

Chapter 14

State Feedback in Sampled Data Systems



The design methods for controllers based on state feedback in the case of CT processes were discussed in Chap. 9. Next this methodology will be summarized for DT systems. For this purpose consider the state equation of a sampled data linear LTI process to be controlled using the results of Sect. 11.4 for the case $d = 0$.

$$\begin{aligned} \mathbf{x}[k + 1] &= \mathbf{F}\mathbf{x}[k] + \mathbf{g}u[k] \\ y[k] &= \mathbf{c}^T\mathbf{x}[k] \end{aligned} \tag{14.1}$$

The block scheme represented by the above equations is seen in Fig. 14.1.

Here $u[k]$ and $y[k]$ are the process input and output, respectively, and \mathbf{x} denotes the state vector. The equivalent pulse transfer function is now

$$G(z) = \mathbf{c}^T(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{g} = \frac{\mathcal{B}(z)}{\det(z\mathbf{I} - \mathbf{F})} = \frac{\mathcal{B}(z)}{\mathcal{A}(z)} = \frac{b_1z^{n-1} + \dots + b_{n-1}z + b_n}{z^n + a_1z^{n-1} + \dots + a_{n-1}z + a_n}. \tag{14.2}$$

A classical closed control loop directly applied to the state equation description is shown in Fig. 14.2, where the reference signal is denoted by $r[k]$. The closed-loop is formed by the feedback from the state vector via the linear proportional feedback vector \mathbf{k}^T in the form

$$u[k] = k_r r[k] - \mathbf{k}^T\mathbf{x}[k] \tag{14.3}$$

Based on Fig. 14.2, the state equation of the complete closed-loop system can be written as

$$\begin{aligned} \mathbf{x}[k + 1] &= (\mathbf{F} - \mathbf{g}\mathbf{k}^T)\mathbf{x}[k] + k_r \mathbf{g}r[k] \\ y[k] &= \mathbf{c}^T\mathbf{x}[k] \end{aligned} \tag{14.4}$$

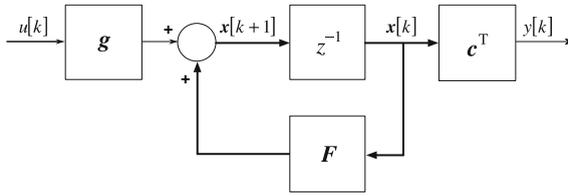


Fig. 14.1 Block scheme of the state equation of the linear time invariant discrete-time system

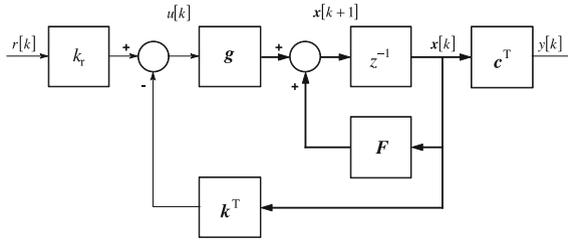


Fig. 14.2 Linear discrete-time control with state feedback

i.e., the dynamics concerning the original system matrix F is modified by the dyadic product gk^T to $(F - gk^T)$.

The transfer function of the closed control loop is

$$\begin{aligned}
 T_{ry}(z) &= \frac{Y(z)}{R(z)} = c^T(zI - F + gk^T)^{-1} gk_r = \frac{c^T(zI - F)^{-1} gk_r}{1 + k^T(zI - F)^{-1} g} \\
 &= \frac{k_r}{1 + k^T(zI - F)^{-1} g} G(z) = \frac{k_r B(z)}{A(z) + k^T \Psi(z) g}
 \end{aligned}
 \tag{14.5}$$

which comes from the comparison of the \mathcal{Z} -transforms $X(z) = (zI - F)^{-1} gU(z)$ [similarly to (3.12)], $U(z) = k_r R(z) - k^T X(z)$ [see (9.3)] and $Y(z) = c^T X(z)$ [see (9.1)], using the matrix inversion lemma (the proof is given in detail in A.9.1 of Appendix A.5 for CT systems). Notice that the state feedback leaves the zeros of the process unchanged, and only the poles of the closed system can be designed by k^T .

Introduce the so-called calibration factor k_r , by means of which the gain of the T_{ry} can be set to unity, i.e., $T_{ry}(1) = 1$. Obviously the open-loop is not an integrating one, so it can not yield zero error and static gain unity. In order to reach this the process parameters have to be known and the condition

$$k_r = \frac{-1}{\mathbf{c}^T (\mathbf{F} - \mathbf{g}\mathbf{k}^T)^{-1} \mathbf{g}} = \frac{\mathbf{k}^T \mathbf{F}^{-1} \mathbf{g} - 1}{\mathbf{c}^T \mathbf{F}^{-1} \mathbf{g}} \quad (14.6)$$

must be fulfilled [see A.9.2 of Appendix A.5]. The above special closed control loop is called state feedback.

14.1 Discrete-Time Pole-Placement State Feedback Regulator

The most natural design method regarding the state feedback is the so-called pole-placement. In this method the feedback vector \mathbf{k}^T has to be chosen to provide a prescribed polynomial $\mathcal{R}(z)$ for the characteristic equation of the closed system, e.g. in DT case,

$$\begin{aligned} \mathcal{R}(z) &= z^n + r_1 z^{n-1} + \dots + r_{n-1} z + r_n = \prod_{i=1}^n (z - z_i) = \det(z\mathbf{I} - \mathbf{F} + \mathbf{g}\mathbf{k}^T) \\ &= \mathcal{A}(z) + \mathbf{k}^T \Psi(z)\mathbf{g} \end{aligned} \quad (14.7)$$

The solution always exists if the process is controllable. If the transfer function of the system to be controlled is known, then it is an exceptional case, because the canonical state equations can be directly written. Based on the controllable control-able canonical forms (3.47) and (11.107) the system matrices can be obtained as

$$\begin{aligned} \mathbf{F}_c &= \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}; \quad \mathbf{c}_c^T = [b_1, b_2, \dots, b_n]; \\ \mathbf{g}_c &= [1, 0, \dots, 0]^T \end{aligned} \quad (14.8)$$

Taking the special forms of \mathbf{F}_c and \mathbf{g}_c it can be easily seen that according to the design equation

$$\begin{aligned}
\mathbf{F}_c - \mathbf{g}_c \mathbf{k}_c^T &= \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mathbf{k}_c^T \\
&= \begin{bmatrix} -r_1 & -r_2 & \dots & -r_{n-1} & -r_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{14.9}
\end{aligned}$$

the choice

$$\mathbf{k}^T = \mathbf{k}_c^T = [r_1 - a_1, r_2 - a_2, \dots, r_n - a_n] \tag{14.10}$$

ensures the characteristic equation Eq. (14.7), i.e., the prescribed poles. The value of the calibration factor can be given by a simple computation:

$$k_r = \frac{a_n + (r_n - a_n)}{b_n} = \frac{r_n}{b_n}. \tag{14.11}$$

It can be easily seen from the Eqs. (14.4) and (14.6) that the overall transfer function of the closed-loop system is

$$T_{ry}(z) = \frac{k_r \mathcal{B}(z)}{\mathcal{R}(z)} \tag{14.12}$$

in the case of state feedback pole placement, as was already mentioned in connection with (14.5).

Example 14.1. Consider an unstable process with transfer function

$$G(z) = \frac{-0.2z}{(z-0.8)(z-2)} = \frac{-0.2z^{-1}}{(1-0.8z^{-1})(1-2z^{-1})} = \frac{-0.2z}{z^2 - 2.8z + 1.6} = \frac{-0.2z}{\mathcal{A}(z)},$$

where $\mathcal{A}(z) = (z-0.8)(z-2) = z^2 - 2.8z + 1.6 = z^2 + a_1z + a_2$. To stabilize the process we should mirror the unstable pole outside the unit circle $p_2^d = 2$ inside the circle, i.e., select $p_2^d = 0.5$. The design polynomial $\mathcal{R}(z) = (z-0.8)(z-0.5) = z^2 - 1.3z + 0.4 = z^2 + r_1z + r_2$ ensures this goal. So the necessary stabilizing feedback vector is

$$\mathbf{k}^T = [r_1 - a_1 \quad r_2 - a_2] = [-1.3 - (-2.8) \quad 0.4 - (1.6)] = [1.5 \quad -1.2].$$

■

The most frequently appearing case of state feedback is when instead of the transfer function, the state space form of the controlled system is given. In connection with (3.67) it has already been discussed that all controllable systems can be written in controllable canonical form by using the transformation matrix $T_c = M_c^c(M_c)^{-1}$. This similarity transformation has an effect on the feedback vector, too:

$$\mathbf{k}^T = \mathbf{k}_c^T T_c = \mathbf{k}_c^T M_c^c M_c^{-1} = \mathbf{g}_c^T M_c^{-1} \mathcal{R}(\mathbf{F}) = [0, 0, \dots, 1] M_c^{-1} \mathcal{R}(\mathbf{F}) \quad (14.13)$$

To compute (14.13), the inverse of the controllability matrix has to be constructed by the system matrices \mathbf{F} and \mathbf{g} , on the one hand. On the other hand, the controllability matrix M_c^c of the controllable canonical form has to be also generated [see (3.61)]. Since this latter depends only on the coefficients a_i in the denominator of the process transfer function, the denominator has to be computed: $\mathcal{A}(z) = \det(z\mathbf{I} - \mathbf{F})$. The same is true for the computation of $\mathcal{R}(\mathbf{F})$ in the second formula. The method of computing the pole placement state feedback vector shown above is named—after its developer—the ACKERMANN method.

Observe that the transformation properties of the CT and DT state equations, their canonical forms and the concepts of controllability and observability are formally completely the same. Deriving from this fact, the state feedback techniques for the control of discrete-time systems also have a great similarity with the CT methods presented above.

14.2 Observer Based Discrete-Time Pole Placement State Feedback Regulator

The method of the state feedback discussed previously requires measuring the state space vector of the state equation describing the process. This is very rarely available, generally only in the case of systems with low order dynamics (for example, mechanical systems described by distance, velocity and acceleration co-ordinates). The usability of the methods depends also on whether measurement or estimation is available on the state vector. For the construction of the state vector, the so-called observer principle has been developed. For this method, the knowledge of the system matrices \mathbf{F} , \mathbf{g} and \mathbf{c}^T is necessary, by means of which an exact model of the process is constructed, and applying the same excitation as for the original process, this model (the observer) provides the estimated values $\hat{\mathbf{x}}[k]$ and $\hat{y}[k]$ of $\mathbf{x}[k]$ and $y[k]$, respectively. The state feedback is performed using $\hat{\mathbf{x}}[k]$. The principle is shown in Fig. 14.3.

Strictly speaking, $\hat{\mathbf{F}}$, $\hat{\mathbf{g}}$ and $\hat{\mathbf{c}}^T$ have to be employed in the observer instead of \mathbf{F} , \mathbf{g} and \mathbf{c}^T . But the particularity of the observer is that besides providing a parallel model, it also constructs an error $\varepsilon[k] = y[k] - \hat{y}[k]$ from the deviation of the original from the estimated output of the process, and feeds it back to the input of

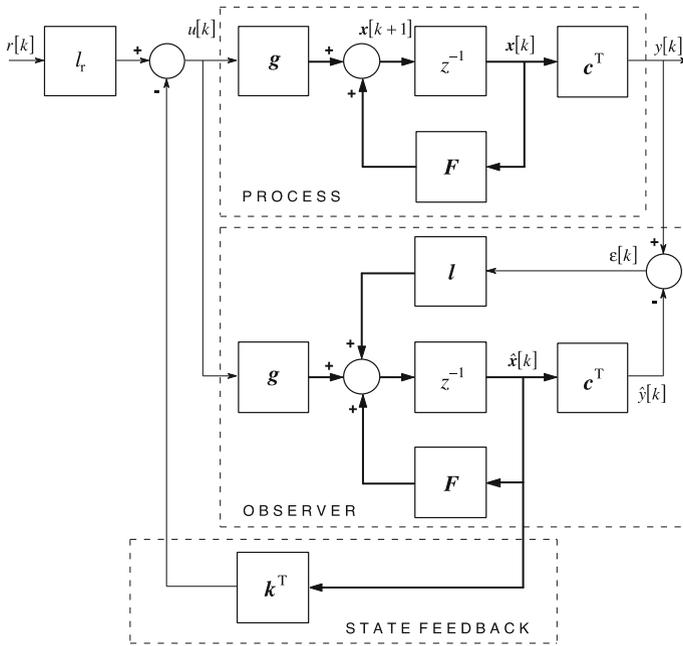


Fig. 14.3 State feedback applying an observer

the observer delay via a proportional feedback vector l . This feedback operates until the error exists, i.e., until the outputs of the process and the observer become the same. With the knowledge of the system matrices this operating mode can compensate relatively large errors. It is also seen in the figure that now the state feedback has the form

$$u[k] = k_r r[k] - k^T \hat{x}[k], \tag{14.14}$$

thus $\hat{x}[k]$ appears instead of $x[k]$. After a long and complex derivation, whose details are not discussed here, the transfer function of the complete closed system can be obtained as

$$\begin{aligned} T_{ry}(z) &= \frac{[c^T(zI - F)^{-1}g] [1 - k^T(zI - F + gk^T + lc^T)^{-1}b] k_r}{1 + [l^T(zI - F + gl^T + lc^T)^{-1}g] [c^T(zI - F)^{-1}g]} \\ &= c^T(zI - F + gk^T)^{-1}gk_r = \frac{c^T(zI - F)^{-1}gk_r}{1 + k^T(zI - F)^{-1}g} = \frac{k_r G(z)}{1 + k^T(zI - F)^{-1}g} = \frac{k_r \mathcal{B}(z)}{\mathcal{R}(z)} \end{aligned} \tag{14.15}$$

which is, perhaps surprisingly, precisely equal to (14.2), i.e., to the case of state feedback without observer. This means that the tracking behavior of the closed system does not depend on the choice of the vector \mathbf{l} . To examine the operation of the observer, let us construct the vector of the state error

$$\tilde{\mathbf{x}}[k] = \mathbf{x}[k] - \hat{\mathbf{x}}[k] \quad (14.16)$$

and also

$$\tilde{\mathbf{x}}[k] = (\mathbf{F} - \mathbf{l}\mathbf{c}^T)\tilde{\mathbf{x}}[k], \quad (14.17)$$

which is very similar to (14.4) without excitation. Very similar methods can be used for the design of observers as were used for state feedback, where the choice of the goal is to ensure the system dynamics (14.17) by the characteristic polynomial

$$\det(z\mathbf{I} - \mathbf{F} + \mathbf{l}\mathbf{c}^T) = \mathcal{F}(z) = z^n + f_1z^{n-1} + \cdots + f_{n-1}z + f_n \quad (14.18)$$

A solution always exists if the process is observable (This is reasonable if the order of \mathcal{F} is equal to that of \mathcal{A}). If the transfer function of the process to be controlled is known, then it is an exceptional case, because then the canonical forms can be directly written. In this case, when the system matrices are based on the observable canonical forms (3.53),

$$\mathbf{F}_o = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix}; \mathbf{c}_o^T = [1, 0, \dots, 0]; \mathbf{g}_o = [b_1, b_2, \dots, b_n]^T \quad (14.19)$$

Taking the special forms of \mathbf{F}_o and \mathbf{c}_o^T into account, it is easily seen that according to the design equation

$$\begin{aligned} \mathbf{F}_o - \mathbf{l}_o\mathbf{c}_o^T &= \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} - \mathbf{l}_o[1, 0, \dots, 0] = \\ &= \begin{bmatrix} -f_1 & 1 & 0 & \cdots & 0 \\ -f_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -f_{n-1} & 0 & 0 & \cdots & 1 \\ -f_n & 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned} \quad (14.20)$$

the choice

$$\mathbf{l} = \mathbf{l}_o = [f_1 - a_1, f_2 - a_2, \dots, f_n - a_n]^T \quad (14.21)$$

ensures the characteristic equation (14.18), i.e., the prescribed poles.

The general case is now, when the state space equation of the process is given instead of its transfer function. It has already been discussed concerning Eq. (3.79) that all observable systems can be written in observable canonical form by the use of the transformation matrix $\mathbf{T}_o = (\mathbf{M}_o^o)^{-1} \mathbf{M}_o$. This similarity transformation has an effect on the feedback vector, too:

$$\mathbf{l} = (\mathbf{T}_o)^{-1} \mathbf{l}_o = \mathbf{M}_o^{-1} \mathbf{M}_o^o \mathbf{l}_o. \quad (14.22)$$

To compute (14.22), the inverse of the observability matrix \mathbf{M}_o has to be constructed by the general system matrices \mathbf{F} and \mathbf{c}^T . On the other hand, the observability matrix \mathbf{M}_o^o of the observable canonical form must be also given (see (3.73)). Since this latter one depends only on the coefficients a_i in the denominator of the transfer function of the process, so to its determination the denominator has to be computed: $\mathcal{A}(z) = \det(z\mathbf{I} - \mathbf{F})$. The computation method of the observer vector shown above is named, after its developer, the ACKERMANN method.

There is an interesting similarity between the design methods of the dynamics of the state feedback and of the observer, a so-called duality, i.e., they correspond to each other under the associations $\mathbf{F} \leftrightarrow \mathbf{F}^T, \mathbf{g} \leftrightarrow \mathbf{c}^T, \mathbf{k} \leftrightarrow \mathbf{l}^T, \mathbf{M}_c^c \leftrightarrow (\mathbf{M}_o^o)^T$.

Based on the state error (14.16) and the equations of the process (14.1), the joint equation of the state feedback and the observer is

$$\begin{aligned} \begin{bmatrix} \mathbf{x}[k+1] \\ \tilde{\mathbf{x}}[k+1] \end{bmatrix} &= \begin{bmatrix} \mathbf{F} - \mathbf{g}\mathbf{k}^T & \mathbf{g}\mathbf{k}^T \\ \mathbf{0} & \mathbf{F} - \mathbf{l}\mathbf{c}^T \end{bmatrix} \begin{bmatrix} \mathbf{x}[k] \\ \tilde{\mathbf{x}}[k] \end{bmatrix} + \begin{bmatrix} k_r \mathbf{g} \\ \mathbf{0} \end{bmatrix} r[k] \\ e[k] &= y[k] - \hat{y}[k] = \mathbf{c}^T \tilde{\mathbf{x}}[k] \end{aligned} \quad (14.23)$$

Since the right hand side system matrix is upper triangular, the characteristic equation of the closed system is

$$\det(z\mathbf{I} - \mathbf{F} + \mathbf{g}\mathbf{k}^T) \det(z\mathbf{I} - \mathbf{F} + \mathbf{l}\mathbf{c}^T) = \mathcal{R}(z) \mathcal{F}(z) \quad (14.24)$$

Thus the polynomial is the product of two factors: one is connected to the state feedback, the other is connected to the observer. It is important to remark that in contrast to (14.24), $\mathcal{F}(z)$ does not appear in the transfer function $T_{ry}(z)$ [see (14.12) and (14.15)].

Equation (14.24) representing the observer based state feedback, according to which the characteristic equations of the state feedback and observer are independent, is called the separation principle.

14.3 Two-Step Design Methods Using Discrete-Time State Feedback

It has been shown in the discussion of the state feedback based control, that the most advantageous (favorable) properties of the method are:

- the applicability of the method does not depend on whether the process is stable or unstable
- the tracking behavior does not depend on the applied observer, thus it can be directly designed
- the method is not very sensitive for the exact knowledge of the parameter matrices of the state equation

There are unwanted, unfavorable properties:

- the state feedback is basically a control of 0-type, therefore the remaining error can be eliminated by the calibration factor, which is never very precise using the model of the process
- the state feedback can not change the zeros of the process
- the noise rejection behavior can not be designed directly.

Mainly due to these latter attributes, usually an extra step is included in the design of control systems using state feedback. The necessity of the calibration factor can be easily eliminated by the construction of a cascade integrating controller according to Fig. 14.4.

The joint state equation of the closed system, which now replaces Eq. (14.4), can be written as

$$\begin{aligned} \dot{\mathbf{x}}^*[k+1] &= \begin{bmatrix} \dot{\mathbf{x}}[k+1] \\ \dot{\delta}[k+1] \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{c}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}[k] \\ \delta[k] \end{bmatrix} + \begin{bmatrix} \mathbf{g} \\ 0 \end{bmatrix} u[k] + \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} r[k] \\ &= (\mathbf{F}^* - \mathbf{g}^* \mathbf{k}_*^T) \mathbf{x}^*[k] + \mathbf{v}^* r[k] \end{aligned} \tag{14.25}$$

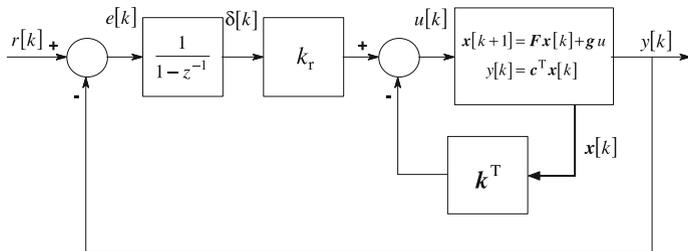


Fig. 14.4 The joint use of the state feedback and the integrating controller

by introducing a new state variable $\delta[k]$, which is the integral of the error $e[k] = r[k] - y[k]$ of the outer loop, where the notations

$$\mathbf{F}^* = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{c}^T & 0 \end{bmatrix}; \quad \mathbf{g}^* = \begin{bmatrix} \mathbf{g} \\ 0 \end{bmatrix}; \quad \mathbf{v}^* = \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} \quad (14.26)$$

and the new extended feedback equation

$$u[k] = -[\mathbf{k}^T \quad k_r] \begin{bmatrix} \mathbf{x}[k] \\ \delta[k] \end{bmatrix} = -\mathbf{k}_*^T \mathbf{x}^*[k] = \frac{k_r}{1 - z^{-1}} e[k] - \mathbf{k}^T \mathbf{x}[k] \quad (14.27)$$

are taken into account.

Equation (14.27) clearly shows the integrating effect. The item $\mathbf{k}^T \mathbf{x}[k]$, however, can be considered as a generalization of the derivative effect.

Thus the closed-loop control having also an integrator can be described by a state equation which has its dimension higher by one than the earlier one, where now k_r has also to be determined besides \mathbf{k}^T . For the design of the extended system the characteristic polynomial $\mathcal{R}^*(z)$ having order greater by one has to be prescribed, then the design Eq. (14.13) of the ACKERMANN method can be directly applied here too. If the process is not given in the transfer function form, then the general state equation has to be rewritten first into a controllable canonical form, as was shown in (10.13).

Notice that the extended task can not be solved sequentially, i.e., by determining first the \mathbf{k}^T belonging to $\mathcal{R}(z)$, and then k_r based on $\mathcal{R}^*(z) = \mathcal{R}(z)(z - z_{n+1})$. The task has to be solved in one step for \mathbf{k}_*^T on the basis of $\mathcal{R}^*(z)$.

An integrating effect can also be included by designing the state feedback for a modified process $G^*(z) = zG(z)/(z - 1)$ instead of the transfer function $G(z)$. Note that the feedback vectors obtained for the earlier case and for this latter approach are not the same!

Obviously, besides the I -controller, a higher order regulator can also be applied. The solution of the pole placement, however, can not be obtained automatically by the ACKERMANN method, and may lead to a complicated system of non-linear equations.

In the case of state feedback applying observer an I or higher order regulator, instead of the regulator of 0-type, can also be applied in the error feedback of the observer using the methods shown above.

The unchanged zeros of the process can be compensated by a serial compensator

$$K_s(z) = G_s(z) \frac{\mathcal{N}(z)}{\mathcal{B}_+(z)} \quad (14.28)$$

where it is assumed, according to the method applied in Chap. 7, that the numerator of the process is $\mathcal{B}(z) = \mathcal{B}_+(z)\mathcal{B}_-(z)$. Here \mathcal{B}_+ contains the stable zeros and \mathcal{B}_- the unstable zeros. For realizability, $\mathcal{N}(z)/\mathcal{B}_+(z)$ has to be proper, thus only as

many zeros can be placed in the transfer function of the closed system as there are stable zeros in the process.

Finally the loop transfer function has the form

$$T_{ry}(z) = \frac{\mathcal{N}(z)}{\mathcal{R}(z)} k_r G_s(z) \mathcal{B}_-(z) \quad (14.29)$$

where the effect of the invariant $\mathcal{B}_-(z)$ can be attenuated optimally by the filter $G_s(z)$. In many cases a simple, but not optimal $G_s(z) = 1$ is chosen.

A favorable design of the disturbance rejection feature can be reached by applying a *YP* controller in the outer cascade loop. This can be done since the state feedback is capable of stabilizing any process, even an unstable one. In general, the control of an unstable process has two steps. In the first step the process is stabilized, then in the second step, via a second outer loop, the required quality goals can be ensured even by a *TDOF* structure.

A stabilizing controller using state feedback can be applied only to delay-free processes. If the process has a significant delay, then the only possibility is to switch to a sampled-data control using the general polynomial method [see Chap. 15].

14.4 Discrete-Time *LQ* State Feedback Regulator

With the method presented in the previous section, arbitrary (stabilizing) pole-placement can be performed via the so-called state feedback from the state vector of the process. A further optimality task can also be solved by the technique of state feedback. The goal of this task to control optimally the DT *LTI* process (11.33–11.34) by the minimization of a complicated optimality criterion

$$I = \frac{1}{2} \sum_{k=0}^{\infty} \{ \mathbf{x}^T[k] \mathbf{W}_x \mathbf{x}[k] + W_u u^2[k] \} \quad (14.30)$$

Here \mathbf{W}_x is a real symmetric positive semi-definite matrix, weighting the DT state vector and W_u is a positive scalar, weighting the DT actuator signal. The solution optimizing the criterion is a state feedback in the form

$$u[k] = -\mathbf{k}_{LQ}^T \mathbf{x}[k] \quad (14.31)$$

[see (9.3)], where \mathbf{k}_{LQ}^T is the feedback vector

$$\mathbf{k}_{LQ}^T = \frac{1}{W_u} \mathbf{g}^T \mathbf{P} \quad (14.32)$$

Here the symmetric positive semi-definite matrix \mathbf{P} is the solution of the algebraic RICCATI equation

$$\mathbf{P}\mathbf{F} + \mathbf{F}^T\mathbf{P} - \frac{1}{W_u}\mathbf{P}\mathbf{g}\mathbf{g}^T\mathbf{P} = -\mathbf{W}_x \quad (14.33)$$

The (algebraic) RICCATI equation is nonlinear in \mathbf{P} , therefore it does not have an explicit algebraic solution. The CAD systems used in control engineering, however, have several numerical algorithms for the solution of the above equation. This controller is called an *LQ* (Linear Quadratic: Linear regulator—Quadratic criterion) regulator.

The state equation of the closed system provided by the *LQ* regulator has the form

$$\mathbf{x}[k+1] = (\mathbf{F} - \mathbf{g}\mathbf{k}_{LQ}^T)\mathbf{x}[k]; \quad \bar{\mathbf{F}} = \mathbf{F} - \mathbf{g}\mathbf{k}_{LQ}^T \quad (14.34)$$

(The derivation of the *LQ* regulator for CT systems can be found in A.9.6 of Appendix A.5, the derivation of the DT controller can be done with a very similar analogy).

If the transfer function of the process is known, then the controllable canonical form can be easily written in analogy with the CT Eq. (9.10) for the special \mathbf{F}_c and \mathbf{g}_c formed by (14.8), according to the design algorithm (14.10) of the classical DT state feedback. The feedback vector \mathbf{k}_{LQ}^T comes from the *LQ* control design (from the solution of the RICCATI equation). So by turning back the derivation of (14.10), the coefficients of the characteristic polynomial $\mathcal{R}(s)$ of the closed-loop system are given by

$$[r_1, r_2, \dots, r_n]^T = \mathbf{k}_{LQ}^T + [a_1, a_2, \dots, a_n]^T \quad (14.35)$$

In the case of *LQ* control it is also possible to apply an observer for the determination of the state vector.

Notice that the state feedback vector \mathbf{k}_{LQ}^T also leaves the zeros of the process unchanged.