Chapter 6
Regulator Design in the Frequency Domain

A closed-loop control system has to meet several prescribed quality specifications. These specifications are:

- stability
- prescribed static accuracy for reference signal tracking and disturbance rejection
- attenuation of the effect of measurement noise
- insensitivity to parameter changes
- prescribed dynamical (transient) behavior
- consideration of constraints resulting from practical realization.

Generally the performance of a control system does not meet all these requirements. The required operation can be ensured by an appropriate design of the control system. The most frequent control scheme is the serial control loop shown in Fig. 6.1. For the plant to be controlled a serially connected controller is to be designed which ensures the stability of the closed-loop control system and fulfills the prescribed quality specifications.

In the control loop $P(s)$ is the transfer function of the plant, while $C(s)$ is the transfer function of the controller (regulator). The controlled output signal y is fed back with a negative sign and compared to the reference signal. The disturbance is taken into account by a signal $y_n(t)$ acting on the output of the plant. This disturbance model is generally satisfactory to handle practical situations.

In the sequel, some considerations are given for controller design based on relationships in the frequency domain.

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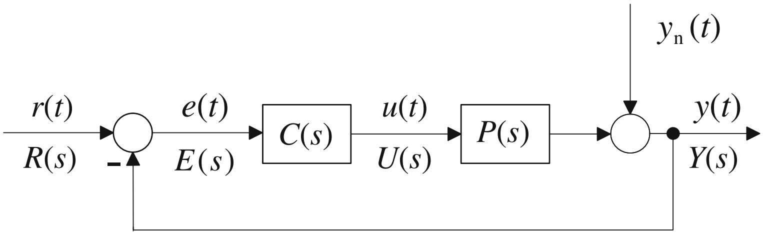

Fig. 6.1 Closed loop control system with serially connected controller

6.1 On the Relationships Between Properties in the Time- and Frequency-Domain

The quality specifications can be demonstrated in the frequency domain, looking as well at the course of the frequency functions of the open-loop and of the closed-loop.

The stability and the susceptibility of the system to oscillations is characterized by the amplification of the Bode amplitude-frequency curve of the closed-loop. Amplification may occur in the vicinity of the cut-off frequency in the open-loop diagram. To determine the stability, the oscillatory behavior, the transient response the course of the frequency function has to be investigated in the middle frequency range (Sect. 4.6). As shown previously the settling time can be estimated from the cut-off frequency. Stability investigation on the basis of the Nyquist and the Bode diagrams has been dealt with in detail in Chap. 5.

Static properties of the control system can be determined from the course of the frequency function in the low frequency range. As seen, the error of the reference signal tracking and of the disturbance rejection in the case of step, ramp and quadratic input signals depends on the type number (the number of integrators in the open-loop) of the control system and on the value of the loop gain. In the case of a type 0 control, the approximate Bode amplitude diagram of the open-loop starts in the low frequency range with a straight line parallel to the 0 dB axis with the value K_0 of the loop gain. For control systems of type 1, the diagram starts with a straight line of slope −20 dB/decade. Extending this line it crosses the 0 dB axis at the frequency K_1 corresponding to the loop gain. In the case of control system of type 2, the Bode diagram starts with a straight line of slope −40 dB/decade, whose extension crosses the horizontal axis at $\sqrt{K_2}$. To ensure the static requirements, the low frequency range of the open-loop Bode diagram has to be shaped appropriately (Fig. 6.2).

Based on the relationship of the open-loop and the closed-loop frequency functions (Sect. 4.6) several statements can be given. These statements have to be considered as criteria to design control systems able to ensure good reference signal tracking, disturbance and noise rejection, and to compensate the effect of parameter uncertainties:

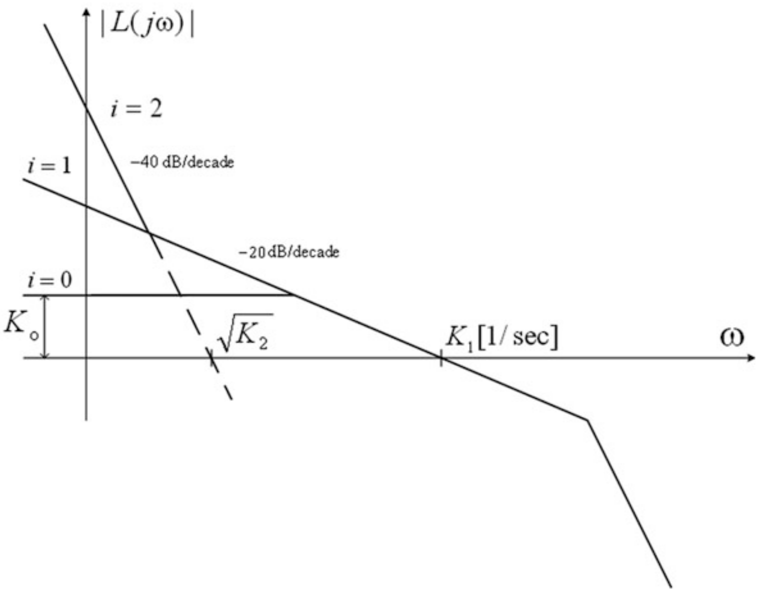

Fig. 6.2 Static accuracy of the control system is characterized by the low frequency range of the open-loop amplitude-frequency curve

- for good reference signal tracking, $|L(j\omega)|$ should be large.
- for effective rejection of the input and output disturbances, $|L(j\omega)|$ should be large.
- for good rejection of measurement noise, $|L(j\omega)|$ should be small.
- to compensate the effect of the parameter uncertainties of the plant, $|L(j\omega)|$ should be large.
- to avoid too high actuating control signals, the acceleration should be moderate and, ω_c should not be too high.

Some requirements are contradictory, and can not be ensured simultaneously for a given frequency range. Therefore different prescriptions have to be given for different frequency ranges. Section 6.2. formulates the quality requirements in the frequency domain in more detail.

6.2 Quality Requirements in the Frequency Domain

As already seen in Chaps. 4 and 5, the quality specifications can also be demonstrated in the frequency domain on the course of the closed-loop and the open-loop frequency functions. The behavior of the closed-loop control system is characterized by the overall frequency function of the closed-loop system. As previously seen, the maximal amplification M_m of the amplitude-frequency function of the closed-loop characterizes the overshoot of the step response in the time domain. An experimental observation is that if $M_m < 1.25$, then there is no significant overshoot in the step response. Many times a closed-loop system can be well approximated by a dominant pair of poles, thus it can be replaced by a second order oscillating element, whose damping factor determines the maximal value of the overshoot. If there is no amplification in the amplitude-frequency curve of the closed-loop, then there is no overshoot in the step response. If the damping factor of the approximating oscillating element is higher than 0.5, the transients will be aperiodic. If the value of the damping factor is around 0.6–0.7, then the overshoot of the step response is about 5–10%.

On the basis of the relationships between the transfer functions of the open and the closed-loop, instead of analyzing the frequency function of the closed-loop we may analyze the amplitude-frequency curve of the open-loop, as well. The overshoot is related to the phase margin. If $\varphi_t \approx 60°$, the damping factor is $\xi \approx 0.7$; ξ also increases by increasing φ_t. With $\varphi_t > 90°$, the transient behavior becomes aperiodic. If the phase margin decreases, the damping factor also decreases. For $\varphi_t \approx 30°$, the damping factor range of $\xi \approx 0.2 - 0.3$, which results in significant, but still decaying oscillations. These simple considerations are satisfactory for orientation.

The control system has to be designed to meet the quality specifications. This can be accomplished by designing an appropriate controller. The Bode diagram of the open-loop has to be shaped to ensure the prescriptions (this procedure is called *loop-shaping*, [see Fig. 6.3]).

The prescriptions for reference signal tracking and for disturbance rejection, and for attenuating the effects of measurement noise, are contradictory in relation to the shape of the Bode amplitude diagram $|L(j\omega)|$. But in practice, generally, the characteristic frequency range of the reference signal and that of the measurement noise are different: the reference signal contains components in a lower frequency range, while the measurement noise mainly contains high frequency components. Thus $|L(j\omega)|$ can be large in the low frequency domain, and small in the high frequency domain.

The course of the amplitude diagram in the low frequency domain determines the static properties. In case of type number 0 the Bode diagram starts horizontally, with type number 1 the initial slope is −20 dB/decade, while in case of type 2 it is −40 dB/decade. The prescribed static accuracy determines the required type number of the control system.

The middle frequency range determines the stability and the dynamical properties of the control system. As shown in Chap. 5, to ensure stability the cut-off frequency ω_c has to be located on a quite long straight line with a slope of −20 dB/decade. This condition is sufficient to ensure stability if the system is of minimum-phase and does not contain dead-time. Otherwise the value of the phase margin or the gain margin has to be calculated. A positive phase margin, or gain margin whose value is higher than 1, ensures stability. Besides stability, the appropriate dynamic behavior (an overshoot less than 10%) can be provided if the

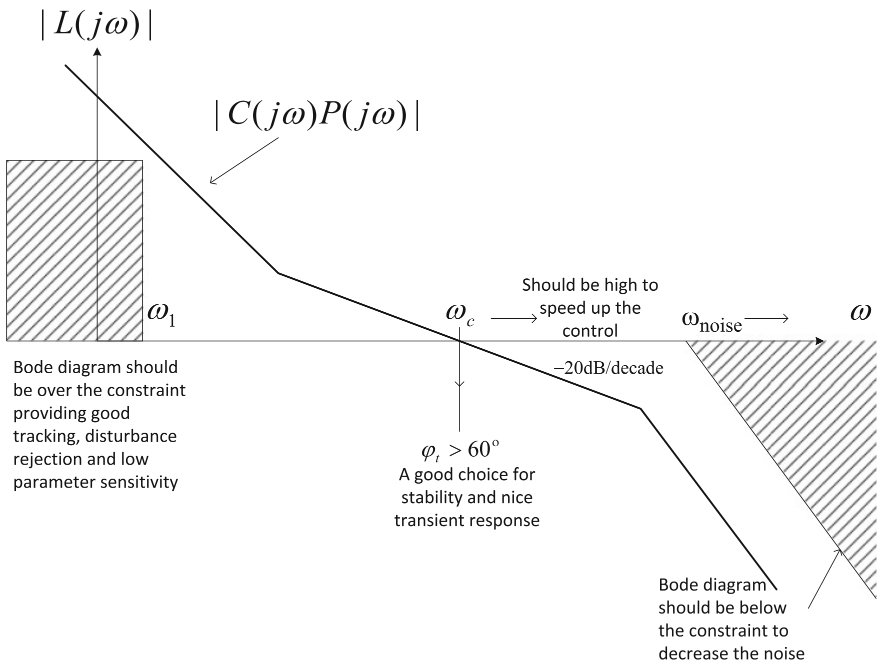

Fig. 6.3 Considerations for loop-shaping on the Bode amplitude-frequency diagram

phase margin is about 60°. Later it will be seen that for minimum-phase systems containing dead-time T_d the phase margin will be about 60° if the cut-off frequency is chosen to be $\omega_c \approx 1/2T_d = 0.5/T_d$. The settling time is related to the cut-off frequency. The higher ω_c is, the shorter the settling time is. As seen in Chap. 4, the settling time can be estimated as $3/\omega_c < t_s < 10/\omega_c$.

In a conventional control structure (Fig. 6.1) in open-loop systems containing dead-time, the cut-off frequency can not be increased beyond one half of the reciprocal of the dead-time. This sets a limit to the acceleration of the system. To achieve a higher ω_c, an advanced control scheme should be considered (e.g. a Smith predictor, see Chap. 12).

By shaping the low frequency and the high frequency sections of the Bode amplitude diagram according to Fig. 6.3 the suppression of the measurement noise and the insensitivity of the control system to the parameter uncertainties can be ensured, as well.

Often the different requirements are contradictory. A satisfactory compromise has to be reached to ensure stability, a good dynamic response, fast performance and the required constraints on the value of the control signal. The control system can be accelerated only by applying a large control action. This high control signal is to be ensured by the actuator which generally has limits due to its physical realization.

In practice the controller is able to provide signals only within a given range. It is important that during the operation the control signal $u(t)$ remains within this range, that is, its maximum value should not exceed the physically reachable maximum value. If a command is given that would require a control signal value exceeding the maximum, the actuator will be saturated: its output will be at the maximum value until—with an increase of the output signal—the error signal reaches such a small value that the output signal of the controller would go out of the saturation domain. The designer of the control system has to consider this phenomenon already in the controller design phase.

Thus during the controller design process the shape of the open-loop frequency function around the point $-1+0j$ has to be formed. In the complex plane, the point $-1+0j$ can be described by two conditions, namely the gain is 1 and the phase angle is $-180°$, that is, $|L(j\omega)|=1$ and $\varphi=-180°$.

In the design procedure these two conditions can be handled as follows: in the case of the fulfillment of one of these conditions, how far is the second one from the value given above? If the gain is 1, then the phase deviation is given by the phase margin at the cut-off frequency, whereas if the phase angle is $-180°$, then the gain margin can be calculated from the value of the gain. If the controller is designed for a prescribed phase margin, then the phase angle of the frequency function has to be set at the frequency belonging to the unity gain (which is the cut-off frequency).

Thus the shape of the open-loop frequency function around the cut-off frequency ω_c is critical. In some cases it is not sufficient to determine the phase angle only at this frequency, but the shape of the frequency function has to be examined also in its vicinity.

Figure 6.4 shows two Nyquist diagrams with identical phase margins. The diagram of L_1 after the cut-off frequency goes quickly to the origin, whereas the diagram of L_2 approximates the point $-1+0j$. For the first system the amplitude of the closed-loop system $|T|=|L/(1+L)|=|L|/|1+L|$ quickly decreases after ω_c, whereas in the second case it may show significant amplification. In this latter case the absolute value of the sensitivity function $|S|=1/|1+L|$ also has amplification in this frequency range.

6.3 Methods to Shape the Open-Loop Frequency Characteristics

Employing the frequency function $H(j\omega)$ for control system analysis let us consider the Bode theorems. The first Bode theorem states that under some conditions (stable and minimum-phase system without dead-time) the amplitude-frequency function of $H(j\omega)$ unambiguously determines the phase-frequency function. That is at a given frequency ω_o

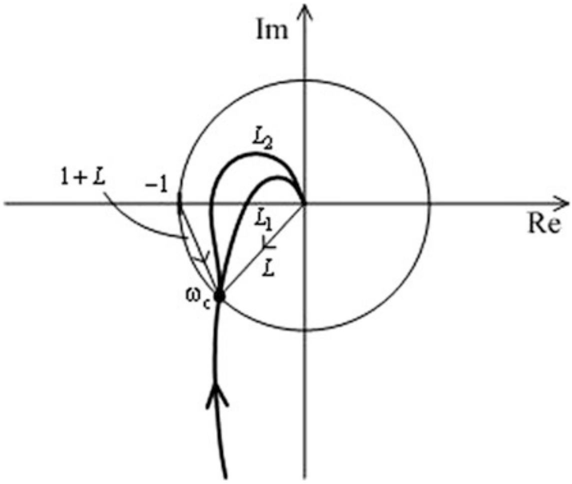

Fig. 6.4 The shape of the Nyquist diagram around ω_c and around $-1+0j$ affects significantly the overshoot

$$\begin{aligned}\arg H(j\omega_o) &= \frac{2\omega_o}{\pi}\int_0^\infty \frac{\log|H(j\omega)| - \log|H(j\omega_o)|}{\omega^2-\omega_o^2}\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}\log|H(j\omega)|}{\mathrm{d}\log\omega}\log\left|\frac{\omega+\omega_o}{\omega-\omega_o}\right|\mathrm{d}\omega \approx \frac{\pi}{2}\frac{\mathrm{d}\log|H(j\omega)|}{\mathrm{d}\log\omega}.\end{aligned} \tag{6.1}$$

According to this relationship, at a given $\omega=\omega_o$, the complete shape of the phase-frequency curve contributes to form the value $\varphi(\omega_o)$ through the expression of the definite integral above. At the same time, because of the inner weighting function

$$\log\left|\frac{\omega+\omega_o}{\omega-\omega_o}\right|, \tag{6.2}$$

the far points have only a barely noticeable effect on the function in the vicinity of ω_o. Equation (6.1) means that if in the vicinity of a point, the slope (in logarithmic scale) of the approximate amplitude curve is $+1$, then the corresponding phase angle is $+\pi/2$. (In decibels +20 dB/decade corresponds to a slope of $+1$).

A similar relationship can be given for how the amplitude-frequency function relates to the phase-frequency function, that is, the relationship is unique and mutual. The relationship is not unique for non-minimum-phase systems.

If the breakpoint frequencies of the frequency function are far enough from each other, then the phase-frequency curve belonging to the long straight lines of the approximated amplitude-frequency curve is horizontal with a value of $\varphi(\omega)\approx\pm j\pi/2$, whereas close to the breakpoints, it is not horizontal and the phase angle can be approximated by $\varphi(\omega)\approx\pm(2j+1)\pi/4$.

In the previous chapters it was shown that the stability and robustness properties, static and dynamic behavior of the closed-loop system all can be designed by the *loop-shaping* of the frequency function of the open-loop. Bode, partly on an empirical basis, came to the conclusion that the optimal (ideal) form of the loop frequency function $L(j\omega)$ is

$$L_{\mathrm{id}}(j\omega)=\frac{1}{(j\omega)^{\eta}}. \tag{6.3}$$

If η is not an integer, then in the complex plane the Nyquist diagram is a straight line going through the origin. Integrators of order n are special cases of this general form when η is an integer. But non-integer forms cannot be realized by lumped parameter linear systems, therefore this expectation is only of theoretical significance and we can only try to approximate it. If at the cut-off frequency ω_c the phase margin φ_t were set according to a characteristic given by $L_{\mathrm{id}}(j\omega)$, then any uncertainty in the gain of $L_{\mathrm{id}}(j\omega)$ would not influence the design condition (i.e., the prescribed φ_t). The ideal case is best approximated if the slope of the phase characteristic $d\varphi/d\omega$ is minimal at the cut-off frequency ω_c. This can be ensured if the breakpoint frequencies in the neighborhood of ω_c are far away from it (see Bode's first theorem).

Discussing the interpretation of the modulus margin, it was seen that its value is the reciprocal of the maximum absolute value of the sensitivity function. According to the geometric representation its value is the closest distance of the Nyquist diagram from the point $-1+0j$. For Nyquist curves which have parts in the third and the fourth quadrant (these are the so-called positive real systems) the absolute value of the sensitivity function can not be higher than 1. But among the non positive real systems there can also be such systems where the sensitivity is less than 1. On the basis of the $M-\alpha$ and $E-\beta$ curves (see Sect. 4.6.) the range where both conditions, $|S(j\omega)|=E(\omega)\leq 1$ and $|T(j\omega)|=M(\omega)\leq 1$ are fulfilled at the same time can be given simply. Figure 6.5 shows the restricted area (crosshatched), where there will already be amplification in $|S(j\omega)|$ or in $|T(j\omega)|$. In the figure the sensitivity function $S(j\omega)$ belonging to $L(j\omega)$ has no amplification. The necessary condition to achieve this is that the pole excess of $L(s)$ be 1, that is, if $\omega\to\infty$ the condition $L(j\omega)\approx -j\omega$ is fulfilled. (In the figure as a curiosity it is also shown, that for systems of integrating type generally $\mathrm{Re}[L(j\omega)]_{\omega\to 0}\neq 0$, that is, the curve starts not from the imaginary axis, a fact not widely known). On the basis of the geometrical interpretation it can easily be understood that for the so-called *positive real* frequency functions ($\mathrm{Re}[H(j\omega)]\geq 0$) (which have less significance in control theory, but more importance in telecommunication) the condition of avoiding amplification in $S(j\omega)$ is automatically fulfilled.

The intersection points of $L(j\omega)$ with the unit circle $E=1$ and the straight line $M=1$ also contain significant information about the course of the frequency characteristics of the open and the closed-loop.

According to Fig. 6.6 the sensitivity function $S(j\omega)$ at low frequencies starts from zero. At ω_1 it reaches the value 1, then at ω_c the absolute value of $L(j\omega)$ will

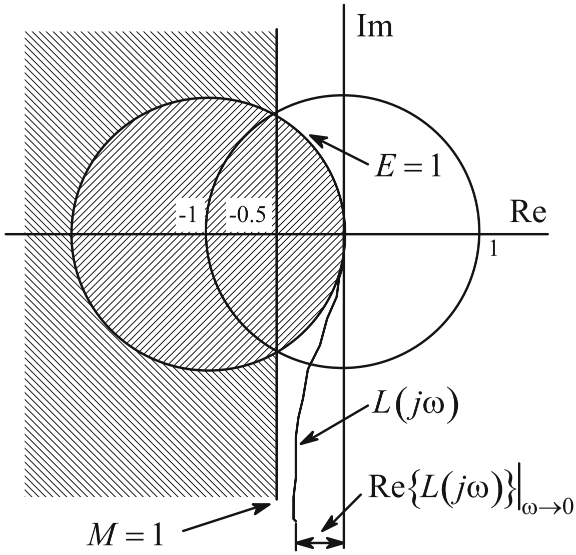

Fig. 6.5 Areas restricting amplification of the sensitivity and the supplementary sensitivity functions

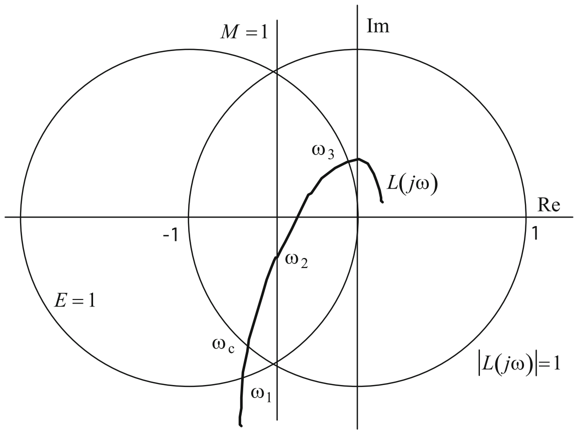

Fig. 6.6 Following the course of $S(j\omega)$ and $T(j\omega)$

be 1. Exceeding its maximum value at ω_3 again it will satisfy $|S(j\omega)| = 1$. If at high frequencies $L(j\omega)$ tends to zero, then $S(j\omega)$ tends to 1. From this analysis we can conclude that the frequency functions of dead-time systems (which may also contain lag elements, integrators, etc.) can never avoid the circle $E = 1$, therefore $|S(j\omega)|$ always has amplification. (The condition $E = 1$ is also called the KALMAN-HO condition).

Based on the construction rules of the $M - \alpha$ and $E - \beta$ curves it is easy to construct the circles (crosshatched areas) shown in Fig. 6.7 to ensure the conditions $|S(j\omega)| \leq 2$ and $|T(j\omega)| \leq 2$. The frequency belonging to $M = \sqrt{2}/2$ is the bandwidth of the closed-loop system.

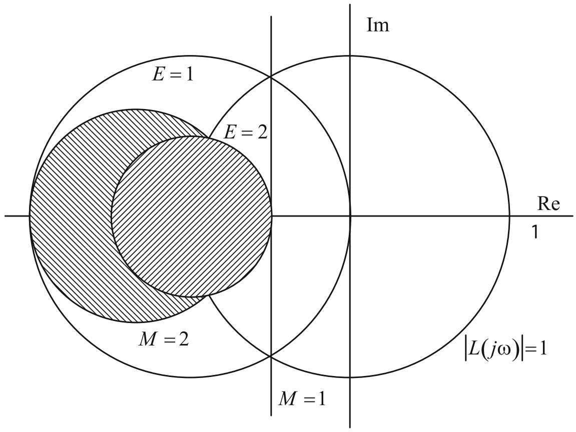

Fig. 6.7 Restricted areas ensuring the conditions $|S(j\omega)| \leq 2$ and $|T(j\omega)| \leq 2$

Unfortunately there are also theoretical limits to the arbitrary shaping of the sensitivity function. According to the second Bode theorem, if $L(j\omega)$ tends to zero more quickly than $1/j\omega$ for high values of ω (that is, the pole excess is at least 2), then the following integral equation is valid:

$$\int_0^\infty \log|S(j\omega)|\mathrm{d}\omega = \int_0^\infty \log\frac{1}{|1+L(j\omega)|}\mathrm{d}\omega = \pi\sum_i \mathrm{Re}\, p_i, \tag{6.4}$$

where the p_i denote the unstable poles. Consider the simplest case, when there are no poles in the right half-plane, so $L(s)$ is stable or is on the boundary of stability. Then

$$\int_0^\infty \log|S(j\omega)|\mathrm{d}\omega = 0. \tag{6.5}$$

This equation can be interpreted more easily, as it represents the so-called water-bed effect, which means that if $|S(j\omega)|$ is pressed at one point, it will bulge at another one. If for example for low frequencies we make an effort to decrease the values of $|S(j\omega)|$, then at high frequencies it will be high. This is because we calculate the integral of the logarithm of an absolute value, thus the area of $|S(j\omega)|$ below 1 will be equal to the area above 1 (in logarithmic scale the area below the zero dB scale will be equal to the area above it).

Note that drawing the Nyquist diagram and also considering the relationships discussed above can convince one, that significant properties, such as the stability and/or robustness of the control system, can be determined from the course of the frequency function. These properties can be recognized in a somewhat more involved way from the Bode diagram. Unfortunately the quantitative properties cannot be concluded exactly from either the Nyquist diagram nor the Bode diagram.

Example 6.1 Let us consider the control system with the open-loop transfer function

$$L(s) = \frac{1 + s\tau}{sT_{\mathrm{I}}(1 + sT_1)}. \tag{6.6}$$

The frequency function is

$$L(j\omega) = \frac{1 + j\omega\tau}{j\omega T_{\mathrm{I}}(1 + j\omega T_1)} = \frac{\tau - T_1}{T_{\mathrm{I}}\left(1 + \omega^2 T_1^2\right)} - j\frac{1 + \omega^2 \tau T_1}{\omega T_{\mathrm{I}}\left(1 + \omega^2 T_1^2\right)} = \mathrm{Re}(\omega) + j\,\mathrm{Im}(\omega). \tag{6.7}$$

The value of the real part at zero frequency is

$$\mathrm{Re}(\omega \to 0) = \frac{\tau - T_1}{T_{\mathrm{I}}} = \frac{\tau}{T_{\mathrm{I}}} - \frac{T_1}{T_{\mathrm{I}}} = \begin{cases} > 0, & \text{if} \quad \tau > T_1 \\ < 0, & \text{if} \quad \tau < T_1 \end{cases} \tag{6.8}$$

and

$$\mathrm{Re}(\omega \to 0) = \frac{\tau - T_1}{T_{\mathrm{I}}} < 1, \quad \text{if} \quad \tau < T_1 + T_{\mathrm{I}} \tag{6.9}$$

which is a necessary condition to avoid amplification in the supplementary sensitivity function $T(j\omega)$. The pole excess of the loop transfer function is 1, which is the necessary condition to avoid amplification in the sensitivity function. Let us examine the sufficient condition as well. The sensitivity function is obtained by a simple calculation according to

$$S = \frac{1}{1 + L} = \frac{sT_{\mathrm{I}}(1 + sT_1)}{sT_{\mathrm{I}}(1 + sT_1) + (1 + s\tau)}. \tag{6.10}$$

The frequency function of the sensitivity function is

$$S(j\omega) = \frac{j\omega T_{\mathrm{I}}(1 + j\omega T_1)}{j\omega T_{\mathrm{I}}(1 + j\omega T_1) + (1 + j\omega\tau)} = \frac{-\omega^2 T_{\mathrm{I}} T_1 + j\omega T_{\mathrm{I}}}{(1 - \omega^2 T_{\mathrm{I}} T_1) + j\omega(T_{\mathrm{I}} + \tau)}. \tag{6.11}$$

The function $S(j\omega)$ has no amplification if

$$|S(j\omega)|^2 = E^2 = \frac{\omega^2 T_{\mathrm{I}}^2\left(1 + \omega^2 T_1^2\right)}{(1 - \omega^2 T_{\mathrm{I}} T_1)^2 + \omega^2 (T_{\mathrm{I}} + \tau)^2} \le 1 \tag{6.12}$$

Solving the inequality above, the following sufficient condition is obtained:

$$\frac{\tau}{T_{\mathrm{I}}} \geq \sqrt{1+\frac{2T_1}{T_{\mathrm{I}}}} - 1. \tag{6.13}$$

The complementary sensitivity function is

$$T = \frac{L}{1+L} = \frac{1+s\tau}{sT_{\mathrm{I}}(1+sT_1)+(1+s\tau)} \tag{6.14}$$

and its frequency function is

$$T(j\omega) = \frac{1+j\omega\tau}{j\omega T_{\mathrm{I}}(1+j\omega T_1)+(1+j\omega\tau)} = \frac{1+j\omega\tau}{(1-\omega^2 T_{\mathrm{I}}T_1)+j\omega(T_{\mathrm{I}}+\tau)}. \tag{6.15}$$

Now, $T(j\omega)$ has no amplification if

$$|T(j\omega)|^2 = M^2 = \frac{1+\omega^2\tau^2}{(1-\omega^2 T_{\mathrm{I}}T_1)^2+\omega^2(T_{\mathrm{I}}+\tau)^2} \leq 1. \tag{6.16}$$

From the solution of the inequality $\tau \geq T_1 - T_{\mathrm{I}}/2$ is obtained as a sufficient condition. So there is no amplification if

$$T_1 - T_{\mathrm{I}}/2 \leq \tau \leq T_1 + T_{\mathrm{I}} \tag{6.17}$$

With the root locus method it can be seen that the closed-loop system is structurally stable. If $\tau > T_1$, then there are only real poles. If $\tau < T_1$, then real poles are obtained only for the ranges $K_{\mathrm{I}} < K_1$ and $K_{\mathrm{I}} > K_2$, and in the range $K_1 < K_{\mathrm{I}} < K_2$ the poles are complex conjugate pairs, where

$$K_{1,2} = \left(\frac{2T_1}{\tau} - 1\right) \pm \sqrt{\left(\frac{2T_1}{\tau} - 1\right)^2 - 1}. \quad \blacksquare \tag{6.18}$$