transfer function again:

```
C=kc*(1+20*s)/s
L=C*P,L=minreal(L)
```

Check the phase margin.

```
[mag,phase]=bode(L,w);
magd=mag(:);
phased=phase(:)-w*Td*180/pi;
margin(magd,phased,w)
```

Figure 8.12 shows the Bode diagram of the open loop and the phase margin. In this simple case, the calculation can also be executed analytically. The transfer function of the open loop is $1/s$, its phase angle is $-90°$ over the whole frequency range. At the prescribed phase margin of 60°, the phase angle is $-120°$. That means that the dead-time can add $-30°$, i.e. $-\omega_c T_d = -\pi/6$. Hence $\omega_c = \pi/6 = 0.5236$. From the condition $1 = |k_c e^{-j\omega}/j\omega|$ the gain is calculated as $k_c = \omega_c = 0.5236$, which is close to the result obtained previously from the table.

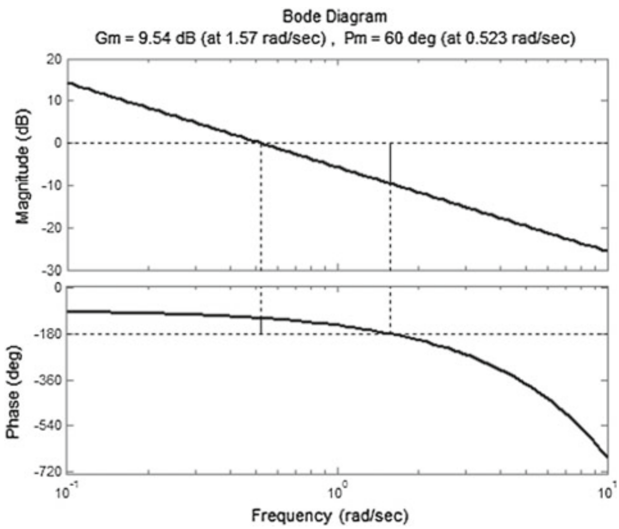



**Fig. 8.12** Bode diagram of the open loop

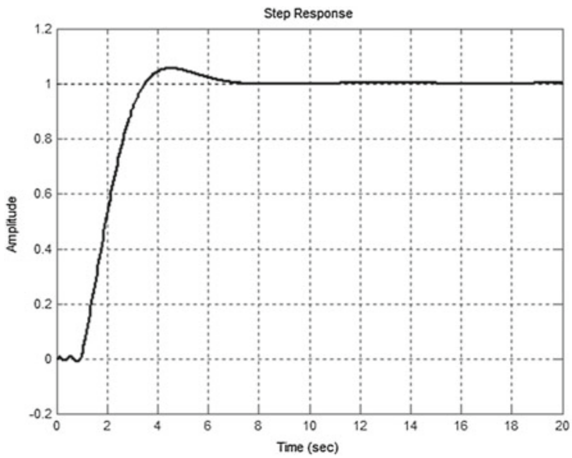



**Fig. 8.13** Step response of the control system

The behaviour of the system can be analysed with the SIMULINK™ model shown in Fig. 8.14. With SIMULINK™, the dead-time can be simulated easily with the *Transport Delay* block. Set its parameter to the value Td, and the transfer function of the process to P. Figure 8.13 shows the step response of the control system.

The result of the simulation can be sent to the MATLAB™ surface for further analysis and graphical representation. This can be done in two ways: with the *To Workspace* block, or with the *Scope* block. In the *To Workspace* block the name of the variable that will be used in MATLAB™ has to be set, its type has to be given as *Matrix*. Then the signals can be plotted using MATLAB™:

```
plot(t,y),grid
```

In MATLAB™ the results of the simulation can also be analysed and plotted from *Scope* blocks. Set the parameters of the graphical window of *Scope* as follows:

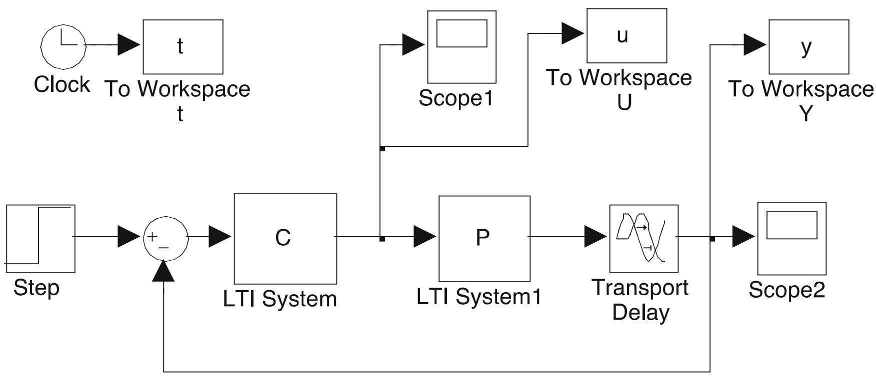



**Fig. 8.14** SIMULINK™ model of a control system with dead time in the process

Under the *properties* menu,

*Data history*: *Save data to workspace*
*Variable name*: ty (tu in the case of the control signal)
*Matrix format*

So the vectors of the time t and the output y can be obtained easily after the simulation. Then the quality characteristics (overshoot, settling time, maximum value of the control signal) can be determined.

```
t=ty(:,1)
y=ty(:,2)
plot(t,y),grid
```

### 8.3.2 Regulator Design for a Dead-Time System Using PADE Approximation

The transfer function of a dead-time element can be approximated by the PADE rational function, $P_{\mathrm{Pade}}(s) \cong e^{-sT_{\mathrm{d}}}$. (The first few terms of the TAYLOR series of the PADE rational function are the same as those of the transfer function of the dead-time element.)

$$P(\mathrm{s}) = P_{+}(\mathrm{s})\, e^{-sT_{\mathrm{d}}} \cong P_{+}(\mathrm{s})\, P_{\mathrm{Pade}}(s)$$

In MATLAB™, the command pade calculates the approximation for the given degree. For example, for the 5th degree:

```
s=zpk('s')
P=1/(1+20*s)
Td=1
kc=1
C=kc*(1+20*s)/s
[numpade,denpade]=pade(Td,5)
Ppade=tf(numpade,denpade)
Ppade=zpk(Ppade)
Pd=P*Ppade
```

From this point on, the steps of the regulator design are the same as for a system without dead-time.

```
L=C*Pd,L=minreal(L)
```

The gain $k_c$ is calculated by

```
[mag,phase,w]=bode(L);
kc= margin(mag,phase-60,w)
        0.5212
C=kc*(1+20*s)/s; L=kc*L;
T=L/(1+L); T=minreal(T)
```

The settling time can be evaluated with the command step. The step response can be compared with the result obtained with SIMULINK™ (Fig. 8.15).

```
step(T,20),grid
```

It can be seen that the step responses obtained by the two methods are approximately the same.

It has to be emphasized that the method given in Sect. 8.3.1 is more accurate than the method using the Pade approximation. With the Pade approximation, the transfer functions of the open and the closed loop are complicated because of the high degree approximation. The advantage of Pade approximation is that the design method is the same as for systems without dead-time.

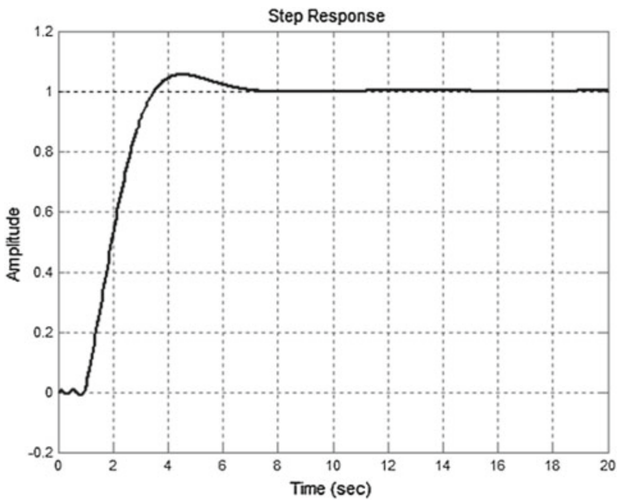



**Fig. 8.15** Step response calculated with Pade approximation of the dead time

## 8.4 Control of an Unstable System

### *8.4.1 Control of an Unstable System with a P Regulator*

An unstable process is given by the transfer function $P(s) = 20/[(s+2)(s-5)]$. Let us analyse whether the process can be stabilized with a proportional regulator $C(s) = k_c$ or not.

```
s=zpk('s')
P=20/((s+2)*(s-5))
figure(1); grid on; nyquist(P);
figure(2); grid on; bode(P);
figure(3);rlocus(P);
```

The transfer function of the process has a pole in the right half-plane. According to the general NYQUIST stability criterion the control system can be stabilized with the given regulator, if the NYQUIST diagram of the open loop encircles $-1+j0$ counter-clockwise as many times as the number of the poles of the process in the right half-plane. Figure 8.16 shows that the control system can not be stabilized, as the NYQUIST diagram encircles $-1+j0$ clockwise. The BODE diagram shows that the phase margin is negative. The same result is obtained by calculating the roots of the characteristic equation $s^2 - 3s - 10 + k_c = 0$. The necessary condition of stability is that the coefficients are of the same sign, which is not fulfilled. The root locus $(0 < k_c < \infty)$ gives the same result. At each value of the gain, at least one root of the characteristic equation lies in the right half-plane.

Let us consider the process given by $P(s) = 5/[(s-2)(s+5)]$. Can this process be stabilized with a proportional $C(s) = k_c$ regulator?

```
clear
s=zpk('s')
```

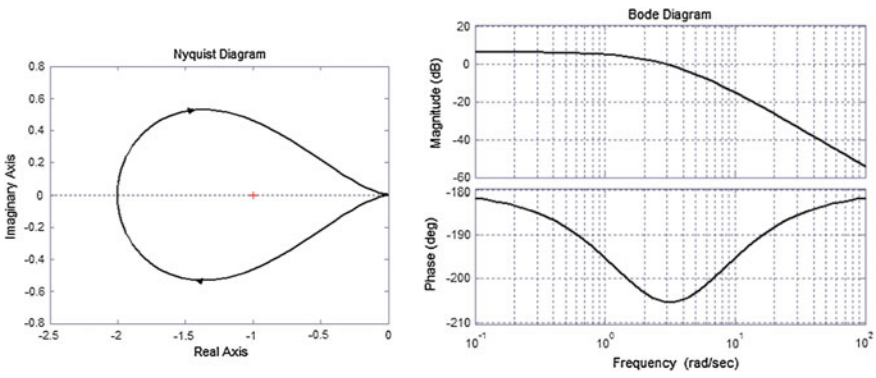



**Fig. 8.16** NYQUIST and BODE diagrams of an unstable process controlled by *P* regulator

```
P=5/((s-2)*(s+5))
figure(1);grid on;nyquist(P);
figure(2); grid on; bode(P);
figure(3); rlocus(P);
```

As can be seen from Fig. 8.17, the Nyquist diagram may encircle the point $-1+j0$ counter-clockwise. This can be arranged by increasing the gain. For $k_c > 2$ the curve encircles the point $-1+j0$, so the control system becomes stable. The Bode diagram shows that the phase margin can be positive.

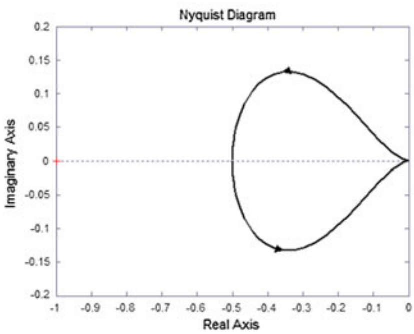
Let us choose the gain $k_c$ to ensure that the cut-off frequency is located where the phase margin is of maximum value. First analyse the behaviour of the open loop for $k_c = 1$.

```
C=1
L=C*P
[mag,phase,w]=bode(L);
```

*Method 1*: using the table

```
T=[phase(:), mag(:), w]
    -158.1444   0.3559   1.7433
    -155.9825   0.3066   2.2122
    -154.7797   0.2530   2.8072
>>  -154.6231   0.2259   3.1623   <<
    -154.7797   0.1994   3.5622
    -155.9825   0.1501   4.5204
```

The Bode phase curve reaches its maximum at `phase` = 154.62, `mag` = 0.2259, $\omega = 3.16$. With a proportional regulator, the maximum reachable phase margin is



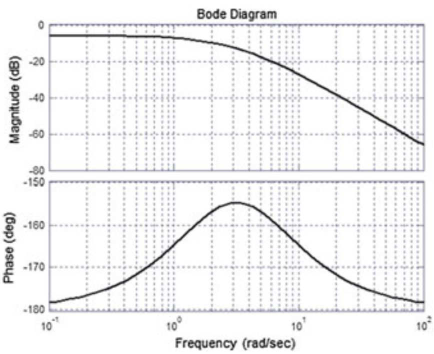




**Fig. 8.17** Nyquist and Bode diagrams of an unstable process controlled by *P* regulator

$\varphi_t = 180° - 154.6° = 25.8°$. In this case the gain has to be chosen as $k_c = 1/0.045 = 4.42$.

```
kc=4.42
```

*Method 2*: The maximum value can be calculated with the command max.

```
[maxphase,index]=max(phase)
kc=1/mag(index)
```

Calculate the regulator again:

```
C=kc;
L=C*P;
```

With margin, the phase margin can be checked graphically.

```
margin(L);
```

The phase margin of the system is small, $\varphi_t = 25.4°$ (60° would be required). Plot the step response of the closed loop (Fig. 8.18.).

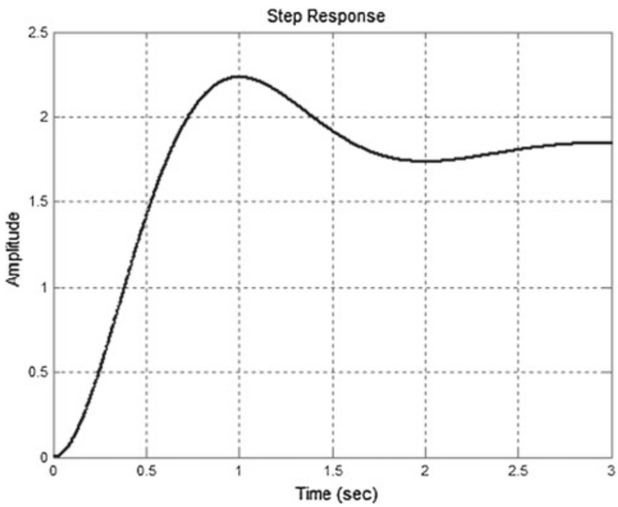



**Fig. 8.18** Step response of *P* control of an unstable process with maximum phase margin

```
T=feedback(L,1)
step(T), grid
```

Stable behaviour has been reached, but there is a quite big overshoot and there is a significant static error. Applying a *PID* regulator the performance of the control system could be improved.

## 8.4.2 Control of an Unstable System with a PID Regulator

To decrease the static error, let us design a *PID* regulator.

$$C(s) = k_c \frac{s+2}{s} \frac{s+5}{s+50}.$$

The unstable pole $p_1 = 2$ can not be cancelled by a zero, as the parameters generally are obtained from measured data, and the system would become unstable even in the case of a small difference between the pole and the cancelling zero. On the other hand with pole cancellation of the unstable pole the inner stability can not be ensured, as the unstable pole would appear in the resulting transfer function between the output and the inner disturbance signals. The unstable pole could be compensated by a stable *PI* element. To accelerate the system the stable pole $p_1 = -5$ is shifted to a higher frequency by a *PD* element ($p = -50$, the pole shift ratio is 10). The gain $k_c$ is chosen again to ensure the maximum phase margin.

Clear all the variables and close the graphic windows.

```
clear all, close all
s=zpk('s')
P=5/((s-2)*(s+5))
C=((s+2)*(s+5))/(s*(s+50))
L=C*P
L=minreal(L)
bode(L)
[mag,phase,w]=bode(L);
```

Determine the gain for the maximum phase:

```
[maxphase,index]=max(phase)
kc=1/mag(index)
```

The gain is $k_c = 152$, and the phase margin $\varphi_t = 180 + \text{maxphase} = 58°$. Check the behaviour of the control system.

```
C=kc*((s+2)*(s+5))/(s*(s+50))
L=C*P, L=minreal(L)
margin(L)
```

The obtained phase margin is really 60° (Fig. 8.19).
Determine the step response of the closed loop and the control signal (Fig. 8.20).

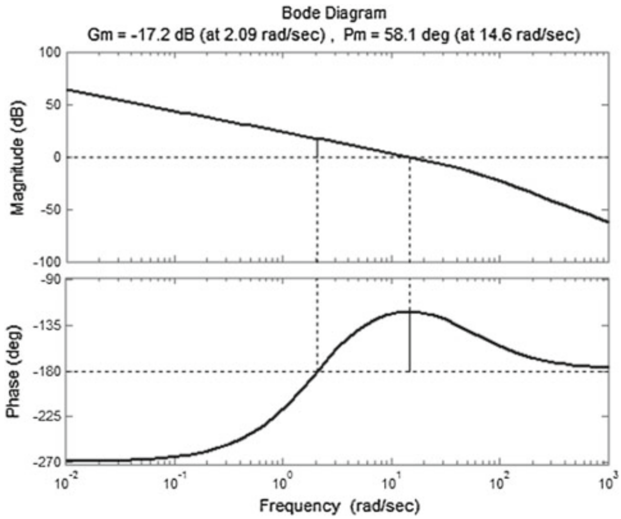



**Fig. 8.19** Bode diagram of *PID* control of an unstable process with phase margin of 60°

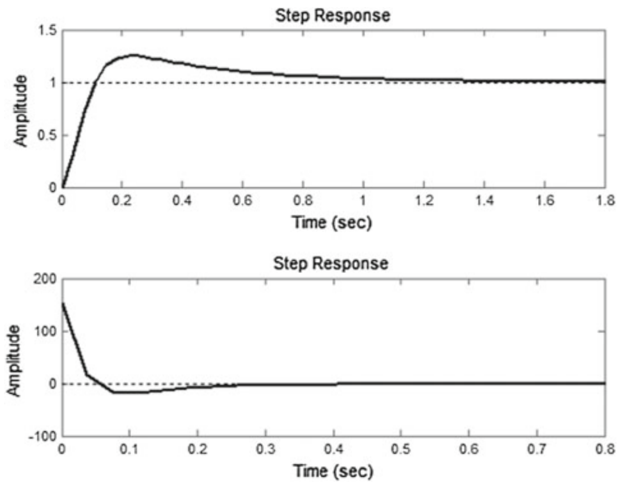



**Fig. 8.20** The output and the control signals

```
T= L/(1+L), T=minreal(T)
```

The static error has been decreased to zero.

```
es=1-dcgain(T)
    es = 1.2212e-015
```

The resulting transfer function between the control signal and the reference signal is

```
U=C/(1+L), U=minreal(U)
subplot(211),step(T)
subplot(212),step(U)
u=step(U)
um=max(u)
```

The maximum value um of the control signal is high. This value can be decreased by decreasing the pole shift ratio.

## 8.5 Handling of Constraints

Let us analyse the behaviour of the *PI* regulator designed in Sect. 8.2.3 when there are constraints.

$$P(s) = \frac{1}{(1+10s)(1+s)(1+0.2s)};\ C(s) = 5.04\frac{1+10s}{10\,s}$$

The maximum value of the control signal is um $= 5.04$. In practical applications limitations do exist for the control signal. Such limitations may originate from several sources. The manipulator which provides the control signal to the process input generally can not produce a higher value than its given maximum. Limitation is applied also at the process input when the process should be protected against too big, harmful interventions.

In the case of *PI* regulators, the FOXBORO regulator provides a simple solution for handling limitations. The regulator is realized by a saturation block fed back with positive feedback by a first order lag element (see Fig. 8.22). Without the saturation the proportional path has gain 1. The resulting transfer function of this circuit provides a *PI* regulator (Fig. 8.21).

If the regulator works in the linear range, then this relationship holds, otherwise its output is limited. Compare the simple limitation at the process input realized by cutting the input signal with the effect of the FOXBORO regulator. The comparison of the regulators is executed in the SIMULINK™ environment.

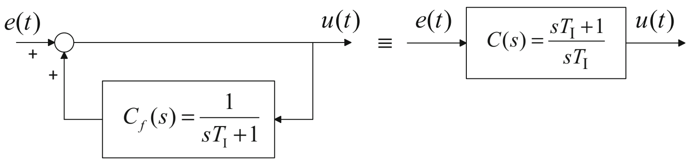



**Fig. 8.21** FOXBORO regulator

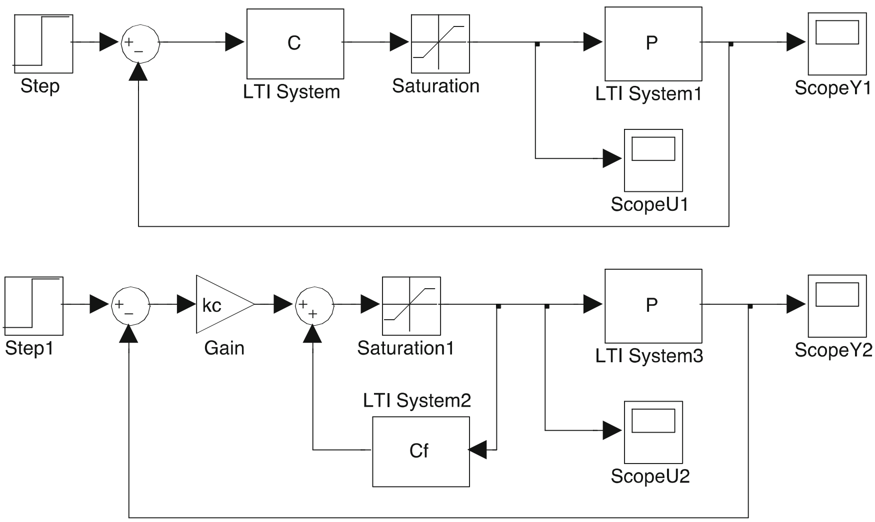



**Fig. 8.22** SIMULINK™ diagram of a control system handling saturation by FOXBORO regulator

```
s=zpk('s')
P=1/((1+10*s)*(1+s)*(1+0.2*s))
kc=5.04
C=kc*(1+10*s)/(10*s)
Cf=1/(1+10*s)
```

Build the SIMULINK™ block diagram shown in Fig. 8.22.

Set the lower and upper limits of blocks *saturation* (SIMULINK™ –> *Discontinuities* –> *Saturation*) (*Upper limit and Lower limit*) to u1 and −u1. Set the simulation time to 50.

```
u1=2
```

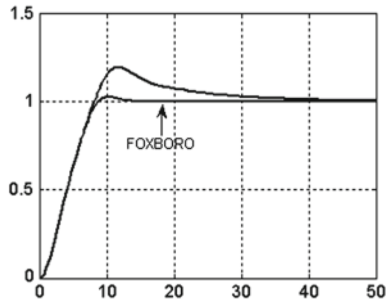



**Fig. 8.23** Step responses in case of saturation

In the case of saturation, the course of the output signal with the FOXBORO regulator is more advantageous, the overshoot is smaller, and the settling time is also smaller (Fig. 8.23).

The reason is that with the conventional *PI* control shown in the upper part of Fig. 8.22 at $t = 0$, a control signal of value 5.04 does appear, the saturating element limits this value, and at the output of the regulator the value of the signal will be 2. The saturation quasi "opens" the circuit until the feedback signal brings the saturating element outside of the range of saturation. The output of the integrating element of the *PI* regulator "winds up", and therefore the saturating element remains in the saturation range for a longer time. In the FOXBORO regulator the saturation acts on the process and on the dynamic part of the regulator located in its feedback path in the same way, therefore the disadvantageous windup phenomenon does not show up here.