Chapter 5
Stability of Linear Control Systems

A general notion of stability says that a system is stable if after being removed from a stable state, the system returns to the original state provided no external input is applied. Another general notion of stability is called *BIBO* stability, i.e. bounded output is obtained as a response to any and all bounded inputs.

Consider the closed-loop system given in Fig. 5.1. The open-loop transfer function is $L(s)$ and a unity feedback is applied around $L(s)$.

The resulting transfer function of the closed-loop system is:

$$T(s) = \frac{L(s)}{1 + L(s)}$$

It is well known that all the components of the transient response will decay once the roots of the closed-loop characteristic equation $1 + L(s) = 0$ are located in the left half-plane side of the complex plane. The characteristic equation contains the denominator polynomial of the closed-loop transfer function above. Consequently, the roots of the characteristic equation are identical to the poles of the closed-loop system. Note that using state-space representations, this statement is only valid for systems which are both controllable and observable.

5.1 *BIBO* Stability

The *BIBO* stability criterion can be used to check the stability of linear systems. In practice a natural choice is to apply a unit step excitation as a bounded input.

Example 5.1.1 Assume we have the following closed-loop transfer function:

$$T(s) = \frac{s+5}{s^5 - 3s^4 + 4s^3 + 10s^2 + 5s - 10}.$$

L. Keviczky et al., *Control Engineering: MATLAB Exercises*,
Advanced Textbooks in Control and Signal Processing,
https://doi.org/10.1007/978-981-10-8321-1_5

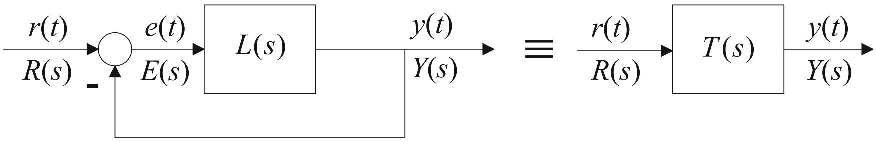


Fig. 5.1 Block diagram of a closed-loop control system

Applying a unit step input, check the stability of the closed-loop system. Use MATLAB™ commands, such as

```
num=[1, 5]
den=[1, -3, 4, 10, 5, -10]
T=tf(num,den)
step(T)
```

It can be seen that the step response will not be bounded, meaning that the closed-loop system is unstable.

5.2 Stability Analysis Based on the Location of the Closed-Loop Poles

One way to obtain the system output in analytical form is to derive the partial fraction expansion form of the Laplace transform of the transfer function. Then the analytical solution of the output signal in the time domain is simply obtained by performing an inverse Laplace transformation. In more detail, the Laplace transform of the output signal is the sum of the components that take the form $\frac{r_i}{s-p_i}$, where p_i denotes a system pole. Consequently, the system's stability can be determined based on the location of the system poles. A system turns out to be stable once all the poles are located in the left half-plane. Moreover, it can be seen whether oscillating components are expected to show up, as is indicated by the existence of complex conjugate poles in the left half-plane.

Example 5.2.1 Check the stability of the system introduced in Example 5.1.1. Check the location of the poles in this analysis:

```
poles=roots(den)
            poles =
             2.1150 + 2.1652i
             2.1150 - 2.1652i
            -0.9824 + 0.7214i
            -0.9824 - 0.7214i
             0.7348
```

or in another form

```
[zeros,poles,KonstGain]=zpkdata(T,'v')
```

Either way, it can be seen that the system has poles in the right half-plane.

Alternatively, the pzmap command shows the location of both the zeros and poles in graphical mode:

```
pzmap(T)
```

Note that complex poles produce oscillations in the output response. However, this can only barely be seen because of the dominant exponential growth.

5.3 Stability Analysis Using the Routh-Hurwitz Criterion

The poles of the closed-loop characteristic equation can be calculated analytically only for polynomials of degree less than five. If MATLAB™ is not available, higher degree equations need to be solved, which can only be solved numerically. If dead-time is included in the open-loop, the equation takes a transcendental form, causing further difficulties for the solution.

If the system is free of dead-time, methods have been developed to judge the stability based on relations between the roots and coefficients of the characteristic equation.

In the sequel assume that the closed-loop characteristic equation is given in the following form:

$$\mathcal{A}(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_o = a_n (s - p_1)(s - p_2)\ldots(s - p_n) = 0$$

5.3.1 Stability Analysis Using the Routh Scheme

Set up the following (so called Routh scheme) from the coefficients of the characteristic equation:

$$\begin{array}{ccccc} a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\ a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\ b_{n-2} & b_{n-4} & b_{n-6} & b_{n-8} & \cdots \\ c_{n-3} & c_{n-5} & c_{n-7} & c_{n-9} & \cdots \\ \vdots & & & & \end{array}$$

where

$$b_{n-2} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}, \quad b_{n-4} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}, \quad b_{n-6} = \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}}, \ldots$$
$$c_{n-3} = \frac{b_{n-2}a_{n-3} - a_{n-1}b_{n-4}}{b_{n-2}}, \quad c_{n-5} = \frac{b_{n-2}a_{n-5} - a_{n-1}b_{n-6}}{b_{n-2}}, \ldots$$

It can be seen that the length of the consecutive rows is getting shorter and shorter. For a characteristic equation of order n, the scheme consists of $n + 1$ rows. Elements with negative indices should be interpreted as elements whose value is zero.

The system is stable if all the coefficients of the characteristic equation are positive and all the elements in the first (leftmost) column of the ROUTH scheme are also positive. If there are changes in sign along the first column, the number of the sign changes equals the number of poles in the right half-plane (i.e. the number of unstable poles).

Example 5.3.1 The transfer function of a loop transfer function is

$$L(s) = \frac{K}{(1+10s)(1+5s)(1+s)(1+0.5s)}.$$

Find the critical value of the gain K (loop gain) yielding a stable closed-loop system. Consider first the characteristic equation. Start with defining $L(s)$:

```
s=tf('s')
den=(1+10*s)*(1+5*s)*(1+s)*(1+0.5*s)
     25 s^4 + 82.5 s^3 + 73 s^2 + 16.5 s + 1
```

The closed-loop characteristic equation becomes

$$1 + L(s) = 0 = 25s^4 + 82.5s^3 + 73s^2 + 16.5s + 1 + K$$

The coefficients in the ROUTH scheme are:

$$a_4 = 25; \quad a_3 = 82.5; \quad a_2 = 73; \quad a_1 = 16.5; \quad a_o = 1 + K;$$
$$b_2 = \frac{a_3a_2 - a_4a_1}{a_3} = \frac{82.5 \cdot 73 - 25 \cdot 16.5}{82.5} = 68;$$
$$b_o = \frac{a_3a_o - a_4a_{-1}}{a_3} = \frac{82.5(1+K) - 0}{82.5} = 1 + K;$$
$$c_1 = \frac{b_2a_1 - a_3b_o}{b_2} = \frac{68 \cdot 16.5 - 82.5(1+K)}{68} = 16.5 - 1.2132(1+K); \text{ and}$$
$$d_o = \frac{c_1b_o - b_2c_{-1}}{c_1} = b_o = 1 + K.$$

The ROUTH scheme can then be constructed:

$$\begin{array}{ccc} 25 & 73 & 1+K \\ 82.5 & 16.5 & \\ 68 & 1+K & \\ 16.5-1.2132(1+K) & 0 & \\ 1+K & & \end{array}$$

For stability, all the values in the first column must be positive. This is fulfilled when $-1 < K < 12.6$.

5.3.2 Stability Analysis Based on the HURWITZ Determinant

Using the coefficients of the characteristic equation construct the following (so called HURWITZ) determinant:

$$\begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\ a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\ 0 & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ 0 & a_n & a_{n-2} & a_{n-4} & \cdots \\ 0 & 0 & a_{n-1} & a_{n-3} & \cdots \\ \vdots & & & & \cdots \end{vmatrix}$$

Elements with negative indices will be taken to have the value zero.

The system is stable if all the coefficients of the characteristic equation are positive and all the sub-determinants along the main diagonal of the HURWITZ determinant are positive: $\Delta_i > 0$.

Example 5.3.2 Solve the problem discussed in Example 5.3.1 using the method of the HURWITZ determinant. Start with building up the HURWITZ determinant:

$$\begin{vmatrix} 82.5 & 16.5 & 0 & 0 \\ 25 & 73 & 1+K & 0 \\ 0 & 82.5 & 16.5 & 0 \\ 0 & 25 & 73 & 1+K \end{vmatrix}$$

The sub-determinants along the main diagonal are

$$\Delta_1 = 82.5 > 0; \quad \Delta_2 = 82.5 \cdot 73 - 16.5 \cdot 25 = 5610 > 0;$$
$$\Delta_3 = -(1+K)82.5^2 + 16.5 \cdot 5610 > 0; \text{ and}$$
$$\Delta_4 = (1+K)\Delta_3 > 0$$

The stability conditions from Δ_3 and Δ_4 are directly read: $-1 < K < 12.6$

Note that $K = -1$ would mean positive feedback and this result is obtained also for $a_o > 0$.

An additional problem:

In a closed-loop control system, the open-loop transfer function is

$$L(s) = \frac{K}{(1 + sT_1)(1 + sT_2)(1 + sT_3)}.$$

(a) Find the critical value (maximum for closed-loop stability) of the gain K, if $T_1 = 1$, $T_2 = 0.4$, $T_3 = 0.1$.
(b) Find the critical value of K, if $T_1 = T_2 = T_3 = T$.
(c) What pair of K and T_3 guarantee closed-loop stability if $T_1 = 1$ and $T_2 = 0.4$?

Plot the function $K_{krit} = f(T_3)$. Show that if $T_3 \to 0$ or if $T_3 \to \infty$, the closed-loop system remains stable even for an infinitely large loop gain.

Solve this problem using either the Routh scheme or the Hurwitz determinant.

5.4 Stability Analysis Based on the Root-Locus Method

The root-locus method is a grapho-analytical method to show the poles of the closed-loop system as one parameter (typically the loop gain) in the system varies from zero to infinity. Note that a zero loop gain means an open-loop system.

If the poles of the characteristic equation are sitting on the imaginary axis, the closed-loop system is just about to be unstable (borderline stability). The root-locus method can not only determine closed-loop stability, but can also yield information on the dynamics of the closed-loop system. Root-locus points on the negative real axis suggest aperiodic transients in the time domain, and root-locus stages with complex poles in the left half-plane indicate oscillatory behaviour with damping.

MATLAB™ offers the rlocus command to draw the root-locus. Another MATLAB™ command (rlocfind) is to be used to find the gain belonging to a given point of the root-locus. rlocfind puts up a crosshair cursor in the graphics window which is used to select a pole location on an existing root locus.

Example 5.4.1 Draw the root-locus of a system given by the open-loop transfer function

$$L(s) = \frac{K}{(1 + 10s)(1 + 5s)(1 + s)(1 + 0.5s)}$$

and find the critical value of the loop gain.

Set up the system with $K = 1$, then draw the root-locus. Then read the gain at the point where the root-locus is crossing the imaginary axis.

```
s=zpk('s')
L=1/((1+10*s)*(1+5*s)*(1+s)*(1+0.5*s))
rlocus(L)
rlocfind(L)
```

Now a left click on the critical point provides the value of the critical gain. Also, the exact coordinates of the selected point are shown. To derive an appropriate result it is worthwhile to zoom on the vicinity of the critical point before rlocfind is employed.

```
selected_point =
    0.0003 + 0.4466i
    ans = 12.5753
```

The root-locus is shown in Fig. 5.2. Show the step response of the closed-loop system at the critical value of the loop gain:

```
K=ans
t=0:0.01:40;
step(K*L/(1+K*L),t)
```

It can be seen that the system output exhibits oscillations with constant amplitude.

Example 5.4.2 Consider a system with the following loop transfer function:

$$L_1(s) = \frac{k}{s(s+2)(s+4)}$$

Sketch the root-locus and find the critical loop gain. Then study the root-locus after an additional zero is introduced in the loop transfer function.

$$L_{2,3}(s) = \frac{k(s+\alpha)}{s(s+2)(s+4)},$$

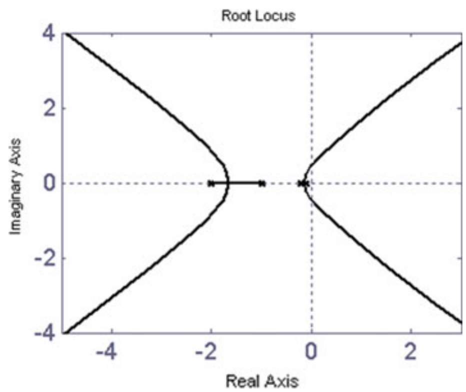


Fig. 5.2 Root-locus of a fourth-order system

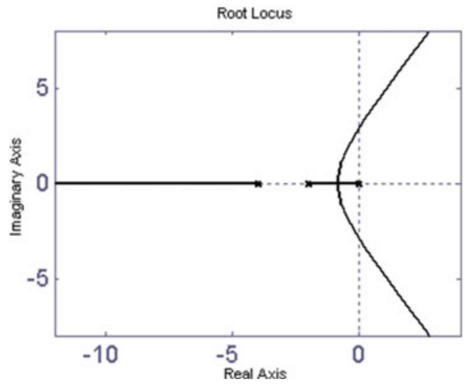


Fig. 5.3 Root-locus of a third-order system

where α takes the values of 3 or 1.

Discuss the stability issues of the extended system.

```
L1=1/(s*(s+2)*(s+4))
L2=L1*(s+3)
L3=L1*(s+1)
figure(1),rlocus(L1)
figure(2),rlocus(L2)
figure(3),rlocus(L3)
```

The root-locus for L_1 is shown in Fig. 5.3. To find the critical value of the loop gain zoom and use the command

```
rlocfind(L1)
```

Apply '*Select a point in the graphics window*' offered by MATLAB™, which results in

```
selected_point =
   0.0006 + 2.8252i
ans =
   47.9006
```

This critical value of k just obtained can also be checked by analytical tools.

It is seen that the inserted zero attracts one branch of the root-locus and the closed-loop becomes structurally stable in both cases ($L_{2,3}$ in Figs. 5.4 and 5.5).

Example 5.4.3 Consider the open-loop transfer function:

$$L(s) = \frac{k(s+4)(s+6)}{s(s+2)(s+8)}$$

Fig. 5.4 Root-locus of a third-order system with a zero

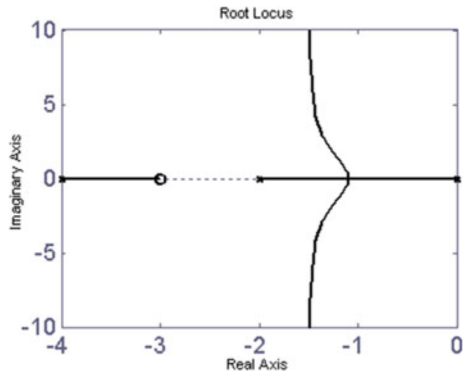
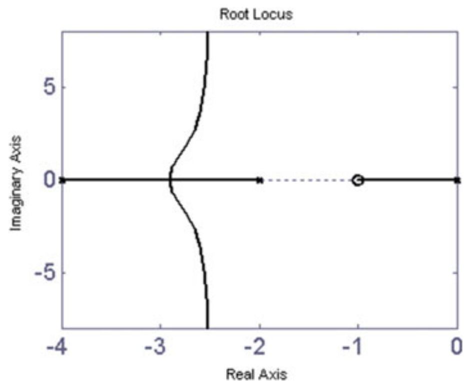


Fig. 5.5 Root-locus of a third-order system with a zero



Sketch the root-locus and evaluate the dynamic behaviour of the closed-loop system:

```
L=((s+4)*(s+6))/(s*(s+2)*(s+8))
rlocus(L)
```

The root-locus is shown in Fig. 5.6. Crossing the real axis can be obtained using rlocfind(L): -1.2 and -4.8865, and the corresponding loop-gain values are 0.4857 and 44.48, respectively. The closed-loop system is structurally stable, specifically for $0.4857 < k < 44.48$ the transient response will be determined by the complex conjugate closed-loop poles. Otherwise the transient response is aperiodic. Having two zeros and three poles, as the loop-gain grows to infinity one branch of the root-locus tends to go to infinity and the other two will converge to the finite zeros. Also, a point on the real axis is part of the root-locus once the total number of the zeros and poles located to the right from this point of the real axis is odd. A set of these point defines a complete region of the real axis.

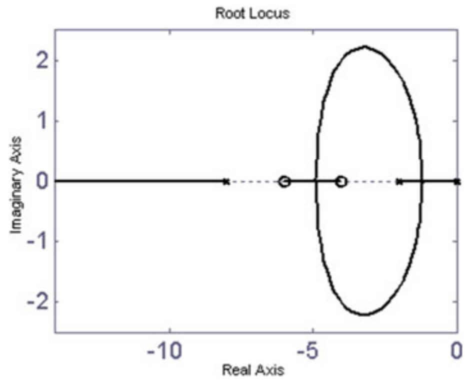


Fig. 5.6 Root-locus of a system with 3 poles and 2 zeros

Example 5.4.4 Let the transfer function of the loop transfer function be: $L(s) = \frac{k(s+2)}{(s-3)(s+5)(s+8)}$.

The open-loop is unstable as a consequence of an open-loop pole in the right half-plane. Can we stabilize the closed-loop system by applying feedback with a proper loop gain?

Draw the root-locus first:

```
L=(s+2)/((s-3)*(s+5)*(s+8))
rlocus(L)
```

The root-locus in Fig. 5.7 shows that all the poles of the closed-loop system will be in the left half-plane if the loop gain exceeds a certain (critical) value. Clearly, the closed-loop system can be stabilized once a sufficiently large loop gain is selected. The critical value of the loop gain can be determined using rlocfind(L) just as before. The critical loop gain will turn out to be 60.

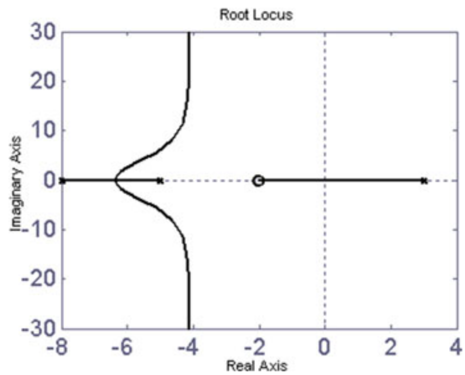


Fig. 5.7 Root-locus of a system with an unstable pole

Note that several root-locus curves can be drawn in the same figure. The root-locus can be drawn also for gain values given in a vector. The root-locus points and the coherent gains also can be obtained. The MATLAB™ commands are

```
rlocus(L,K)
rlocus(L1,L2,...)
rlocus(L1,'r',L2,'g:',L3,'mx')
[R,K]=rlocus(L)
```

5.5 Nyquist Stability Criterion

Having designed a closed-loop control system, stability is the most important attribute to be checked. Typically, we check the stability of a closed-loop system based on the behaviour of the open-loop system. Several methods exist to perform this step. The Nyquist stability criterion clarifies the stability issues of the closed-loop system given by $T(s) = \frac{Y(s)}{R(s)} = \frac{L(s)}{1+L(s)}$ based on the analysis of the frequency function of the open-loop transfer function $L(s) = \frac{Y(s)}{E(s)}$ (Fig. 5.8).

(a) To check the closed-loop stability, a simplified version of the Nyquist criterion can be employed if the open-loop transfer function has no unstable pole (pole with positive real part). The closed-loop system is then stable if the complete Nyquist diagram of the open-loop system does not encircle the point $(-1+0j)$ of the complex plane.

(b) To check the closed-loop stability, the generalized Nyquist criterion is to be employed if the open-loop transfer function has unstable poles (poles with positive real part). The closed-loop system is then stable if the number times $(-1+0j)$ is encircled by the complete Nyquist diagram of the open-loop system is equal to the number of unstable poles of the open-loop transfer function. The number of times a point encircled (*the winding number*) is considered to be positive when the path is traversed counter-clockwise. Note that the simplified Nyquist criterion is a special case of the generalized Nyquist criterion.

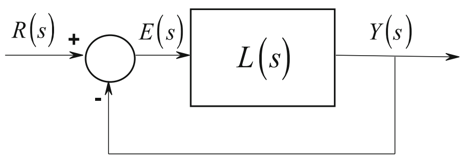


Fig. 5.8 Scheme of a closed-loop system

5.5.1 The Simplified Nyquist Stability Criterion

Example 5.5.1 Assume the open-loop transfer function is given by $L(s) = \frac{10}{(1 + 10s)(1+s)}$.
Use negative unity feedback. Check the stability of the closed-loop system using the simplified Nyquist criterion:

```
s=zpk('s')
L=10/((1+10*s)*(1+s))
[z,p,k]=zpkdata(L,'v')
```

It can be seen that the open-loop system has no unstable pole, thus the simplified Nyquist criterion is applicable.

```
nyquist(L),grid
```

The Nyquist diagram does not encircle the point $(-1+0j)$, so the closed-loop system is stable. Furthermore, we can conclude that the closed-loop system is structurally stable.

5.5.2 The Generalized Nyquist Stability Criterion

Example 5.5.2 Suppose given the open-loop transfer function $L(s) = \frac{-5}{(1-10s)(1 + 0.1s)}$.
Use negative unity feedback. Check the stability of the closed-loop system using the generalized Nyquist criterion:

```
s=zpk('s')
L=-5/((1-10*s)*(1+0.1*s))
[z,p,k]=zpkdata(L,'v')
```

Note that the open-loop system is unstable $(p_2 = 0.1)$.

```
nyquist(L)
```

The complete Nyquist diagram winds around $(-1+0j)$ in the positive sense (i.e., counter-clockwise). Consequently, the closed-loop system is stable.

Check this result by calculating the closed-loop poles:

```
T=feedback(L,1)
step(T)
[z,p,k]=zpkdata(T,'v')
pzmap(T)
```

Repeat this analysis when changing the sign of the open-loop poles:

$$L(s) = \frac{-5}{(1 + 10s)(1 - 0.1s)}$$

Look how the NYQUIST diagram will encircle $(-1 + 0j)$. Will the closed-loop system be stable?

Example 5.5.3 Consider the open-loop transfer function $L(s) = k\frac{1-s}{(1 + s)(1 + 0.5s)}$.

Negative unity feedback is applied. Find those values of the loop gain k which results in stable closed-loop system.

(a) To start with, assume $k = 1$:

```
L=(1-s)/((1+s)*(1+0.5*s))
[z,p,k]=zpkdata(L,'v')
```

All poles being stable allows us to use the simplified NYQUIST criterion.

```
nyquist(L); grid
```

Find k as the NYQUIST diagram crosses the real axis $(k = -0.666)$. Apply the `zoom` command or use the `zoom` option from the menu to read off this value. Increasing k will magnify the NYQUIST plot in the sense that all the points of the NYQUIST diagram will have an increased distance from the origin. The closed-loop system comes to borderline stability if the NYQUIST diagram crosses the real axis at -1. To achieve this, $k = 1/0.666 = 1.5$ is to be applied. So the closed-loop system will be stable for $0 < k < 1.5$ (if k is positive).

(b) Assume $k = -1$:

```
nyquist(-L), grid
```

Find again the stability region as before. Here $k > -1$ will be obtained.

Summing up the two conditions, we have $-1 < k < 1.5$ for the closed-loop stability.

5.6 Phase Margin, Gain Margin, Modulus Margin, Delay Margin

Beyond the fact that a system is stable, we are also interested in seeing how far we are from the borderline of stability. Several measures exist to characterize how far a stable system is from being unstable.

5.6.1 Phase Margin, Gain Margin

The phase margin defines the value of the phase angle needed to decrease the phase at the cut-off frequency to achieve borderline stability. The phase margin can be expressed analytically:

$$\varphi_t = \varphi(\omega_c) + 180^\circ$$

where ω_c is the cut-off frequency defined by

$$\omega_c \text{ is such that } |L(j\omega)|_{\omega=\omega_c} = 1$$

A closed-loop system is stable if the phase margin is positive. For example, if $\varphi = -120^\circ$ at the cut-off frequency, the phase margin is $\varphi_t = \varphi(\omega_c) + 180^\circ = -120^\circ + 180^\circ = 60^\circ$. This means that the closed-loop system is stable. For design purposes, 60° for the phase margin is a typical prescription.

The gain margin g_t is the factor by which the loop gain is to be multiplied to push a closed-loop system to borderline stability, i.e.

$$g_t = \frac{1}{|L(\omega_\pi)|}, \quad \text{where } \omega_\pi\text{: } \varphi(\omega)_{\omega=\omega_\pi} = -180^\circ$$

Here ω_π is that frequency where the phase angle is -180°. The closed-loop system is stable if the gain margin exceeds 1. A reasonable design prescription for the gain margin is around 2. MATLAB™ offers the margin command both to calculate and plot the phase and gain margin values.

Example 5.6.1 Given the open-loop transfer function:

$$L(s) = \frac{1}{(0.5 + s)(s^2 + 2s + 1)},$$

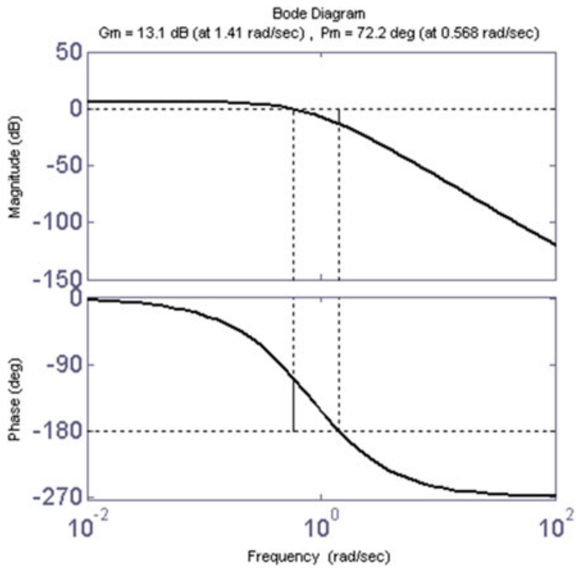
Apply negative unity feedback. What can we say about the stability? Find the phase margin (pm) and the gain margin (gm), as well as the cut-off frequency wc.

```
s=zpk('s')
L=1/((0.5+s)*(s^2+2*s+1))
```

(a) Method_1: Use the margin command

```
[gm,pm,wg,wc]=margin(L)
    gm=4.5001
    pm=72.227
    wg=1.4142
    wc=0.5675
```

Fig. 5.9 Characteristic points of the BODE diagram



Here wg is the frequency at which the phase shift of the open-loop frequency function is $-180°$.

To get a graphical evaluation (see Fig. 5.9) we have:

```
margin(L)
```

Note that if a graphical evaluation is selected, the gain margin Gm is given in decibels.

```
Gm=20*log10(gm)
```

(b) Method_2: use the BODE and NYQUIST diagrams

```
nyquist(L)
```

Read the crossing of the NYQUIST plot with the negative real axis. The gain margin is $g_t = \mathrm{gm} = \frac{1}{|L(j\omega_\pi)|} = \frac{1}{0.22} = 4.5$

```
bode(L)
```

Read the phase angle at the 0 dB (unity) gain (it is $-108°$). The phase margin is then obtained as $\varphi_t = \varphi(\omega_c) + 180° = -108° + 180° = 72°$.

The gain margin can be read off from the BODE diagram, just check the gain at ω_π. Clicking on the white background of the BODE diagram, select, using the right button *Properties->Units->Magnitude in—absolute*. The gain margin can be seen to be $g_t = \mathrm{gm} = \frac{1}{|L(j\omega_\pi)|} = \frac{1}{0.22} = 4.5$.

(c) Method_3: Read from a frequency-amplitude-phase table

Store the calculated points in a table then read the margins:

```
w=logspace(-1,1,100);
[num,den]=tfdata(L,'v')
[mag,phase]=bode(num,den,w);
Tabl=[mag, phase,w']
```

```
      Mag          phase           w
     1.1123      -99.5242        0.5094
     1.0643     -103.0406        0.5337
>>   1.0158     -106.6104        0.5591<<   ≈wc
     0.9669     -110.2286        0.5857

     0.2449     -176.7848        1.3530
>>   0.2211     -180.1658        1.4175<<   ≈wg
     0.1991     -183.4774        1.4850
     0.1789     -186.7160        1.5557
```

```
[mag,phase]=bode(L,w);
Tabl=[mag(:), phase(:), w']
```

LTI structure is interpreted for *MIMO* linear systems. This is why mag and phase are variables of three dimensions. In the case of *SISO* systems, the ":" operator converts the three dimensional structures to vector structures.

The phase margin can be calculated from the data in the table. To get the phase margin first the phase at the cut-off frequency should be read (the cut-off frequency is $w = wc = 0.5591$), then this phase value should be added to 180°: $pm = 180 - 106.6 = 73.4$. The gain margin turns out to be $gm = 1/0.221 = 4.52$, where 0.221 is the gain value belonging to the ω_π frequency.

Now investigate how a change in the loop gain will change the open-loop and closed-loop properties. To start, multiply the loop-transfer function by the gain margin 4.5.

```
Lk=4.5*L
nyquist(Lk)
Tk=Lk/(1+Lk)
step(Tk)
pzmap(Tk)
```

Modify the loop gain to 4.4 and then 4.6. For a loop gain like 4.6, one pole moves to the right half-plane, and the step response will diverge to infinity.

As far as the root-locus is concerned

```
rlocus(L)
```

allows sketching the critical value for the loop gain.

5.6.2 Delay Margin

The delay margin is the smallest value of the dead-time needed to push the system to borderline stability. It can be calculated as follows:

$$T_{\min} = \frac{\varphi_t}{\omega_c} = \frac{\mathsf{pm}}{\mathsf{wc}} \frac{[\text{rad}]}{[\text{rad/s}]}.$$

Example 5.6.2 Find the delay margin for the system discussed in Example 5.6.1.

```
Tmin=(72*pi/180)/0.56
   Tmin = 2.24
```

So a dead-time of 2.24 s can be inserted into the open-loop system to get borderline stability for the closed-loop system.

5.6.3 Modulus Margin

The modulus margin ρ_m is the minimum distance between the NYQUIST diagram and the point $(-1+0j)$. In other words, drawing a circle around the point $(-1+0j)$, the modulus margin will be the radius of the circle just touching the NYQUIST diagram. A practical specification for the modulus margin is $\rho_m > 0.5$. Alternatively, the modulus margin is identical to the reciprocal of the maximum of the absolute value of the sensitivity function.

$$\rho_m = \frac{1}{\max\limits_{\omega}|S(j\omega)|} = \min_{\omega}|1 + L(j\omega)|$$

Example 5.6.3 Find the modulus margin of the system discussed in Example 5.6.1.

```
M=bode(L+1)
ro=min(M)
   ro = 0.6317
```

5.7 Robust Stability

A closed-loop is robustly stable if it remains stable in spite of uncertainities in the process to be controlled.

For a process with nominal model $\hat{P}$, suppose that its real transfer function is P. Then the relative uncertainty is $\ell = \frac{P-\hat{P}}{\hat{P}}$. Robust closed-loop stability is achieved if $|\ell(j\omega)| < \frac{1}{|\widehat{T}_{\mathrm{m}}|}$ holds for all frequencies, where $\widehat{T}_{\mathrm{m}}$ is the maximum value of the complementary sensitivity function of the nominal closed-loop system. Note that the complementary sensitivity function equals the overall transfer function of the closed-loop system between the process output and the reference input. (See Eq. 5.44 in the textbook [1].)

Example 5.7.1 Consider the transfer function of a nominal process: $\hat{P}(s) = \frac{1}{(1+s)(1+5s)(1+10s)}$.

The time constants and the gain of the real process, however differ from those of the nominal transfer function:

$$P(s) = \frac{1.2}{(1+2s)(1+6s)(1+12s)}.$$

The following series regulator has been designed for the nominal process:

$$C(s) = 2.5\frac{1+10s}{10s}\frac{1+5s}{1+s}.$$

Check the stability of the closed-loop system containing the real process driven by the series regulator designed for the nominal process. The open-loop transfer function with the nominal process is determined by

$$L(s) = C(s)\hat{P}(s) = \frac{0.25}{s(1+s)^2}.$$

Check whether the condition of robust stability is satisfied:

```
s=zpk('s')
Pk=1/((1+s)*(1+5*s)*(1+10*s))
P=1.2/((1+2*s)*(1+6*s)*(1+12*s))
L=0.25/(s*(1+s)*(1+s))
T=L/(1+L)
T=minreal(T)
l=(P-Pk)/Pk
l=minreal(l)
w=logspace(-2,1,200);
[magT,phaseT]=bode(T,w);
```

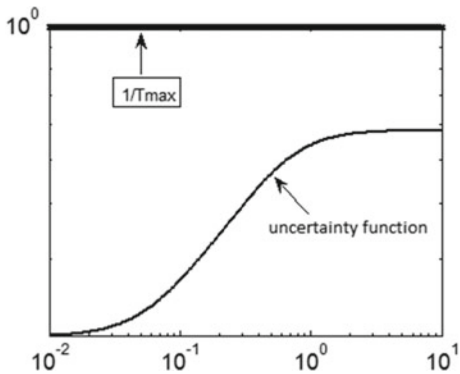


Fig. 5.10 Uncertainty function

```
Tmax=max(magT(:))
[magl,phasel]=bode(l,w);
loglog(w,magl(:),'r',w,1/Tmax,'bx')
```

Figure 5.10 shows that the frequency function of the relative uncertainty runs below the value of $1/\mathrm{Tmax}$ in the whole frequency range, so the condition for robust stability holds. The step responses of the nominal (solid line) and real (dotted line) system (Fig. 5.11) differ. The transient of the real system exhibits a higher overshoot, but it remains stable.

The MATLAB™ commands to calculate the step responses are:

```
C=2.5*(1+10*s)*(1+5s)/((10*s)*(1+s))
L1=C*P
T1=L1/(1+L1)
t=0:0.1:40;
y=step(T,t);
y1=step(T1,t);
plot(t,y,t,y1), grid
```

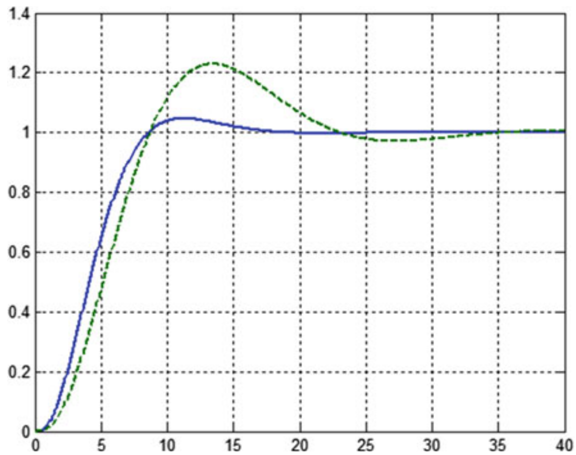


Fig. 5.11 Step responses of the nominal and the real closed-loop system

5.8 Internal Stability

A closed-loop system is internally stable (in the *BIBO* sense) if upon applying any bounded external excitation, all the internal signals in the system remain bounded. The reference signal, the disturbance acting at the input and the disturbance at the output of the process and the measurement noise signal are considered as external excitations.

For internal stability, all the entries in the transfer function matrix T_t must be stable:

$$T_t = \begin{bmatrix} \frac{CP}{1+CP} & \frac{P}{1+CP} \\ \frac{C}{1+CP} & \frac{1}{1+CP} \end{bmatrix}$$

Example 5.8.1 Consider the unstable process $P(s) = \dfrac{10}{s-1}$

Can we stabilize (in the internal sense) the closed-loop system using a regulator given by $C(s) = \dfrac{s-1}{s}$?

Find the transfer functions involved by using the T_t matrix:

```
s=zpk('s')
P=10/(s-1)
C=(s-1)/s
L=C*P, L=minreal(L)
T11=L/(1+L); T11=minreal(T11)
T12=P/(1+L); T12=minreal(T12)
T21=C/(1+L); T21=minreal(T21)
T22=1/(1+L); T22=minreal(T22)
```

The results give:

```
  10
------;
(s+10)

    10 s
------------;
(s-1)(s+10)

 (s-1)
------;
(s+10)

  s
------.
(s+10)
```

It can be seen that one of the transfer functions (T12) has a pole in the right half-plane, thus the system is internally unstable.

Note that direct cancellation of an unstable pole is not an accepted design procedure, as it results in unstable internal behaviour.

Example 5.8.2 Recap the design problem introduced in the previous example: $P(s) = \dfrac{10}{s-1}$.

Can we stabilize (in the internal sense) the closed-loop system using a regulator by $C(s) = \dfrac{s+1}{s}$?

Find the transfer functions involved by using the T_t matrix.

```
s=zpk('s')
P=10/(s-1)
C=(s+1)/s
L=C*P;L=minreal(L)
T11=L/(1+L);T11=minreal(T11)
T12=P/(1+L);T12=minreal(T12)
T21=C/(1+L);T21=minreal(T21)
T22=1/(1+L);T22=minreal(T22)
```

The elements of the transfer function matrix are:

```
       10 (s+1)
-------------------;
(s+1.298) (s+7.702)

        10 s
-------------------;
(s+1.298) (s+7.702)

    (s+1)(s-1)
-------------------;
(s+1.298) (s+7.702)

       s (s-1)
-------------------.
(s+1.298) (s+7.702)
```

The poles of each transfer function have negative real parts, so the closed-loop system has become internally stable.