

Chapter 3

State-Space Representation of Continuous Systems



In the time domain, linear systems can be characterized by their input and output signals. Far more information is obtained, however, if there are also considered those (mostly internal) signals whose value remains unchanged in case a step-like change is applied in the input signal. These signals represent information determined by the history of the development of the system and they do not exhibit sudden changes in their character. The set of these signals are called state variables (or states for short) and the models employing states are called state-space models (*SSM*). Using an *SSM*, the set $\{A, b, c^T, d\}$ exhibits an *SISO* system representation with input u , output y and state vector x in the model

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= c^T x + du\end{aligned}$$

where the matrices $\{A, b, c^T, d\}$ are parameter matrices describing the system. Formally, *SSM* describe linear systems with n first-order differential equations and the *SSM* consists of a set of these first-order differential equations as one single first order vector differential equation and an output equation. The state equation is a differential equation (so it needs to be solved), while the output equation is only a linear combination using the states and the input signal (Note that in the state-space model A is an $n \times n$ quadratic matrix, b is an $n \times 1$ column vector, c^T is a $1 \times n$ row vector, and d is an 1×1 dimension constant.).

3.1 State Transformation

It is well known that there are infinitely many of input-output equivalent *SSM* associated with a given transfer function. MATLAB™ offers one possible way to get a *SSM*, using the command `tf2ss` ('transfer function to state space').

Example 3.1.1 Start our discussion with the transfer function

$$H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+2)(s+1)}$$

and transform it to an *SSM*:

```

s = zpk('s')
H = 1/(s*s + 3*s + 2)
H = ss(H)
a =
      x1 x2
x1  -1  1
x2   0 -2
b =
      u1
x1   0
x2   1
c =
      x1 x2
y1  1  0

d =
      u1
y1  0

```

The step response can be obtained by

step(H);

The parameter matrices can also be retrieved from the *LTI* form:

[A,b,c,d] = ssdata(H)

and the step response can also be obtained from the parameter matrices:

step(A,b,c,d);

Further on, the parameter matrices can be retrieved from the polynomial form, as well:

```

num = 1, den = [1 3 2]
[A1, b1, c1, d1] = tf2ss(num, den)
  A1 =
      -3  -2
       1   0
  b1 =
       1
       0
  c1 =
       0   1
  d1 =
       0

```

It can be seen that the two state space representations are different from each other. Still they give the same input-output transfer function. Just to check this, derive the transfer functions from the state models:

```

Hchk = ss(A, b, c, d); Hchk = zpkm(Hchk)
Hchk1 = ss(A1, b1, c1, d1); Hchk1 = zpkm(Hchk1)

```

The polynomial form of the transfer function can be obtained from the state space form as

```

[num, den] = ss2tf(A1, b1, c1, d1)

```

From a given *SSM*, another different representation can be generated using a coordinate transformation in the state space. The state variables in the new coordinate-system can be obtained by a linear transformation (called a *similarity transformation*). In more detail, assuming T to be a non-singular quadratic transformation matrix and z the new state variable of the transformed system, the similarity transformation can be summarized in the following set of equations:

$$\begin{aligned}\dot{z} &= \tilde{A}z + \tilde{b}u \\ y &= \tilde{c}^T z + \tilde{d}u\end{aligned}$$

where

$$\begin{aligned}z &= Tx \quad \Rightarrow \quad x = T^{-1}z \\ \tilde{A} &= TAT^{-1}, \tilde{b} = Tb, \tilde{c}^T = c^T T^{-1}, \tilde{d} = d\end{aligned}$$

Note, that if T is constructed so that the column vectors of T^{-1} are the eigenvectors of A , the resulting matrix \tilde{A} will be a diagonal. In this case, the

transformation is called a canonical transformation and the transformed state model is said to be in parallel canonical form.

Example 3.1.2 Find the canonical form of the system introduced in Example 3.1.1.

To properly build up the transformation matrix first determine the eigenvectors and deposit them in a V matrix:

```
[V, ev] = eig(A1)
V =
    -0.8944    0.7071
     0.4472   -0.7071

ev =
    -2     0
     0    -1
```

Then apply the relations derived for the similarity transformation:

```
Ti = V; T = inv(V)
Ap = T*A1*T
bp = T*b1
cp = c1*T
dp = d1
```

It can be seen that A_p is a diagonal, as expected.

The above steps can be replaced by the following more compact commands:

```
[Ap, bp, cp, dp] = ss2ss(A1, b1, c1, d1, inv(V))
```

or

```
[Ap, bp, cp, dp] = canon(A1, b1, c1, d1, 'modal')
Ap =
    -2     0
     0    -1

bp =
    -2.2361
    -1.4142

cp =
     0.4472   -0.7071

dp =
     0
```

3.2 Solution of the State Equation by Analytical Methods

3.2.1 Solution of the State Equation in the Time Domain

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}\mathbf{b}u(\tau)d\tau = e^{At}\mathbf{x}(0) + \mathbf{x}_u(t)$$

where $\mathbf{x}_0(t)$ is the response to the initial conditions and $\mathbf{x}_u(t)$ the response to the input signal.

As e^{At} is defined by its TAYLOR series by

$$e^{At} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}(\mathbf{A}t)^2 + \dots + \frac{1}{n!}(\mathbf{A}t)^n + \dots$$

it can be obtained in closed analytical form if \mathbf{A} is a diagonal matrix. In this case, for typical input signals, like a unit step, the system response can be easily calculated. So it is worthwhile to transform both the system equations and the initial conditions to canonical form, then to derive the solution in this form, and finally to transform the results back into the original coordinate system.

Example 3.2.1 Let the initial conditions for the system introduced in Example 3.1.1

be given by $x_1(0) = 1; x_2(0) = 2$, or in vector form $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{x}_0$. Assume an input as a unit step: $u(t) = 1(t)$. Find $\mathbf{x}(t)$ for $t > 0$ in analytical form. Using a canonical transformation, the initial conditions for the state variables can be obtained as

$$\mathbf{x}_0 = [1 \ 2]'$$

$$\mathbf{z}_0 = \mathbf{T}*\mathbf{x}_0$$

where the transformation matrix calculated in Example 3.1.2 has been applied

$$\mathbf{T} = \begin{matrix} & -2.2361 & -2.2361 \\ -1.4142 & & -2.8284 \end{matrix}$$

The transformed initial conditions are

$$\mathbf{z}_0 = \begin{matrix} -6.7082 \\ -7.0711 \end{matrix}$$

Then for the first component of the state vector of the canonical form we have:

$$\mathbf{z}_0(t) = e^{Ap \cdot t} \begin{bmatrix} -6.7082 \\ -7.0711 \end{bmatrix} = e^{\begin{bmatrix} -2t & 0 \\ 0 & -t \end{bmatrix}} \begin{bmatrix} -6.7082 \\ -7.0711 \end{bmatrix} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -6.7082 \\ -7.0711 \end{bmatrix} = \begin{bmatrix} -6.7082e^{-2t} \\ -7.0711e^{-t} \end{bmatrix}$$

while the first component of the state vector of the original system turns out to be:

$$\mathbf{x}_0(t) = \mathbf{T}^{-1}\mathbf{z}(t) = \begin{bmatrix} -0.8944 & 0.7071 \\ 0.4472 & -0.7071 \end{bmatrix} \begin{bmatrix} -6.7082e^{-2t} \\ -7.0711e^{-t} \end{bmatrix} = \begin{bmatrix} 6e^{-2t} - 5e^{-t} \\ -3e^{-2t} + 5e^{-t} \end{bmatrix}.$$

The canonical state response for the unit step input becomes:

$$\mathbf{z}_0(t) = \int_0^t e^{A p(t-\tau)} \mathbf{b} \mathbf{p} u(\tau) d\tau = \int_0^t \begin{bmatrix} e^{-2(t-\tau)} & 0 \\ 0 & e^{-(t-\tau)} \end{bmatrix} \begin{bmatrix} -2.2361 \\ -1.4142 \end{bmatrix} 1(\tau) d\tau$$

which gives

$$\mathbf{z}_0(t) = \begin{bmatrix} -2.2361 \int_0^t e^{-2(t-\tau)} d\tau \\ -1.4142 \int_0^t e^{-(t-\tau)} d\tau \end{bmatrix} = \begin{bmatrix} -2.2361e^{-2t} \int_0^t e^{2\tau} d\tau \\ -1.4142e^{-t} \int_0^t e^{\tau} d\tau \end{bmatrix} = \begin{bmatrix} -1.11805(1 - e^{-2t}) \\ -1.4142(1 - e^{-t}) \end{bmatrix}$$

Transforming back to the original state coordinates yields

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{T}^{-1}\mathbf{z}(t) = \begin{bmatrix} -0.8944 & 0.7071 \\ 0.4472 & -0.7071 \end{bmatrix} \begin{bmatrix} -1.11805(1 - e^{-2t}) \\ -1.4142(1 - e^{-t}) \end{bmatrix} \\ &= \begin{bmatrix} 1 - e^{-2t} - 1 + e^{-t} \\ -0.5(1 - e^{-2t}) + 1 - e^{-t} \end{bmatrix} \end{aligned}$$

which leads to

$$\mathbf{x}(t) = \begin{bmatrix} e^{-2t} + e^{-t} \\ 0.5 + 0.5e^{-2t} - e^{-t} \end{bmatrix}$$

The overall state response then is the sum of the response to the initial conditions and the response to the input signal. The output signal of the system is simply determined by $y = \mathbf{c}^T \mathbf{x}$. Note the output can be calculated from the original or from the transformed state variables.

Example 3.2.2 For the problem discussed in Example 3.1.1, find the value of the state variables at $t = 5$ s. The initial values of the state variables are $x_1(0) = 1$ and $x_2(0) = 2$, or in vector form, $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{x}_0$. The input signal is zero. Using the MATLAB™ commands

```
A = [-3 -2; 1 0]
t = 5;
x0 = [1 2]'
x5 = expm(A*t) * x0
```

results in

```
x5 =
    -0.0334
     0.0336
```

The state response for the initial conditions can also be calculated using the initial command:

```
b = [1; 0]; c = [0 1]; d = 0;
Hv = ss(A, b, c, d)
[y, t, x] = initial(Hv, x0)
plot(t, x(:, 1), t, x(:, 2)), grid
```

Beyond the state variables as function of time they can be plotted as state trajectories, e.g. x_1 as a function of x_2 :

```
plot(x(:, 1), x(:, 2)), grid
```

A state trajectory allows us to think about the dynamic behaviour of the system.

3.2.2 Solution of the State Equation in the LAPLACE Operator Domain

The LAPLACE transform of the state vector is given by

$$\mathbf{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}U(s)$$

while the output signal is

$$Y(s) = [\mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d]U(s).$$

The transfer function then turns out to be

$$P(s) = \frac{Y(s)}{U(s)} = \mathbf{c}^T (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{b} + d$$

Remark No matter which state representation form of the system we start from, the transfer function is the same.

Example 3.2.3 Find the transfer function of the system introduced in Example 3.1.1.

The parameter matrices of the system are

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \\ \mathbf{b} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \mathbf{c} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \\ \mathbf{d} &= 0 \end{aligned}$$

Calculating the transfer function results in

$$P(s) = \mathbf{c}^T (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{b} + d = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 = \frac{1}{s^2 + 3s + 2}.$$

MATLAB™ simply follows the analytical expression:

```
I = [1 0; 0 1]  
H = c1*inv(s*I-A1)*b1 + d1
```

or simply employs the `ss2tf` command:

```
[num,den] = ss2tf(A,b,c,d)
```

Problem Show that the same result is obtained if the canonical forms are used.

Problem Solve Example 3.2.1 in the LAPLACE operator domain.

3.3 Controllability, Observability

Controllability (more precisely, state controllability) is a notion to answer the question whether all the states can or can not be controlled by the input signal arbitrarily and independently from each other. The output controllability gives an answer to the question if the output signal can be controlled by the input signal arbitrarily or not. Observability tells if the initial value of the state variables can or can not be determined based on an input-output record of a certain time period.

3.3.1 Determination of Controllability and Observability Assuming Canonical Form

Controllability and observability can be determined directly if the canonical forms are available. An *SISO* linear system is (state) controllable if in its diagonal parameter matrix \mathbf{A}_p all the values along the main diagonal (the eigenvalues) are different from each other and all the values in the vector \mathbf{b}_p are different from zero. An *SISO* linear system is observable if in its diagonal parameter matrix \mathbf{A}_p all the values along the main diagonal (the eigenvalues) are different from each other and all the values in the vector \mathbf{c}_p are different from zero.

Example 3.3.1 Define the system parameter matrices as follows:

$$\begin{aligned} \mathbf{A} &= [-1 \ -0.5 \ 0.5; \ 2 \ -3 \ 0; \ 2 \ -1 \ -2] \\ \mathbf{b} &= [2; 3; 1] \\ \mathbf{c} &= [0 \ 0 \ 1] \\ \mathbf{d} &= 0 \\ \mathbf{H} &= \text{ss}(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \end{aligned}$$

Find the canonical form of the system and check its controllability and observability.

To get the canonical form, first find the eigenvectors of the parameter matrix \mathbf{A} and collect them into a matrix \mathbf{V} :

$$[\mathbf{V}, \mathbf{eV}] = \text{eig}(\mathbf{A})$$

where \mathbf{eV} gives the eigenvalues.

A canonical transformation applies the inverse of \mathbf{V} as a transformation matrix:

$$\begin{aligned} \mathbf{T}_i &= \mathbf{V}; \quad \mathbf{T} = \text{inv}(\mathbf{V}) \\ \mathbf{A}_p &= \mathbf{T} * \mathbf{A} * \mathbf{T}_i \\ \mathbf{b}_p &= \mathbf{T} * \mathbf{b} \\ \mathbf{c}_p &= \mathbf{c} * \mathbf{T}_i \\ \mathbf{d}_p &= \mathbf{d} \end{aligned}$$

Note that \mathbf{A}_p is diagonal, as was expected.

The above results can also be obtained using more compact MATLAB™ commands:

```
[Ap, bp, cp, dp] = ss2ss(A, b, c, d, inv(V))
```

or

```
[Ap, bp, cp, dp] = canon(A, b, c, d, 'modal')
```

As a result the canonical representation becomes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1.73 \\ 0 \\ -2.23 \end{bmatrix} u$$

$$y = [0.57 \quad -0.707 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

Draw the block diagram of the system (Fig. 3.1). This is a *parallel representation* of the system. Notice that the gain is obtained by $b_1c_1 = 1.73 \cdot 0.57 = 1$. It is

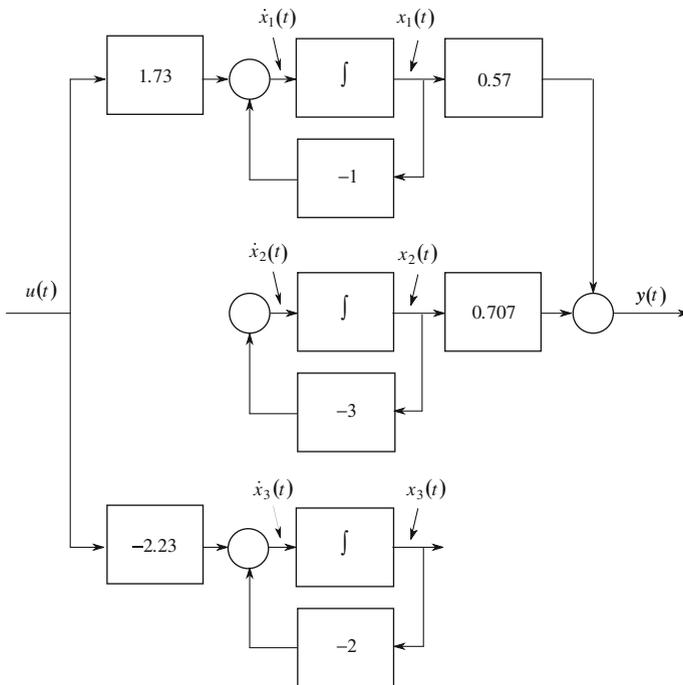


Fig. 3.1 Block diagram of a not fully state controllable and non-observable system

clear that any b_1 and c_1 satisfying $b_1c_1 = 1$ gives identical I/O equivalent state space representations.

The above parallel structure allows us to directly read the observability and controllability conditions. Namely x_3 is not observable (there is no information on x_3 in y) and x_2 is not controllable. The output signal is controllable as it is influenced by the input signal u through the state x_1 .

3.3.2 Determination of Controllability and Observability from Arbitrary (Non-canonical) Representations

Controllability and observability are analytically handled by checking the rank of the appropriate KALMAN controllability and observability matrices (see Sect. 3.4 of the textbook [1]).

The controllability matrix is built up as follows:

$$\mathbf{M}_c = [\mathbf{b} \quad \mathbf{A}\mathbf{b} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{b}].$$

A state-space representation is state controllable if the rank of the above matrix is n . Also, a state-space representation is output controllable if the row vector

$$\mathbf{m}_c^T = [\mathbf{c}^T\mathbf{b} \quad \mathbf{c}^T\mathbf{A}\mathbf{b} \quad \dots \quad \mathbf{c}^T\mathbf{A}^{n-1}\mathbf{b}] = \mathbf{c}^T\mathbf{M}_c$$

has at least one nonzero element (the rank of this matrix is equal to the number of the output signals).

The observability matrix is built up as follows: $\mathbf{M}_o = \begin{bmatrix} \mathbf{c}^T \\ \mathbf{c}^T\mathbf{A} \\ \vdots \\ \mathbf{c}^T\mathbf{A}^{n-1} \end{bmatrix}$.

A state-space representation is observable if the rank of the above matrix is n .

Example 3.3.2 Check the controllability and observability of the system introduced in Example 3.3.1. Note that this is a third order state-space representation ($n = 3$).

The controllability matrix can be obtained by MATLAB[™] using command `ctrb` as

$$\mathbf{M}_c = \text{ctrb}(\mathbf{A}, \mathbf{b})$$

or using the *LTI* structure:

$$\mathbf{M}_c = \text{ctrb}(\mathbf{H})$$

The system is state controllable if the rank of \mathbf{M}_c turns out to be $n = 3$.

$$\text{rank}(\mathbf{C}_o)$$

It is seen that this representation is not controllable, as $\text{rank}(M_c) = 2 < n = 3$. To check the output controllability use the following command:

rank(c*M_c)

As this value is 1, which is equal to the number of outputs (1), the system is output controllable.

To check the observability with MATLAB™ use the command **obsv**:

M_o = obsv(A, c)

M_o = obsv(H)

rank(M_o)

As $\text{rank}(M_o) = 2 < n = 3$, the system is not observable.

Find the transfer function of the system.

[num, den] = ss2tf(A, b, c, d)

or in zero-pole form:

H_{zpk} = zpk(H)

[z, p, k] = zpkmdata(H, 'v')

Retrieve the zero-pole-gain information from the zero-pole form:

```
z =
    -3.0000
    -2.0000
p =
    -3.0000
    -2.0000
    -1.0000
k =
    1.0000
```

Note the double pole-zero cancellation in the transfer function.

H_{zpk} = minreal(H_{zpk})

Using transfer function representation information gets lost, because the transfer function only reflects information on the controllable and observable subsystem of the complete state-space representation.

The **residue** command allows us to get the parallel transfer function form:

[num, den] = tfdata(H, 'v')

[r, p, k] = residue(num, den)

$$H(s) = \frac{0}{s+3} + \frac{0}{s+2} + \frac{1}{s+1}$$

Note that the transfer function $\frac{1}{s+1}$ only partly describes the system represented by the state-space equations. This subsystem, is the (state) controllable and observable part of the system.

Example 3.3.3 Consider the following third order system:

$$H(s) = \frac{8}{(s+2)^3}$$

The number of the states will be 3, however, as the system has repeated poles, these poles are not independent from each other, it is expected that the state model will not be controllable and will not be observable.

```

num = 8;
den = [1 6 12 8];
[A,b,c,d] = tf2ss(num,den)
[V,ev] = eig(A)
V =
    0.8729    -0.8729    -0.8729
   -0.4364     0.4364     0.4364
    0.2182    -0.2182    -0.2182
eV =
   -2.0000     0         0
         0   -2.0000     0
         0     0    -2.0000
    
```

It is seen the V eigenvectors are not linearly independent, consequently no canonical form can be obtained using similarity transformation. The following block diagram (Fig. 3.2) of the system helps us to come up with a close-to-parallel state-space representation.

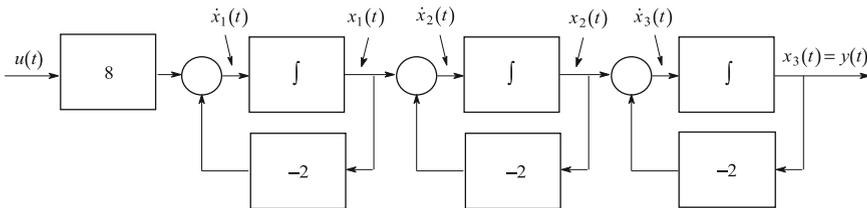


Fig. 3.2 State block diagram of a system with multiple poles

The related state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0 u$$

It is seen that in matrix A the poles show up along the main diagonal, however additional unity values accompany the pure diagonal form. This form is called JORDAN form.

Example 3.3.4 Consider the following closed-loop system (Fig. 3.3).

The process to be controlled is a first-order lag given by the transfer function:

$$H_2(s) = \frac{5}{10s + 1} = \frac{0.5}{s + 0.1}$$

This first-order lag can be equivalently redrawn as an integrator with a constant feedback (Fig. 3.4). Define the state variables as the output of the integrators in the complete system. The state equations can then be easily derived as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5.1 & 0.5 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 u$$

Check the controllability and observability of the above state-space representation of the closed-loop system.

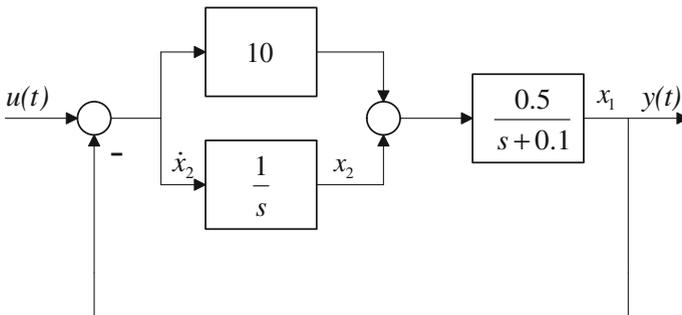


Fig. 3.3 Block diagram of a control system

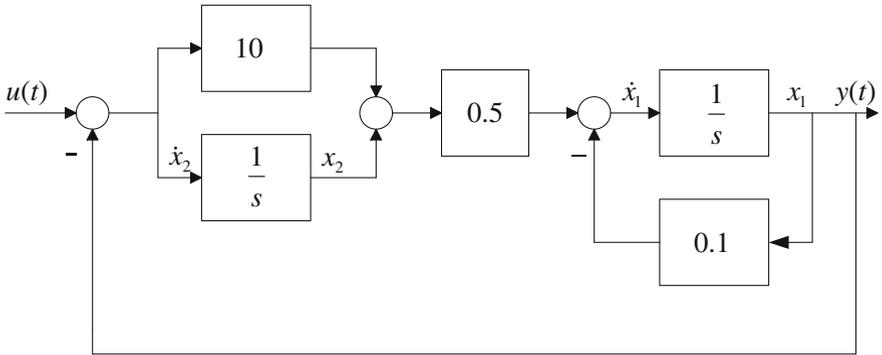


Fig. 3.4 Block diagram of the control system showing the state variables

The parameter matrices in MATLAB™ are given as

```
A = [-5.1 0.5; -1 0]
b = [5; 1]
c = [1 0]
d = 0
```

The rank of the controllability and observability matrices is calculated as

```
rank(ctrb(A,b))
ans = 1
```

The system is not (state) controllable,

```
rank(c*ctrb(A,b))
ans = 1
```

but the system is output controllable.

```
rank(obsv(A,c))
ans = 2
```

The system is observable.

The reason why the system is not controllable can be seen by analysing the dynamics of the regulator, namely the zero of the regulator cancels the pole of the process, so one state variable becomes “invisible”.

$$\mathbf{H} = \mathbf{ss}(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$$

$$\mathbf{H} = \mathbf{zpk}(\mathbf{H})$$

$$5 (s + 0.1)$$

$$(s + 5) (s + 0.1)$$

$$\mathbf{H} = \mathbf{minreal}(\mathbf{H})$$

$$5$$

$$(s + 5)$$