Chapter 12
Sampled Data Controller Design for Stable Discrete-Time Processes

Comparing Chaps. 7 and 9 it is worth distinguishing stable and unstable processes at the design step of a CT controller. For stable processes precise mathematical methods—based on or derived from YOULA-parameterization—are available. Even the conventional *PID* regulator—under the usual tuning—corresponds to a rough approach of the YOULA-controller. The unstable processes require different methods, where the most important task is to stabilize the process. For this task, in Chap. 9, a general polynomial method based on state-feedback and observer was presented, and in Chap. 10 another method based on DIOPHANTINE-equations (*DE*) was presented. For sampled data systems, a similar logic will be followed to design the controllers. The main difference is that the establishment of the different signal-forming items realizing the different methods for DT control loops in computer based control systems is significantly simpler.

In this section the transfer function operators $G(\ldots)$ are functions of z and z^{-1} or q and q^{-1} depending on the character of the DT description. The design methods presented here are practically polynomial (algebraic) methods, so they can be applied to all models discussed for the sampled systems, only the chosen model form has to be applied correspondingly.

12.1 The YOULA Controller for Sampled Data Systems

In Chap. 7 a general control parameterization method and a design method based on that was shown. The so-called YOULA-parameterization method was suggested for the design of one- or two degree of freedom (*ODOF*, *TDOF*) control loops. The advantage of the method is that the design of the closed-loop is assigned to two reference models, namely the reference signal tracking behavior may be assigned to R_{r}, the disturbance rejection to R_{n}, and the design of the controller can be given in a

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relatively simple closed form. The disadvantage of the method is that it can be applied only to stable processes.

The presentation of the Youla-parameterization, its relationship to the *IMC* principle, the optimality and the best reachable control, were discussed in a very general way, so—in many cases—it is enough to replace the transfer functions with the pulse transfer functions, and all the relations are valid here, too. Therefore the general statements of Chap. 7 are not repeated here, instead the differences and deviations are emphasized. Consider the DT process

$$G(z^{-1}) = G_+(z^{-1})\bar{G}_-(z^{-1}) = G_+(z^{-1})G_-(z^{-1})z^{-d} \quad \text{or} \\ G = G_+\bar{G}_- = G_+G_-z^{-d}, \tag{12.1}$$

where G_+ is stable, its inverse is also stable and realizable (*ISR*). The inverse of $\overline{G}_-$ is unstable (*Inverse Unstable*: *IU*) and non-realizable (*IUNR*). In general the inverse of the delay z^{-d} is unrealizable, because it would mean an ideal predictor.

The optimal controller obtained for the general case [see (7.14)] is

$$C_{\text{opt}} = \frac{R_nK_n}{1 - R_nK_nG} = \frac{Q_{\text{opt}}}{1 - Q_{\text{opt}}G} = \frac{R_nG_nG_+^{-1}}{1 - R_nG_nG_-z^{-d}} = R_nG_nC'_{\text{opt}}, \tag{12.2}$$

where the optimal Youla parameter is

$$Q_{\text{opt}} = R_nG_nG_+^{-1} = R_nK_n \quad \text{and} \quad K_n = G_nG_+^{-1} \tag{12.3}$$

and

$$Q_r = R_rG_rG_+^{-1} = R_rK_r \quad \text{and} \quad K_r = G_rG_+^{-1}. \tag{12.4}$$

For sampled systems the equivalent optimal control system corresponding to the generalized *IMC* principle is completely the same as seen in Fig. 7.9, whose simplified version is shown in Fig. 12.1.

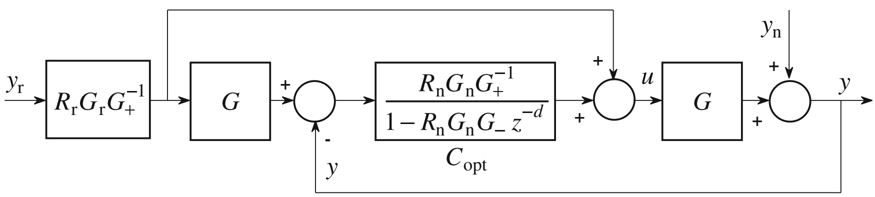

Fig. 12.1 Optimal sampled-time control system based on the generalized *IMC* principle

The most important signals of the *TDOF* closed-control loop are

$$\begin{aligned}
u_{\rm opt} &= R_{\rm r}G_{\rm r}G_{+}^{-1}y_{\rm r} - R_{\rm n}G_{\rm n}G_{+}^{-1}y_{\rm n} \\
e_{\rm opt} &= \left(1 - R_{\rm r}G_{\rm r}G_{-}z^{-d}\right)y_{\rm r} - \left(1 - R_{\rm n}G_{\rm n}G_{-}z^{-d}\right)y_{\rm n} = \left(1 - T_{\rm r}^{\rm opt}\right)y_{\rm r} - S_{\rm n}^{\rm opt}y_{\rm n} \\
y_{\rm opt} &= R_{\rm r}G_{\rm r}G_{-}z^{-d}y_{\rm r} + \left(1 - R_{\rm n}G_{\rm n}G_{-}z^{-d}\right)y_{\rm n} = T_{\rm r}^{\rm opt}y_{\rm r} + \left(1 - T_{\rm n}^{\rm opt}\right)y_{\rm n} = T_{\rm r}^{\rm opt}y_{\rm r} + S_{\rm n}^{\rm opt}y_{\rm n}
\end{aligned} \tag{12.5}$$

where the equalities $T_{\rm r}^{\rm opt} = R_{\rm r}G_{\rm r}G_{-}z^{-d}$ and $T_{\rm n}^{\rm opt} = R_{\rm n}G_{\rm n}G_{-}z^{-d}$ are valid. The further equivalent forms of the best reachable optimal control systems are shown in Fig. 12.2. (These figures are for illustration only, their realizability has to be investigated in each case!).

As mentioned earlier, the theory of the optimality of $G_{\rm r}$ and $G_{\rm n}$ will not be discussed here. The choice $G_{\rm r} = G_{\rm n} = 1$ employed in this simple case leaves the invariant process factor G_{-} unchanged, so it appears unchanged in the signals of the system.

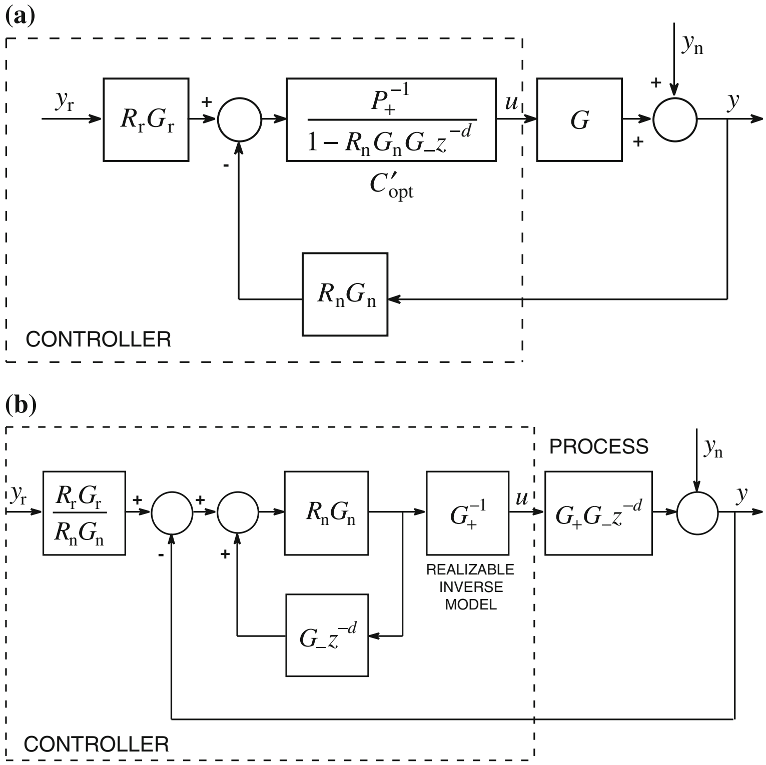

Fig. 12.2 The equivalent forms of the best reachable optimal sampled data control loop

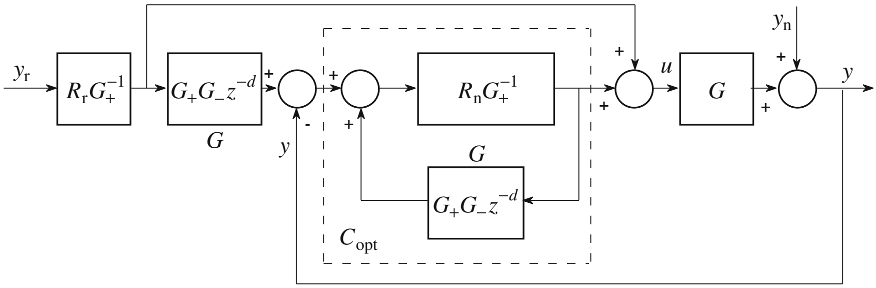

Fig. 12.3 YOULA-parameterized realizable sampled control loop with the choice of $G_r = G_n = 1$

$$\begin{aligned} u &= R_r G_+^{-1} y_r - R_n G_+^{-1} y_n \\ e &= \left(1 - R_r G_- z^{-d}\right) y_r - \left(1 - R_n G_- z^{-d}\right) y_n = (1 - T_r) y_r - S_n y_n \\ y &= R_r G_- z^{-d} y_r + \left(1 - R_n G_- z^{-d}\right) y_n = T_r y_r + (1 - T_n) y_n = T_r y_r + S_n y_n \end{aligned} \quad (12.6)$$

so the realizability of the transfer functions $R_r G_+^{-1}$, $R_n G_+^{-1}$, and $R_n G_-$ has to be ensured, respectively. It can be clearly seen that the realizability can be simply handled by the appropriate choice of the order and pole excess of the reference models R_r and R_n. A realizable but not optimal control system is shown in Fig. 12.3.

In the case of sampled data systems it is worth noting, that using the *SRE* transformation it is always true for the delay-free part $(G_+ G_-)$ in the pulse transfer function of the process—independently from the pole excess of the CT process—that the pole excess is equal to one. So the realizability of the items $R_r G_+^{-1}$ and $R_n G_+^{-1}$ is ensured even for first order reference models R_r and R_n.

Example 12.1 Let the controlled system be a first order process with delay

$$G = \frac{0.2z^{-1}}{1 - 0.8z^{-1}} z^{-3} = \frac{0.2z^{-4}}{1 - 0.8z^{-1}} \quad \text{i.e.} \quad G_+ = \frac{0.2z^{-1}}{1 - 0.8z^{-1}} \quad \text{and} \quad G_- = 1 \quad (12.7)$$

and the goal of the control is to make it faster. Let the tracking and disturbance rejection reference models be

$$R_r = \frac{0.8z^{-1}}{1 - 0.2z^{-1}} \quad \text{and} \quad R_n = \frac{0.5z^{-1}}{1 - 0.5z^{-1}}. \quad (12.8)$$

Since $G_- = 1$ there is nothing to be compensated optimally, i.e., $G_r = 1$ and $G_n = 1$ can be chosen. The optimal controller is

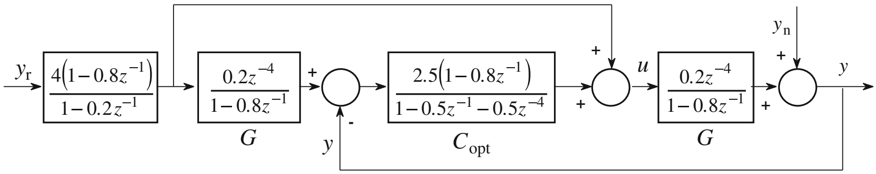

Fig. 12.4 The optimal control loop of Example 12.1

$$
\begin{aligned}
C_{\text{opt}} &= \frac{R_n G_n G_+^{-1}}{1 - R_n G_n G_- z^{-d}} = \frac{1}{1 - R_n z^{-d}} R_n G_+^{-1} \\
&= \frac{1}{1 - \frac{0.5z^{-1}}{1-0.5z^{-1}} z^{-3}} \frac{0.5z^{-1}}{1 - 0.5z^{-1}} \frac{1 - 0.8z^{-1}}{0.2z^{-1}} = \frac{2.5(1 - 0.8z^{-1})}{1 - 0.5z^{-1} - 0.5z^{-4}}
\end{aligned}
\tag{12.9}
$$

and the serial compensation has the form

$$
R_r G_+^{-1} = \frac{0.8z^{-1}}{1 - 0.2z^{-1}} \frac{1 - 0.8z^{-1}}{0.2z^{-1}} = \frac{4(1 - 0.8z^{-1})}{1 - 0.2z^{-1}} \tag{12.10}
$$

so the optimal *TDOF* control loop has the scheme shown in Fig. 12.4 (see Fig. 12.2b). Note that $C_{\text{opt}}(z = 1) = \infty$, i.e., the controller has an integrating character, which comes from the condition $R_n(z = 1) = 1$.

It can be easily checked that the output of the closed-loop is

$$
\begin{aligned}
y_{\text{opt}} &= R_r z^{-d} y_r + \left(1 - R_n z^{-d}\right) y_n = \frac{0.8z^{-1}}{1 - 0.2z^{-1}} z^{-3} y_r + \left(1 - \frac{0.5z^{-1}}{1 - 0.5z^{-1}} z^{-3}\right) y_n \\
&= \frac{0.8z^{-4}}{1 - 0.2z^{-1}} y_r + \left(1 - \frac{0.5z^{-4}}{1 - 0.5z^{-1}}\right) y_n
\end{aligned}
\tag{12.11}
$$

which completely corresponds to the designed *TDOF* control system. ■

12.2 The Smith Controller for Sampled Data System

Let us consider a simple process with delay based on (12.1) in the sampled data control system

$$
\begin{aligned}
G\left(z^{-1}\right) &= G_+\left(z^{-1}\right)\bar{G}_-\left(z^{-1}\right) = G_+\left(z^{-1}\right)G_-\left(z^{-1}\right)z^{-d} \quad \text{or} \\
G &= G_+\bar{G}_- = G_+ G_- z^{-d},
\end{aligned}
\tag{12.12}
$$

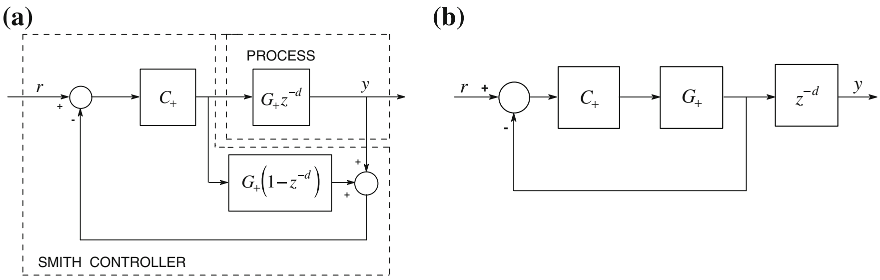

Fig. 12.5 The scheme of the sampled data SMITH controller

where G_+ is stable. For the DT process of (12.12) the SMITH-predictor principle discussed in Chap. 7.1. is shown by the control system given in Fig. 12.5a. Since this control loop is equivalent with the scheme given in Fig. 12.5b, the goal of the control is clearly seen, to separate the original closed-loop containing the delay to a delay-free closed-loop and the delay appears serially connected. So the controller C_+ for the process G_+ can also be designed by a conventional method (not taking the delay into account).

Figure 12.5a can be redrawn for the equivalent forms (a) and (b) of Fig. 12.6 by simple block-manipulations.

The *IMC* structure of Fig. 12.6a clearly shows that the SMITH controller is a YOULA-parameterized special controller with YOULA parameter

$$Q_+ = \frac{C_+}{1 + C_+ G_+} = \frac{C_+ G_+}{1 + C_+ G_+} G_+^{-1} = \frac{L_+}{1 + L_+} G_+^{-1} = R_+ G_+^{-1}, \qquad (12.13)$$

if the controller C_+ stabilizes the delay-free part G_+ of the process. Here $L_+ = C_+ G_+$ is the loop transfer function of the closed-loop shown in Fig. 12.5b, furthermore the complementary sensitivity function

$$T_+ = R_+ = \frac{L_+}{1 + L_+} \qquad (12.14)$$

will be the reference model R_+.

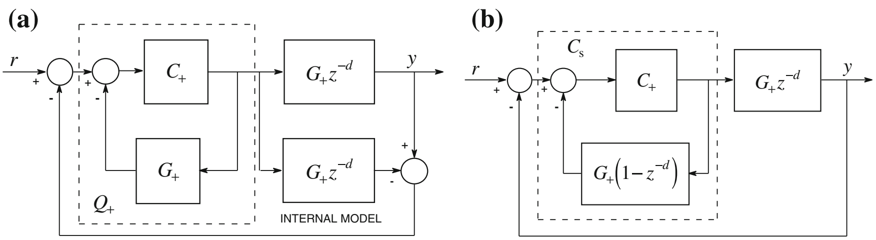

Fig. 12.6 Schemes of the equivalent sampled data SMITH controller

Figure 12.6b shows the equivalent complete closed-loop, where the *YP* sampled data serial controller is

$$C_s = \frac{Q_+}{1 - Q_+ G_+ z^{-d}} = \frac{C_+}{1 + C_+ G_+ (1 - z^{-d})} = C_+ K_S \tag{12.15}$$

the form of which, at the same time, suggests the realization of the inner closed-loop representing the mode of the realizability. Here K_S represents the serial transfer function by means of which the SMITH controller modifies the effect of the original controller C_+. Thus

$$K_S = \frac{1}{1 + C_+ G_+ (1 - z^{-d})} = \frac{1}{1 + L_+ (1 - z^{-d})}. \tag{12.16}$$

Contrary to the CT systems, the realization of the sampled-time SMITH controller does not involve any difficulty in practice, since C_S can be easily realized in part or completely by computer aided systems (see the statements in the previous section about the linear DT filters).

12.3 The TRUXAL-GUILLEMIN Regulator for Sampled Data Systems

The TRUXAL-GUILLEMIN method can be applied to the design of the controller of *ODOF* sampled data control systems. According to this method, the prescribed design goal has to be formulated for the transfer function of the closed system, which is a process with delay

$$T = \frac{CG}{1 + CG} = \frac{CG_+ z^{-d}}{1 + CG_+ z^{-d}} = R_n z^{-d}, \tag{12.17}$$

where it is assumed that in the formula (12.1) of the DT process $G_- = 1$. Now, based on this condition the following simple algebraic equation is obtained for C

$$CG_+ = R_n + CG_+ z^{-d} R_n. \tag{12.18}$$

From this the controller can be chosen according to

$$C = \frac{R_n}{1 - R_n z^{-d}} (G_+)^{-1} = C_{TG}. \tag{12.19}$$

Observe that this form is equal to the basic case ($G_n = 1$, $G_- = 1$) of the sampled data YOULA controller (12.2). The controller can be realized according to

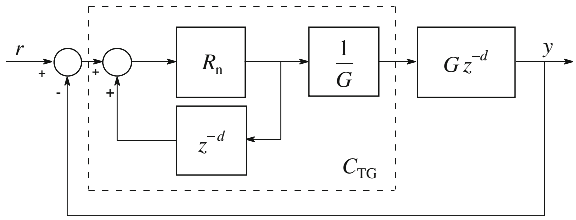

Fig. 12.7 The realization of the TRUXAL-GUILLEMIN regulator

Fig. 12.7, but there is no problem even with the complete realization of the formula (12.19) in computer aided control systems.

Thus R_n corresponds to one of the reference models of the YOULA method. For the *ODOF* case $R_n = R_r$. Let the reference model and the process be given in the forms $R_n = \mathcal{B}_n/\mathcal{A}_n$ and $G = \mathcal{B}/\mathcal{A}$, respectively. In this case the polynomial form of the controller is

$$C_{TG} = \frac{\mathcal{B}_n}{\mathcal{A}_n - \mathcal{B}_n} \frac{\mathcal{A}}{\mathcal{B}}. \tag{12.20}$$

The controller is realizable if the pole excess of R_n is greater than or equal to that of the process. It was seen in Chap. 11 for the DT case that the pole excess of the pulse transfer function of the process is one (in practice generally for zero order hold, thus in the case of unit step equivalent (*SRE*) transformation). Thus the controller (12.20) can be realized, in general, because R_n is usually chosen to ensure the necessary pole excess. If R_n has unit gain ($R_n(1) = 1$), then the controller is of 1-type.

12.4 Design of Regulators Providing Finite Settling Time

In the case of DT systems it is possible to design a controller which is able to track exactly the unit step reference signal within finite steps, or make the error signal zero in finite steps. This controller is called a *Dead-Beat* (*DB*) controller which provides finite settling time. Let us assume that in an *ODOF* control system the process is a relative prime $G = \mathcal{B}/\mathcal{A}$, and C_{DB} is the "*deadbeat*" controller to be designed. Assuming a unit step reference signal, its z-transform is $R(z) = z/(z-1) = 1/(1-z^{-1})$. The dead-beat control requires that the z-transform of the error must be a finite order polynomial $\mathcal{P}_e(z)$, i.e.,

$$E(z) = S(z)R(z) = \frac{1}{1 + C_{DB}G} R(z) = \left[1 - \frac{C_{DB}G}{1 + C_{DB}G}\right] R(z) = \mathcal{P}_e(z). \tag{12.21}$$

It follows from this that both the sensitivity function and the complementary sensitivity transfer functions must also be finite order polynomials, i.e., the polynomials

$$S = \left(1 - z^{-1}\right)\mathcal{P}_{\mathrm{e}}\left(z^{-1}\right) \quad ; \quad T = \frac{C_{\mathrm{DB}}G}{1 + C_{\mathrm{DB}}G} = 1 - \left(1 - z^{-1}\right)\mathcal{P}_{\mathrm{e}}(z) = \mathcal{P}_{\mathrm{y}}(z) \tag{12.22}$$

have finite order. Similarly it has to be required that the complementary sensitivity transfer function referring to the output of the controller must be a finite order polynomial

$$\frac{C_{\mathrm{DB}}}{1 + C_{\mathrm{DB}}G} = \mathcal{P}_{\mathrm{u}}(z). \tag{12.23}$$

This kind of transfer function is usually said to be a *finite impulse response* (*FIR*) type, also known as a *moving-average filter*. Based on the above, we may write

$$T = \frac{C_{\mathrm{DB}}G}{1 + C_{\mathrm{DB}}G} = \mathcal{P}_{\mathrm{y}}(z) = \mathcal{P}_{\mathrm{u}}(z)\frac{\mathcal{B}}{\mathcal{A}}, \tag{12.24}$$

whence the condition for the dead-beat control is

$$\frac{\mathcal{B}(z)}{\mathcal{A}(z)} = \frac{\mathcal{P}_{\mathrm{y}}(z)}{\mathcal{P}_{\mathrm{u}}(z)} = G, \tag{12.25}$$

which—in the case of relative prime process polynomials—can be fulfilled if

$$\mathcal{P}_{\mathrm{y}}(z) = \mathcal{M}(z)\mathcal{B}(z) \quad \text{and} \quad \mathcal{P}_{\mathrm{u}}(z) = \mathcal{M}(z)\mathcal{A}(z). \tag{12.26}$$

Since in steady state the error is zero, the condition $\mathcal{P}_{\mathrm{y}}(1) = 1$ must be fulfilled. As a consequence the gain of the design polynomial $\mathcal{M}(z)$ must be

$$\mathcal{M}(1) = \frac{1}{\mathcal{B}(1)}. \tag{12.27}$$

Finally, based on (12.23), (12.24) and (12.26), the controller has the form

$$C_{\mathrm{DB}} = \frac{\mathcal{P}_{\mathrm{u}}}{1 - \mathcal{P}_{\mathrm{u}}G} = \frac{\mathcal{M}\mathcal{A}}{1 - \mathcal{M}\mathcal{B}}. \tag{12.28}$$

Thus the most important step of the design of a dead-beat control is the choice of the design polynomial $\mathcal{M}(z)$. The dead-beat behavior for the input and the output of the process is given by (12.26). It is worth investigating the forms of the signals on the basis of (12.21) and (12.28):

$$\mathcal{P}_e(z) = \frac{z(1-\mathcal{M}\mathcal{B})}{z-1} = \frac{1-\mathcal{M}\mathcal{B}}{1-z^{-1}} = \mathcal{N}. \tag{12.29}$$

Here, (12.27) is taken into account, according to which the factor $(1-\mathcal{M}\mathcal{B})$ has always the root $z=1$, since $1-\mathcal{M}(1)\mathcal{B}(1)=0$.

Equation (12.28) has also the forms

$$C_{DB} = \frac{\mathcal{M}\mathcal{A}}{1-\mathcal{M}\mathcal{B}} = \frac{\mathcal{P}_u}{1-\mathcal{P}_y} = \frac{\mathcal{P}_y}{1-\mathcal{P}_y}\frac{\mathcal{A}}{\mathcal{B}} = \frac{R_n}{1-R_n}\frac{\mathcal{A}}{\mathcal{B}}, \tag{12.30}$$

where the substitution $R_n = \mathcal{P}_y$ is applied. Thus the same form is obtained as the Truxal-Guillemin regulator (12.19) or the basic case of the Youla regulator (7.9). The significant difference is that now R_n is a *FIR* filter, thus it is a polynomial and (12.29) must also be fulfilled.

Let us summarize the applied restrictions concerning the design of a dead-beat controller:

- it is assumed that the process to be controlled is stable
- the reference signal of the closed-loop is assumed to be unit step
- the dead-beat behavior is valid only at the sampling instants.

Note that if the above conditions are not fulfilled, the dead-beat controller design can still be performed in certain cases (e.g., by polynomial or state-space techniques), but it can not be made by the simple and clear design methods to be presented next.

Example 12.2 The method is presented for a second order CT process with dead-time. Let the transfer function of the CT process be

$$P(s) = \frac{e^{-s}}{(1+10s)(1+5s)}. \tag{12.31}$$

The first step is to discretize the CT process by a zero-order hold term under the sampling time $T_s = 1$ s

$$G(z) = \frac{\mathcal{B}(z)}{\mathcal{A}(z)} = \frac{0.0091(z+0.9048)}{(z-0.9048)(z-0.8187)z} \tag{12.32}$$

(*SRE* transformation). Notice that the factor z represents the delay supposing a sampling time of $T_d = 1$ s, since $T_s = T_d$. The effect of the delay can be better seen from the form

$$G(z) = \frac{0.0091(z+0.9048)}{(z-0.9048)(z-0.8187)} z^{-1}. \tag{12.33}$$

Regarding the goal of the design, the following considerations can be made. Without a delay, the effect of the input signal $u[0] \neq 0$, appearing at the sampling time $k = 0$ at the input of the zero order hold, will appear in the output—at the earliest—at the sampling time $k = 1$ due to the order of the process. Consequently, if the delay is $T_d = 1$ s, then the effect of the input $u[0] \neq 0$ will appear—at the earliest—at the sampling time $k = 2$. As a consequence, the best tracking control that can be constructed for the unit step reference signal $y_r[k] = 1[k]$ turns out to be, in discrete form, $y[k] = 1[k-2]$. Thinking in terms of the pulse transfer function, in this example the condition

$$T = \frac{C(z)G(z)}{1+C(z)G(z)} = \mathcal{P}_y(z) = z^{-2}$$

is required for the transfer function of the closed system (12.24), from which the controller is:

$$C(z) = \frac{\mathcal{P}_y}{1-\mathcal{P}_y}\frac{\mathcal{A}}{\mathcal{B}} = \frac{z^{-2}}{(1-z^{-2})}\frac{1}{G(z)} = \frac{1}{G(z)(z^2-1)} = C_{DB}.$$

Expressing the controller in terms of the polynomials of the pulse transfer function of the process:

$$C(z) = \frac{1}{G(z)(z^2-1)} = \frac{\mathcal{A}(z)}{\mathcal{B}(z)(z^2-1)} = \frac{109.9(z^3-1.7236z^2+0.7408z)}{z^3+0.9048z^2-z-0.9048}. \tag{12.34}$$

It can be clearly seen that for a unit step reference signal, the steady-state output can be expected to be error-free, since the loop transfer function $L(z) = C(z)G(z)$ has a pole at $z = 1$, or in other words the controller contains an integrator. The

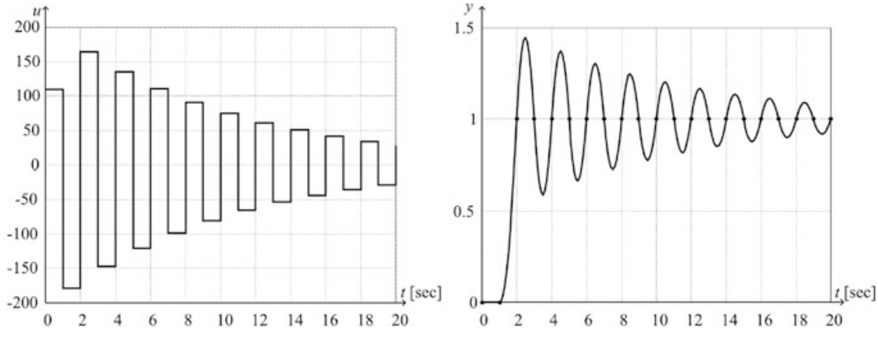

Fig. 12.8 Dead-beat control (2 steps)

behavior of the closed-loop is shown in Fig. 12.8. Considering the discrete-time instants, the result is the same as is expected for the output, but in the case of sampled-time systems the quality of the closed control loop is determined by the continuous output signal, which, however, shows an unacceptable oscillation. Furthermore the actuator signal does not have a dead-beat character: its changes have extreme dynamics, because the condition (12.26) is not fulfilled.

Investigate the hidden oscillations between the sampling times, which is called in the English language literature "*intersampling ripples*". It can be seen in the time diagrams that the oscillation of the CT output is caused by the oscillation of the step-like inputs generated by the zero-order hold term. These values are produced by the regulator due to the fact that it has a so-called slightly (or under) damped pole $p_1 = -0.9048$. The relevance of this qualification can be explained in two ways. Strictly investigating the effect of the specific pole, consider the pulse transfer function

$$G_1(z) = \frac{z+1}{z+0.9048} \tag{12.35}$$

and its step response is shown in Fig. 12.9.

Based on the figure it can be asked, considering only the pulse transfer function itself of the controller, where are those poles which will produce a stable, well attenuated step response, i.e., they do not generate an oscillating output from a unit step input. In the case of CT systems, the regions of the well- and under-damped poles are separated by lines belonging to constant attenuations (damping factors) in the complex frequency domain of s. These lines make a constant angle φ (which depends on the damping) with the negative real axis: $\cos(\varphi) = \xi$. Now the mapping of these lines have to be found in the z-plane. In the s-plane the points $s = \sigma + j\omega$ are on the lines of constant damping with the condition $\omega = \sigma\sqrt{1-\xi^2}\big/\xi$. For a given damping, the mapping of $z = e^{sT_s}$ to $z = e^{\sigma + j\sigma T_s\sqrt{1-\xi^2}/\xi}$ can be calculated

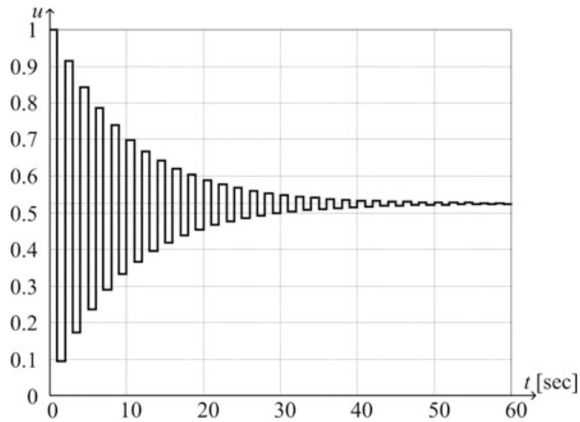

Fig. 12.9 The unwanted dynamics of the actuator signal

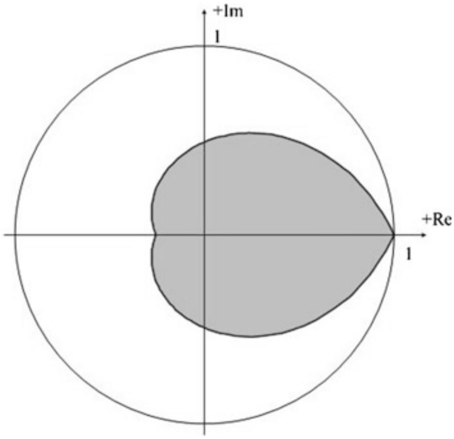

Fig. 12.10 The region of the well damped poles in the z-plane

and drawn for different values of σ. For example, the curve in the z-plane corresponding to the constant line $\xi = 0.4$ can be seen in Fig. 12.10. This well damped ($\xi > 0.4$) region shown in the figure is also called the "heart form curve" in the literature. The formula $\xi = 1 \Big/ \sqrt{1 + \pi^2 \Big/ [\ln(|p_1|)]^2}$ is obtained after some long calculations showing how ξ depends on a root p_1 falling on the negative real axis.

Based on the previous investigations it can be stated that the oscillation is generated by the controller itself, because it can be clearly seen from

$$C(z) = \frac{\mathcal{A}(z)}{\mathcal{B}(z)(z^2 - 1)} \tag{12.36}$$

that $C(z)$ has the roots of the polynomial $\mathcal{B}(z)$ as its poles (in this case $\mathcal{B}(z)$ is of first order, i.e., it has only one root). The oscillating effect of the roots depends on their position relative to the heart-shaped figure belonging to a given damping. In the present case the root of $\mathcal{B}(z)$ is $z = -0.9048$, which is outside of the well damped region. In order to avoid oscillation the direct compensation of the slightly damped roots of $\mathcal{B}(z)$, i.e., simply saying their cancellation with the corresponding poles of the controller, has to be avoided. Separate the roots of $\mathcal{B}(z)$ in such a way that $\mathcal{B}_+(z)$ contains the well damped roots of $\mathcal{B}(z)$ (they are inside of the heart-shaped region) and $\mathcal{B}_-(z)$ contains the slightly damped roots (outside of the heart-shaped region)

$$\mathcal{B}(z) = \mathcal{B}_+(z)\mathcal{B}_-(z) \tag{12.37}$$

and the condition $\mathcal{B}_-(z)|_{z=1} = \mathcal{B}_-(1) = 1$ must be fulfilled. For the example

$$\mathcal{B}(z) = \mathcal{B}_+(z)\mathcal{B}_-(z) = 0.01733(0.525z + 0.475), \tag{12.38}$$

where $\mathcal{B}_+(z) = 0.01733$ (the polynomial $\mathcal{B}_+(z)$ has no well damped pole) and $\mathcal{B}_-(z) = 0.525z + 0.475$ (the polynomial $\mathcal{B}(z)$ has one slightly damped pole). Next let m_+ and m_- be the degrees of the polynomials $\mathcal{B}_+(z)$ and $\mathcal{B}_-(z)$, respectively.

During the design process, take the applied separation into account, and modify our expectation for the pulse transfer function of the closed discrete-time system:

$$T = \frac{C(z)G(z)}{1 + C(z)G(z)} = \mathcal{P}_y(z) = \mathcal{B}_-(z)z^{-2}z^{-m_-} = \mathcal{B}_-(z)z^{-m_- -2}, \tag{12.39}$$

so the assumption (12.26) is also fulfilled. From the above condition the controller becomes:

$$C(z) = \frac{\mathcal{A}(z)}{\mathcal{B}_+(z)[z^3 - \mathcal{B}_-(z)]} = \frac{57.7(z^3 - 1.7236z^2 + 0.7408z)}{z^3 - 0.525z - 0.475}. \tag{12.40}$$

From the above form it is easily seen why the condition $\mathcal{B}_-(z)|_{z=1} = \mathcal{B}_-(1) = 1$ has to be assumed during the separation of the polynomial $\mathcal{B}(z)$. For in this case, due to the fact that $[z^3 - \mathcal{B}_-(z)]_{z=1} = 0$, $z = 1$ is still the pole of the loop transfer function $L(z)$, i.e., it is of 1-type. The results obtained with the modified controller are illustrated in Fig. 12.11. The closed system is slowed down, the settling time increased from 2 to 3 s, in other words, from 2 steps to 3 steps, but, at the same time, the moderate dynamics of the actuator signal can be observed. The oscillation is completely eliminated, but the magnitude of the initial value of the actuator signal can not be considered acceptable for any kind of application.

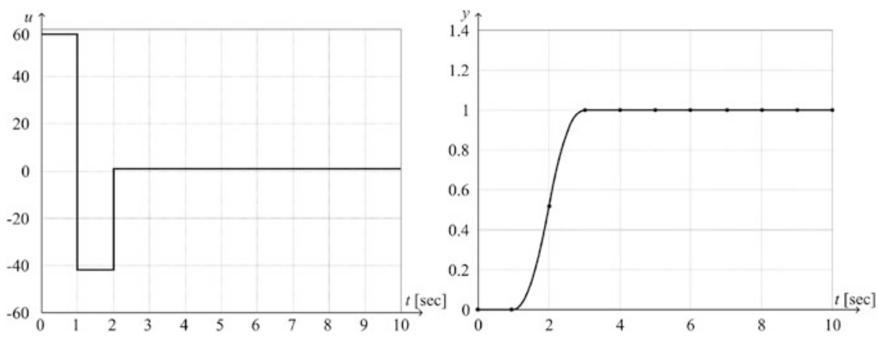

Fig. 12.11 Dead-beat controller (3 steps)

The slowing down can be performed in many different ways, from which such a solution is shown next, where the dead-beat character is kept, but the settling time is further increased depending on the degree of the design polynomial. Let the slowing design polynomial be

$$T(z) = \mathcal{P}_{\mathrm{y}}(z) = 0.2z^2 + 0.3z + 0.5, \tag{12.41}$$

of degree $m_{\mathrm{T}} = 2$, and the condition $T(z)|_{z=1} = T(1) = 1$ is used in the specification of the closed-loop pulse transfer function according to

$$\frac{C(z)G(z)}{1 + C(z)G(z)} = \mathcal{P}_{\mathrm{y}}(z)\mathcal{B}_{-}(z)z^{-2-m_{-}-m_{\mathrm{T}}} = \mathcal{P}_{\mathrm{y}}(z)\mathcal{B}_{-}(z)z^{-5}. \tag{12.42}$$

The controller becomes

$$C(z) = \frac{\mathcal{A}(z)\mathcal{P}_{\mathrm{y}}(z)}{\mathcal{B}_{+}(z)\left[z^5 - \mathcal{B}_{-}(z)\mathcal{P}_{\mathrm{y}}(z)\right]} = \frac{11.54\left(z^5 - 0.2235z^4 + 0.6555z^3 - 3.198z^2 + 1.852z\right)}{z^5 - 0.105z^3 - 0.2525z^2 - 0.405z - 0.2375}$$

The operation of the closed-loop in the time domain is seen in Fig. 12.12.

In connection with the controller of (12.30) it has already been mentioned that the design equation of the dead-beat controller actually corresponds to the basic case (7.9) of the Youla regulator. For comparison with the general case (12.2), consider the formula (12.1) of the DT process according to the above separation (12.37),

$$G = G_{+}G_{-}z^{-d} = \frac{\mathcal{B}_{+}\mathcal{B}_{-}}{\mathcal{A}}z^{-2}, \tag{12.43}$$

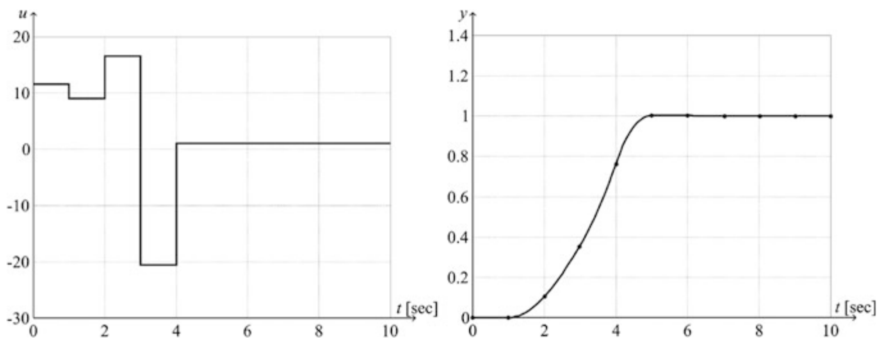

Fig. 12.12 Dead-beat controller (5 steps)

where

$$\mathcal{B}_+(z) = 0.01733 \quad \text{and} \quad \mathcal{B}_-(z) = 0.525z + 0.475 \tag{12.44}$$

Here $\mathcal{B}_+$ is the factor having acceptable inverse form, $\mathcal{B}_-$ is the factor having non-acceptable inverse form in the numerator of the pulse transfer function. Notice that the inverse of $\mathcal{B}$ is not unstable now, but is underdamped, therefore it is also considered an unwanted factor. The form of the optimal controller obtained for the general case according to (12.2) with the choice $G_r = G_n = 1$ is

$$\begin{aligned} C_{\text{opt}} &= \frac{R_n G_+^{-1}}{1 - R_n G_- z^{-d}} = \frac{\mathcal{P}_y G_+^{-1}}{1 - \mathcal{P}_y G_- z^{-d}} = \frac{\mathcal{P}_y \mathcal{A}}{\mathcal{B}_+\left(1 - \mathcal{P}_y \mathcal{B}_- z^{-d}\right)} \\ &= \frac{z^{-2}\mathcal{A}}{\mathcal{B}_+\left(1 - z^{-2}\mathcal{B}_- z^{-1}\right)}, \end{aligned} \tag{12.45}$$

which is completely the same as the controller of (12.40),

$$C(z) = \frac{\mathcal{A}(z)}{\mathcal{B}_+(z)[z^3 - \mathcal{B}_-(z)]} = \frac{57.7(z^3 - 1.7236z^2 + 0.7408z)}{z^3 - 0.525z - 0.475}. \tag{12.46}$$

It is confirmed again that the Youla-controller is generally valid for stable processes.

■

12.5 Predictive Controllers

Assume that the pulse transfer function of a control system in an *ODOF* loop is

$$G\left(z^{-1}\right) = \frac{\mathcal{B}(z^{-1})}{\mathcal{A}(z^{-1})} z^{-d} = G_+\left(z^{-1}\right) G_-\left(z^{-1}\right) \quad ; \quad G_-\left(z^{-1}\right) = z^{-d}, \tag{12.47}$$

which corresponds to a CT process with dead-time. A relationship is sought by which the value of the output signal at the sampling time $k + d$ can be estimated from the information available up to the sampling time k. To achieve this let us introduce a special polynomial equations

$$1 = \mathcal{A}\mathcal{F} + \mathcal{P}z^{-d} \tag{12.48}$$

whose solution is unambiguous, seeking $\mathcal{F}$ of degree $(d - 1)$ and $\mathcal{P}$ of degree $(n - 1)$, if $\mathcal{A}$ has degree n. Equation (12.48) is a special form of the *DE* discussed in Chap. 10. Using equivalent rewriting $G(z^{-1})$ can be decomposed as

$$G=\frac{\mathcal{B}}{\mathcal{F}}z^{-d}=\frac{\mathcal{BAF}+\mathcal{BP}z^{-d}}{\mathcal{A}}z^{-d}=\left(\mathcal{BF}+\frac{\mathcal{BP}z^{-d}}{\mathcal{A}}\right)z^{-d}$$
$$=\mathcal{BF}z^{-d}+\mathcal{P}\left(\frac{\mathcal{B}}{\mathcal{A}}z^{-d}\right)z^{-d}=\mathcal{BF}z^{-d}+\mathcal{P}Gz^{-d} \quad (12.49)$$

Apply both sides to the series of the input signal $u[k]$

$$y[k]=\mathcal{BF}z^{-d}u[k]+\mathcal{P}z^{-d}Gu[k]=\mathcal{BF}u[k-d]+\mathcal{P}z^{-d}y[k]$$
$$=\mathcal{BF}u[k-d]+\mathcal{P}y[k-d]=y[k|k-d] \quad (12.50)$$

The equation can also be written for sampling time $k+d$

$$y[k+d]=\mathcal{BF}u[k]+\mathcal{P}y[k]=y[k+d|k], \quad (12.51)$$

where $y[k+d|k]$ is the estimate or prediction of the series $y[k]$ for the sampling time $k+d$. Notice that the prediction is error-free and it uses only the information available at the time instant k concerning both the input and output. Both polynomials $\mathcal{BF}$ and, $\mathcal{P}$ are functions of z^{-1} and in (12.51) their coefficients weight only the current and the previous values of the signals u and y. Based on the d-step predictor a special, so-called predictive controller can be constructed. If the goal is to track the output of a reference model R_r, then the equation of the controller is

$$R_r y_r[k]=y[k+d|k]=\mathcal{BF}u[k]+\mathcal{P}y[k] \quad (12.52)$$

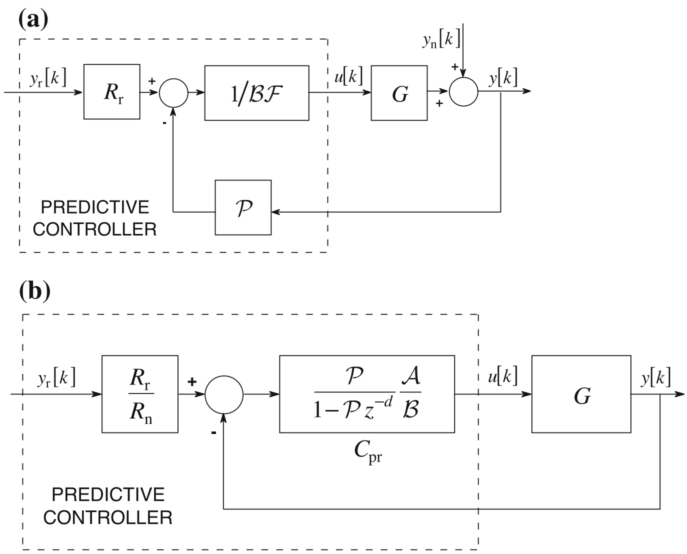

Fig. 12.13 Forms of predictive controllers

from which the input signal is

$$u[k] = \frac{R_r y_r[k] - \mathcal{P} y[k]}{\mathcal{B}\mathcal{F}} \tag{12.53}$$

The direct mapping of (12.53) can be seen in Fig. 12.13a. The equivalent block scheme of the Fig. 12.13b, however, shows the conventional closed-loop control.
Thus the predictive controller has the form

$$C_{\mathrm{pr}} = \frac{\mathcal{P}}{1 - \mathcal{P} z^{-d}} \frac{\mathcal{A}}{\mathcal{B}} = \frac{\mathcal{P}}{\mathcal{B}\mathcal{F}}, \tag{12.54}$$

i.e. the reference model for the disturbance rejection is now

$$R_n = \mathcal{P}, \tag{12.55}$$

which does not depend on the designer, but comes from (12.48). The predictive controller is formally equal to a YOULA regulator where $G_{-} = z^{-d}$. The transfer characteristic of the complete control loop is

$$y = R_r z^{-d} y_r + \left(1 - R_n z^{-d}\right) y_n = R_r z^{-d} y_r + \left(1 - \mathcal{P} z^{-d}\right) y_n \tag{12.56}$$

by means of which the predictive controller completely solves the problem for reference signal tracking, but R_n can not be designed. The d-step predictor of (12.51) is linear in its parameters, therefore it is easy to apply in parameter estimation (identification) techniques to determine the parameters of the controller. From the above control design principle, a new, widely applied computer controlled method has been developed, which is called *Model Predictive Control* (*MPC*).

For the noise rejection behavior of a closed system, a method is introduced which penalizes the change or variance of the input. So the dynamics of the closed-loop, though in a restricted way, can be acceptable by the proper choice of the penalty weights.

Example 12.3 Let the controlled system be a first order process with delay $d = 2$

$$G\left(z^{-1}\right) = \frac{\mathcal{B}(z^{-1})}{\mathcal{A}(z^{-1})} = \frac{0.7z^{-1} - 1.0z^{-2}}{1 - 1.5z^{-1} + 0.2z^{-2}} z^{-1} = \frac{0.7 - 1.0z^{-1}}{1 - 1.5z^{-1} + 0.2z^{-2}} z^{-2} \tag{12.57}$$

Compute the d-step ahead predictor by solving the DIOPHANTINE equation

$$1 = \mathcal{A}\mathcal{F} + \mathcal{P} z^{-d} \tag{12.58}$$

where $n = 2$ and the polynomials $\mathcal{F}$ and $\mathcal{P}$ are of degrees $d - 1$ and $n - 1$ respectively. The equation is

$$1 = \left(1 - 1.5z^{-1} + 0.2z^{-2}\right)\left(1 + f_1 z^{-1}\right) + \left(p_o + p_1 z^{-1}\right) z^{-2} \tag{12.59}$$

The solution is: $f_1 = 1.5$, $p_o = 2.05$ and $p_1 = -0.3$. So the predictive regulator is given by

$$C_{pr} = \frac{\mathcal{P}}{1 - \mathcal{P}z^{-d}}\frac{\mathcal{A}}{\mathcal{B}} = \frac{\mathcal{P}}{\mathcal{B}\mathcal{F}} = \frac{p_o + p_1 z^{-1}}{(0.7 - 1.0z^{-1})(1 + f_1 z^{-1})} = \frac{2.05 - 0.3z^{-1}}{0.7 + 0.05z^{-1} - 1.5z^{-2}} \tag{12.60}$$

■

12.6 The Best Reachable Discrete-Time Control

12.6.1 General Theory

The decomposition of the control error discussed in Sect. 7.5 is completely valid for DT systems, so all relationships can be applied in unchanged form.

It is worth noting that in the DT case the fastest reachable first order reference model can be easily determined under the amplitude restriction of the output of the controller

$$|u(t)| = U_{max} \tag{12.61}$$

if the YP controller is applied. Let the first order reference model with unit gain be

$$R_n = \frac{(1 + a_n)z^{-1}}{1 + a_n z^{-1}} = \frac{(1 + a_n)}{z + a_n}. \tag{12.62}$$

Let the first (so-called leading) coefficient in the numerator of the pulse transfer function of the process be b_1. Then the following restriction

$$\frac{1 + a_n}{b_1} \le U_{max} \tag{12.63}$$

must be fulfilled by the first biggest jump of the step response series, from which the maximum value of the coefficient a_n of the reference model is

$$a_n \le b_1\, U_{max} - 1. \tag{12.64}$$

12.6.2 Empirical Rules

It has already been seen in the recent discussion of the best reachable control, that the basic restriction derives from the saturation of the actuator or from the process dynamics itself. It can even be supplemented with the noise of the measurements. This noise may derive from the physical operation of the sensor, but also from the electronics, and in DT control from the A/D converter. The measurement noise usually appears in high frequency regions, therefore the uncertainty caused by them restricts the high frequency gain A_∞ of the controller. For simplicity, assume the A/D and D/A converters are those generally used, with 12 bits, which corresponds to 4096 levels. Thus a conversion error or measurement error of 1 bit, by being increased 4096 times, reaches the whole signal region.

In practice measurement changes greater than 5% are not allowed. This means that the high frequency gain of the controller must be smaller than 200, i.e., $A_\infty < 200$.