Chapter 14
State Feedback in Sampled Systems

Let us analyse state feedback control in sampled systems. In continuous systems the poles of the closed loop system can be set to prescribed values by feeding back the state variables of the process to the input of the process (see Chap. 9). State feedback can be applied similarly to sampled systems.

14.1 State Feedback with Pole Placement

The state equation of the sampled system is (see Chap. 11 in the textbook [1]):

$$\begin{aligned} \boldsymbol{x}[k+1] &= \boldsymbol{F}\,\boldsymbol{x}[k] + \boldsymbol{g}\,u[k] \\ y[k] &= \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}[k] + d\,u[k] \end{aligned}$$

The poles of the system in the z domain are the roots of the characteristic equation

$$\det(z\boldsymbol{I} - \boldsymbol{F}) = 0.$$

The aim of the control is the acceleration of the dynamic behaviour (or stabilization of an unstable process). One method of compensation is state feedback. Prescribing the poles of the closed loop defines the rate of acceleration.

The control signal is obtained by feedback of the discrete state variables:

$$u[k] = -\boldsymbol{k}^{\mathrm{T}}\boldsymbol{x}[k]$$

L. Keviczky et al., *Control Engineering: MATLAB Exercises*,
Advanced Textbooks in Control and Signal Processing,
https://doi.org/10.1007/978-981-10-8321-1_14

The state equation of the closed loop system is

$$\boldsymbol{x}[k+1] = \left(\boldsymbol{F} - \boldsymbol{g}\,\boldsymbol{k}^{\mathrm{T}}\right)\boldsymbol{x}[k]$$

The prescribed characteristic polynomial of the closed loop is:

$$\mathcal{R}_{\mathrm{d}}(z) = \det\left[z\boldsymbol{I} - \left(\boldsymbol{F} - \boldsymbol{g}\,\boldsymbol{k}^{\mathrm{T}}\right)\right] = (z - p_{\mathrm{d}1})\,(z - p_{\mathrm{d}2})\ldots(z - p_{\mathrm{d}n})$$

where $p_{\mathrm{d}1}, p_{\mathrm{d}2}, \ldots, p_{\mathrm{d}n}$ are the prescribed poles of the closed loop system in the z domain.

The controllability matrix of the process is

$$\boldsymbol{M}_{\mathrm{c}} = \begin{bmatrix}\boldsymbol{g} & \boldsymbol{F}\boldsymbol{g} & \ldots & \boldsymbol{F}^{n-1}\boldsymbol{g}\end{bmatrix}.$$

According to the ACKERMANN formula the state feedback vector $\boldsymbol{k}$ is calculated from the state matrices $\boldsymbol{F}$, $\boldsymbol{g}$ of the process and from the characteristic polynomial belonging to the prescribed poles of the closed loop, as follows:

$$\boldsymbol{k}^{\mathrm{T}} = [\,1 \quad \ldots \quad 0 \quad 0\,]\,\boldsymbol{M}_{\mathrm{c}}^{-1}\,\mathcal{R}_{\mathrm{d}}(\boldsymbol{F}),$$

where $\mathcal{R}_{\mathrm{d}}(\boldsymbol{F})$ is the characteristic polynomial of the closed loop by the substitution $z = \boldsymbol{F}$.

The state feedback vector $\boldsymbol{k}$ is calculated in MATLAB™ with the command acker(F, g, Rd).

```
k=acker(F,g,Rd)
```

Rd is a vector containing the prescribed poles of the feedback system [the roots of the equation $\mathcal{R}_{\mathrm{d}}(z) = 0$]. If the prescribed poles are given in continuous time, their corresponding discrete values can be determined by the transformation $z = e^{sT_{\mathrm{s}}}$, where T_{s} is the sampling time.

Example 14.1 The continuous process is given by a third order lag element. Its transfer function is

$$P(s) = \frac{6}{(s+1)\,(s+2)\,(s+3)} = \frac{1}{(1+s)\,(1+0.5s)\,(1+0.333s)}$$

(a) Give the continuous state equations of the process, then with sampling time $T_{\mathrm{s}} = 0.2$ determine the discrete state equation supposing zero order hold.
(b) Design a state feedback control, prescribing the poles of the discrete closed loop. The poles are given in continuous time, then they are transformed to discrete time with the transformation $z = e^{sT_{\mathrm{s}}}$.
Let the prescribed continuous poles be $\mathsf{Rc} = [\,-6 \quad -3+4j \quad -3-4j\,]$.
(c) Analyse the behaviour of the system for initial conditions, and for reference signal tracking and disturbance rejection.

Solution The state equation of the continuous process:

```
po=[-1 -2 -3]
[A,b,c,d]=tf2ss(6,poly(po))
Hc=ss(A,b,c,d)
     A =
        -6  -11   -6
         1    0    0
         0    1    0
     b =
         1
         0
         0
     c =
         0    0    6
     d =
         0
```

The sampling time:

```
Ts = 0.2
```

Transformation to discrete state equation:

```
Hd=c2d(Hc,Ts,'zoh')
[F,g,cd,dd]=ssdata(Hd)
          F  =
               0.1977    -1.2693    -0.6483
               0.1081     0.8461    -0.0807
               0.0135     0.1888     0.9940
          g  =
               0.1081
               0.0135
               0.0010
          cd  =
                  0         0         6
          dd  =   0
```

The prescribed continuous poles for the closed loop system:

```
Rc=[-6; -3+i*4; -3-i*4]
```

Their corresponding discrete values with the given sampling time:

```
Rd=exp(Rc*Ts)
```

The obtained values are

```
Rd =
    0.3012
    0.3824 + 0.3937i
    0.3824 - 0.3937i
```

Apply the ACKERMANN formula to determine the state feedback vector.

```
k=acker(F,g,Rd)
k =
     4.2463    32.4319    77.4220
```

The parameter matrices of the state equation of the closed loop:

```
Fc=F-g*k; gc=g;cc=cd;dc=dd;
Tk=ss(Fc,gc,cc,dc,Ts)
```

The static gain:

```
kr=1/dcgain(Tk)
kr = 13.9037
```

The state parameter matrices of the closed-loop compensated with the static gain:

```
Fck=Fc;gck=kr*gc;cck=cc;dck=dc;
Tk1=ss(Fck,gck,cck,dck,Ts)
```

The step response of the closed loop (Fig. 14.1.):

```
step(Tk1,3)
```

Remark The static gain can be calculated also according to the following considerations. With state feedback we would like to arrange that for a step reference signal r, the output signal y in steady state is equal to the constant value of the reference signal. Then the derivatives of the state variables are zeros. The reference signal acts on the input of the control system through the correction factor k_r. The relation is given for the case of a *single input—single output* (*SISO*) system. In steady state the values of the state variables at the sampling point $n+1$ is the same

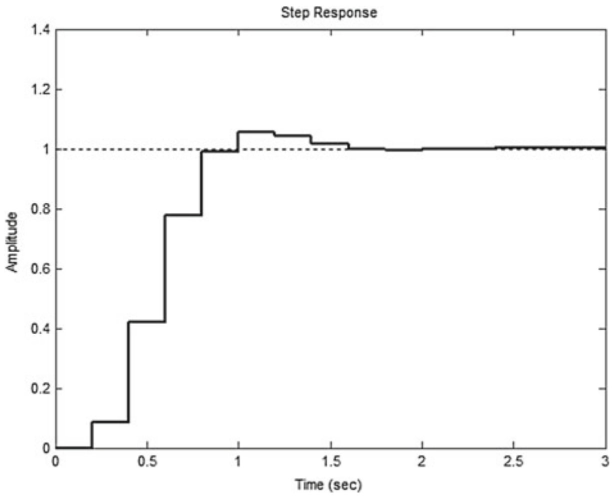

Fig. 14.1 Step response of the sampled control system

as at the point n, and the steady value of the output signal is the same as the reference signal.

$$\begin{aligned} \boldsymbol{x}_\infty &= \boldsymbol{F}\boldsymbol{x}_\infty + \boldsymbol{g}\,u_\infty \\ u_\infty &= k_\mathrm{r} r - \boldsymbol{k}\boldsymbol{x}_\infty \\ y_\infty &= \boldsymbol{c}^\mathrm{T}\boldsymbol{x}_\infty = r\,; \quad r \neq 0 \end{aligned}$$

whence

$$\begin{aligned} \boldsymbol{x}_\infty &= \left(\boldsymbol{I} - \boldsymbol{F} + \boldsymbol{g}\boldsymbol{k}^\mathrm{T}\right)^{-1}\boldsymbol{g}\,k_\mathrm{r} r \\ y_\infty &= \boldsymbol{c}^\mathrm{T}\left(\boldsymbol{I} - \boldsymbol{F} + \boldsymbol{g}\boldsymbol{k}^\mathrm{T}\right)^{-1}\boldsymbol{g}\,k_\mathrm{r} r = r \end{aligned}$$

and the correction factor is expressed as

$$k_\mathrm{r} = 1 \Big/ \left[\boldsymbol{c}^\mathrm{T}\left(\boldsymbol{I} - \boldsymbol{F} + \boldsymbol{g}\boldsymbol{k}^\mathrm{T}\right)^{-1}\boldsymbol{g}\right]$$

In our example

```
kr=1/(cd*inv(eye(3)-F+g*k)*g)
          kr = 13.9037
```

The result is the same as obtained before.

Problem Build the SIMULINK™ diagram of the control system. Analyse the behaviour of the system for initial conditions, for step reference signal and for output disturbance. Analyse the behaviour between the sampling points also.

Remark The static compensation ensures the accurate tracking of the reference signal in steady state, but does not eliminate the static error of disturbance rejection. To ensure this the state model should be enhanced with the state variables of the disturbance, and state feedback should be designed again for the enhanced system. Another possibility is extension of the system with an integrator and designing state feedback to the enhanced system.

14.2 State Feedback with Extension with Integrator

In order to track accurately the step reference signal and to decrease the effect of the disturbance it is expedient to include an integrator in the control circuit. Extend the state space model of the process with an additional state variable which is the integral of the output signal (Fig. 14.2). (A discrete equivalent of the system given in Fig. 9.6 is created.)

The difference equation of the integrator is

$$x_{\mathrm{i}}[k+1] = x_{\mathrm{i}}[k] + T_{\mathrm{s}} y[k] = x_{\mathrm{i}}[k] + T_{\mathrm{s}}\,\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}[k].$$

The extended state equation is

$$\begin{bmatrix} \boldsymbol{x}[k+1] \\ x_{\mathrm{i}}[k+1] \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} & 0 \\ T_{\mathrm{s}}\,\boldsymbol{c}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}[k] \\ x_{\mathrm{i}}[k] \end{bmatrix} + \begin{bmatrix} \boldsymbol{g} \\ 0 \end{bmatrix} u[k] = \boldsymbol{F}_{\mathrm{b}}\boldsymbol{x}_{\mathrm{b}}[k] + \boldsymbol{g}_{\mathrm{b}}\,u[k]$$

$$y[k] = \begin{bmatrix} \boldsymbol{c}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}[k] \\ x_{\mathrm{i}}[k] \end{bmatrix} + d\,u[k] = \boldsymbol{c}_{\mathrm{b}}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{b}}[k] + d\,u[k]$$

(The indices b refer to the extended state variables and parameter matrices.)

The state feedback of the extended system can be built according to Fig. 9.7, with the discrete system and the discrete integrator. As the number of the state variables has been increased by one, the number of the prescribed poles has to be also increased by one. The state feedback vector $\boldsymbol{k}_{\mathrm{b}}^{\mathrm{T}}$ which ensures the prescribed poles $\boldsymbol{p}_b$, the roots of $\det\left(s\boldsymbol{I} - \boldsymbol{F}_{\mathrm{b}} + \boldsymbol{g}_{\mathrm{b}}\boldsymbol{k}_{\mathrm{b}}^{\mathrm{T}}\right) = 0$, is calculated by the ACKERMANN formula with the extended parameter matrices $\boldsymbol{F}_{\mathrm{b}}$ and $\boldsymbol{g}_{\mathrm{b}}$.

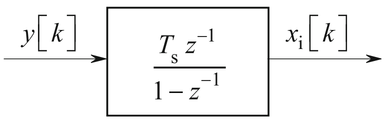

Fig. 14.2 Discrete integrator

The vector $\boldsymbol{k}_{\mathrm{e}}^{\mathrm{T}}$ containing the first n elements of $\boldsymbol{k}_{\mathrm{b}}^{\mathrm{T}}$ gives the state feedback coefficients of the original state variables. The constant k_{i} which gives the feedback of the integrator is the last, $n+1$-th element of $\boldsymbol{k}_{\mathrm{b}}^{\mathrm{T}}$.

For an *SISO* system supposing $d=0$ the state equation of the *closed loop* system can be given by the following vector-matrix equation:

$$\begin{bmatrix} \boldsymbol{x}[k+1] \\ x_{\mathrm{i}}[k+1] \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} - \boldsymbol{g}\,\boldsymbol{k}_{\mathrm{e}}^{\mathrm{T}} & \boldsymbol{g}\,k_{\mathrm{i}} \\ -T_{\mathrm{s}}\,\boldsymbol{c}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}[k] \\ x_{\mathrm{i}}[k] \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ T_{\mathrm{s}} \end{bmatrix} r[k]$$

$$y[k] = \begin{bmatrix} \boldsymbol{c}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}[k] \\ x_{\mathrm{i}}[k] \end{bmatrix} + 0 \cdot r[k]$$

Example 14.2 Let us extend the system given in Example 14.1 with an integrator and realize state feedback to the extended system. Analyse the course of the output signal for step reference input signal.

The MATLAB™ program is

```
clear
clc
po=[-1 -2 -3]
[A,b,c,d]=tf2ss(6,poly(po));
Hc=ss(A,b,c,d);
Ts=0.2
% The discrete state equation
Hd=c2d(Hc,Ts,'zoh');
[F,g,cd,dd]=ssdata(Hd)
% Extension by integrator
Fb=[F zeros(3,1);Ts*cd 1];
gb=[g;0];
cb=[c 0];
Hdb=ss(Fb,gb,cb,d,Ts)
% The prescribed poles
Rb=[-9 -6 -3+i*4 -3-i*4]
Rd=exp(Rb*Ts)
% The state feedback vector
k=acker(Fb,gb,Rd)
kk=k(1:3);
ki=k(4)
% State equation of the closed loop
Fc=[F-g*kk g*ki;-Ts*c 1]
gc=[zeros(3,1);Ts]
cc=cb
dc=0
Hdc=ss(Fc,gc,cc,dc,Ts)
step(Hdc)
```

The extended state equation:

```
Fb = 0.1977   -1.2693   -0.6483        0
     0.1081    0.8461   -0.0807        0
     0.0135    0.1888    0.9940        0
          0         0    1.2000   1.0000
gb = 0.1081
     0.0135
     0.0010
          0
cb = 0    0     6      0
```

The state feedback vector:

```
k = 6.7401      60.9170     260.8297      58.0271
```

The step response of the closed loop is shown in Fig. 14.3.

The SIMULINK™ block diagram of the control system extended with the integrator is shown in Fig. 14.4. The state equation block is taken from the *discrete* block library. The state variables have to be measurable. In the figure the setting of

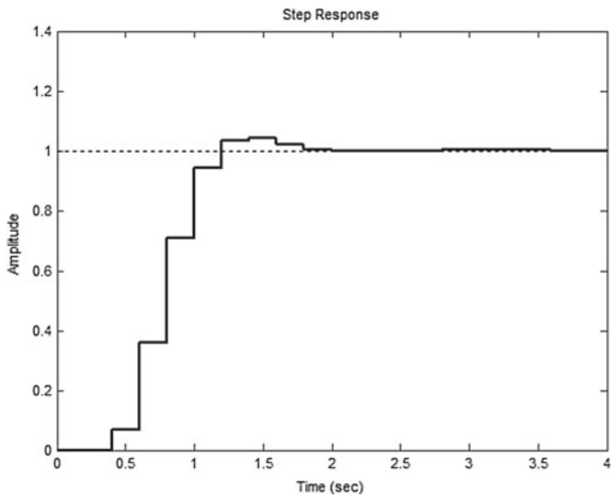

Fig. 14.3 Step response of the sampled state feedback control system extended with integrator

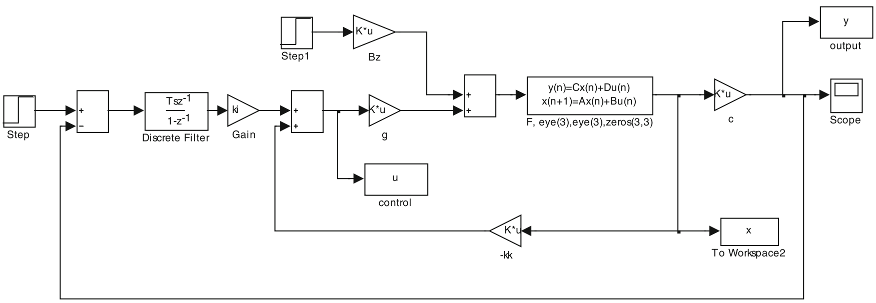

Fig. 14.4 SIMULINK™ block diagram of the discrete control system

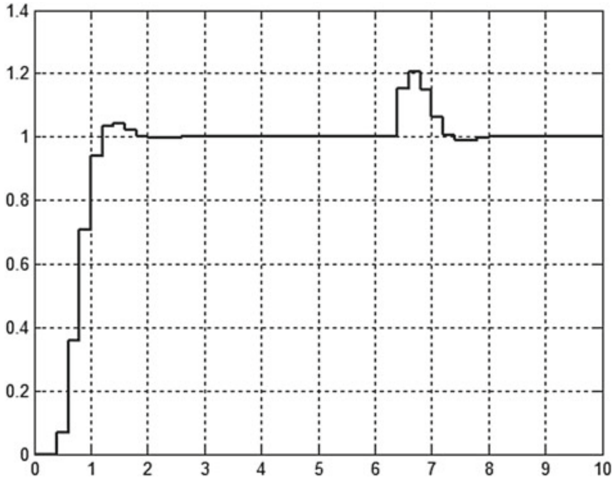

Fig. 14.5 Time course of the output signal

the parameters is indicated. Simulate the behaviour of the system by running the SIMULINK™ program for the initial conditions, for a step reference signal and for a disturbance acting at the input of the process. Let `Bz=[0 1 0]'`. In Fig. 14.5 it can be seen, that the control system tracks the reference signal without steady error, and rejects the effect of the step disturbance of amplitude 0.2 acting at the time point t = 6 s.

Problem Supplement the SIMULINK™ model with the state space model of the continuous process. Analyse the course of the output signal also between the sampling points.

14.3 State Estimation

If the state variables are not measurable, they have to be estimated. The observer can be applied for state estimation. The discretized form of the circuit shown in Fig. 9.9 is realized. If the process is known, its model is built. Figure 14.6 shows the SIMULINK™ block diagram of the state estimation of the discrete system.

The process and its model are excited by the same input signal. Comparing the output signals of the process and the model an error signal is obtained which is used to set the state variables of the model through the parameter $\boldsymbol{l}$ (see textbook [1]). The values of the estimated state variables will approach quickly and follow the values of the real state variables, if the dynamics of the estimation circuit is much faster than the dynamics of the process. The poles of the estimation circuit can be prescribed and then applying the Ackermann formula vector $\boldsymbol{l}$ can be determined.

Example 14.3 Consider the process of the proportional system with three time lags investigated in Example 14.1. The initial values of all the three state variables are 1. The reference signal and the disturbance signal are zero. Suppose the prescribed poles of the estimation circuit are real and of the same value, and ensure faster transients as the smallest time constant of the state feedback system (in case of conjugate complex poles let us consider the reciprocal of the absolute value).

Prescribe the poles of the continuous closed loop system:

```
Rc=[-6; -3+i*4; -3-i*4]
```

Set the poles of the continuous state estimation circuit:

```
Fc=[-7 -7 -7]
```

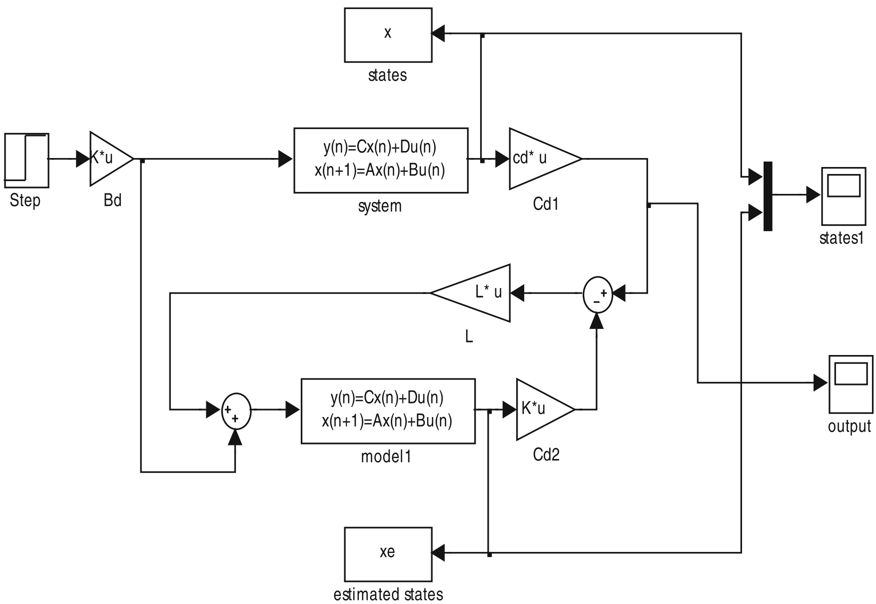

Fig. 14.6 SIMULINK™ diagram for state estimation in discrete system

The poles of the discrete state estimation circuit are

```
Fd=exp(Ts*Fc)
```

The parameters of the estimation circuit (the elements of vector $\boldsymbol{l}$, in MATLAB™ L) in the discrete system are determined by the command

```
L=acker(F',c',Fd)'
```

The MATLAB™ program for the discrete version of the algorithm of Fig. 9.10 is

```
po=[-1,-2,-3]
[A,b,c,d]=tf2ss(6,poly(po))
Hc=ss(A,b,c,d)
Ts=0.2
Hd=c2d(Hc,Ts,'zoh')
[F,g,cd,dd]=ssdata(Hd)
% prescribed poles of the continuous estimation
Pc=[-7 -7 -7]
% poles of the discrete state estimation
Pd=exp(Pc*Ts)
% parameters of estimation circuit (elements of vector L)
L=acker(F',cd',Pd)'
Fest=F-L*cd
sysest=ss(Fest,L,cd,dd,Ts)
x0=[1;1;1]
t=0:Ts:6;
[y,t,x]=initial(Hd,x0,t);
figure(1)
stairs(t,x),grid
x0est=[0;0;0]
[yest,t,xest]=lsim(sysest,y,t,x0est);
figure(2)
stairs(t,xest),grid
figure(3)
stairs(t,x(:,1))
hold
stairs(t,xest(:,1)),grid
figure(4)
stairs(t,y),grid
hold
stairs(t,yest)
```

The state estimation vector:

```
L' = -0.7709     0.2062     0.2163
```

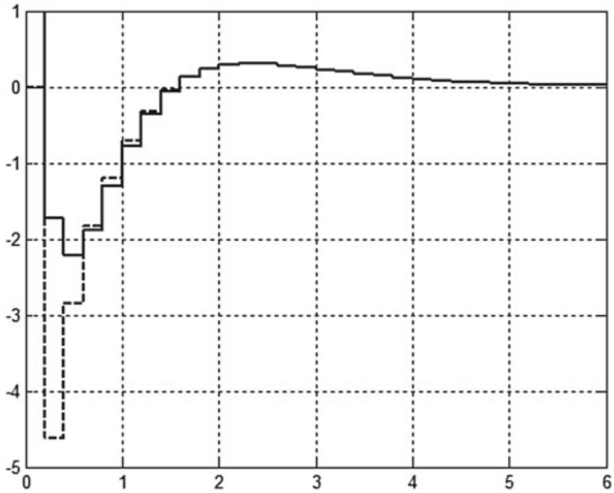

Fig. 14.7 The time course of the real and estimated first state variables

The simulation shows the fast settling of the state variables. Plot the real and the estimated first state variable in the same diagram (Fig. 14.7).

Running the SIMULINK™ model yields a similar result.

Problem Supplement the SIMULINK™ model with the state space model of the continuous process. Analyse the course of the output signal also between the sampling points.

14.4 State Feedback from the Estimated State Variables

State feedback control can be realized from the estimated state variables with the same feedback constants that are calculated for feedback from the original, real state variables (the separation principle). The control system operates well if the prescribed poles of the estimation circuit ensure faster behaviour of the estimation circuit than that of the feedback control circuit. The SIMULINK™ diagram of state feedback from the estimated state variables is shown in Fig. 14.8.

Example 14.4 The proportional process with three time lags given in Example 14.1. is sampled with sampling time **`Ts=0.2`**. The initial value of all the three state variables is 1. The state variables are estimated, then the control signal is

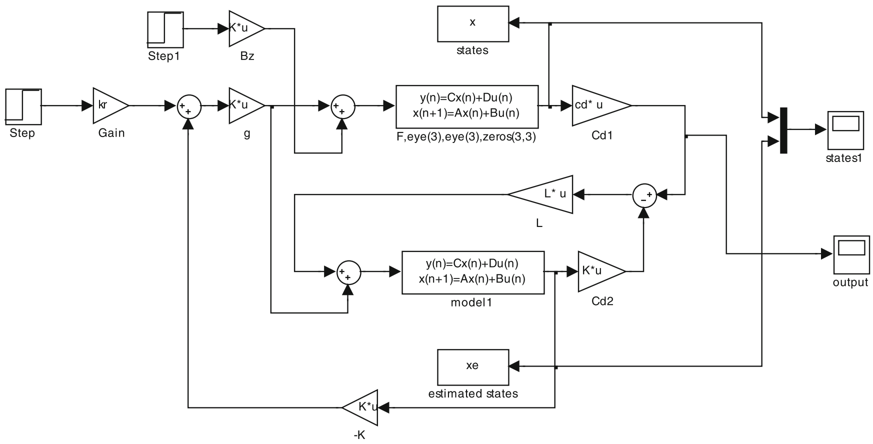

Fig. 14.8 Discrete state feedback from the estimated state variables

produced by feeding back the estimated state variables. Static compensation is applied. The prescribed poles for estimation in the continuous system are set by **`Fc=[-7-7 -7]`**. The prescribed poles for the closed loop continuous system are set by **`Rc=[-6; -3+i*4; -3-i*4]`**. The reference signal is a unit step. Let us simulate the output signal.

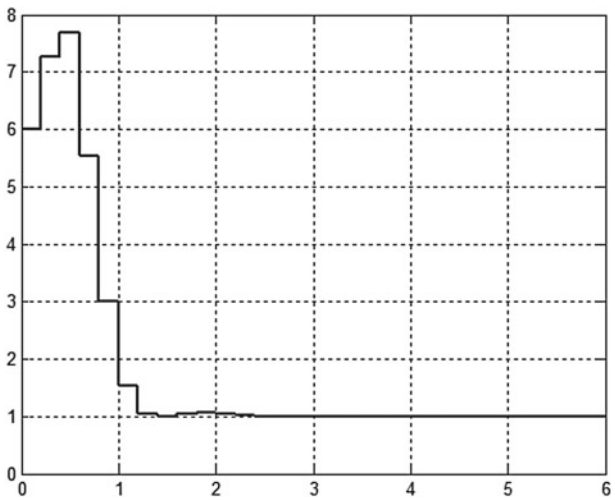

Fig. 14.9 Settling of the output signal for step reference signal and initial conditions

The values of the parameters are:

```
kr=13.9037; k=4.2463   32.4319   77.422;
L=-0.7709 0.2062   0.2163
```

The result of running SIMULINK™ is shown in Fig. 14.9.

Problem Compare the result with the time response obtained for the case without state estimation. Analyse the behaviour of the system if there is also an input disturbance. Extend the SIMULINK™ block scheme with the continuous state model of the process. Analyse the behaviour of the output signal also between the sampling points. Write a MATLAB™ program for analysing the behaviour of the system with state feedback from the estimated state variables (for the continuous case see Example 9.5).