Chapter 40
Adiabatic Spherical Pulsations

40.1 The Eigenvalue Problem

The functions $P_0(m), r_0(m)$, and $\varrho_0(m)$ are supposed to belong to a solution of the stellar-structure equations (10.1)–(10.4) for the case of complete equilibrium. Let us assume that we perturb the hydrostatic equilibrium, say by compressing the star slightly and releasing it again suddenly. It will expand and owing to inertia overshoot the equilibrium state: the star starts to oscillate. The analogy to the oscillating piston model (see Sect. 6.6) is obvious. More precisely we assume the initial displacement of the mass elements to be only radially directed ($d\vartheta = d\varphi = 0$) and of constant absolute value on concentric spheres. This leads to purely *radial oscillations* (or radial pulsations) during which the star remains spherically symmetric all time. For the perturbed variables at time t we write

$$
\begin{aligned}
P(m,t) &= P_0(m) + P_1(m,t) = P_0(m)\left[1 + p(m)\mathrm{e}^{\mathrm{i}wt}\right], \\
r(m,t) &= r_0(m) + r_1(m,t) = r_0(m)\left[1 + x(m)\mathrm{e}^{\mathrm{i}wt}\right], \\
\varrho(m,t) &= \varrho_0(m) + \varrho_1(m,t) = \varrho_0(m)\left[1 + d(m)\mathrm{e}^{\mathrm{i}wt}\right], \qquad (40.1)
\end{aligned}
$$

where the subscript 1 indicates the perturbations for which we have made a separation ansatz with an exponential time dependence [as in (25.17)]. The relative perturbations p, x, d are assumed to be $\ll 1$.

We now insert these expressions into the equation of motion (10.2), linearize, and use the fact that P_0, r_0 obey the hydrostatic equation (10.2). Then with $g_0 = Gm/r_0^2$ we obtain

$$
\frac{\partial}{\partial m}(P_0 p) = (4g_0 + r_0\omega^2)\frac{x}{4\pi r_0^2}\ . \qquad (40.2)
$$

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Using (10.2) again for $\partial P_0/\partial m$ and the relation

$$\frac{\partial}{\partial r_0} = 4\pi r_0^2 \varrho_0 \frac{\partial}{\partial m} , \tag{40.3}$$

we find

$$\frac{P_0}{\varrho_0}\frac{\partial p}{\partial r_0} = \omega^2 r_0 x + g_0(p + 4x) . \tag{40.4}$$

Quite similarly (40.1) introduced into (10.1) yields with (40.3)

$$r_0 \frac{\partial x}{\partial r_0} = -3x - d . \tag{40.5}$$

Note that the transformation (40.3) does not mean that we go back to an Eulerian description. The partial derivative $\partial/\partial t$ describes time variations at constant r_0. But since $r_0 = r_0(m)$ is given by the equilibrium solution, $\partial/\partial t$ also refers to a fixed value of m.

We know already that perturbations of hydrostatic equilibrium proceed on a timescale $\tau_{\rm hydr} \ll \tau_{\rm adj}$. We therefore assume here that the oscillations are adiabatic, which means that

$$p = \gamma_{\rm ad} d . \tag{40.6}$$

This shows again the advantage of using Lagrangian variables: the adiabatic condition has the simple form (40.6) only if p and d are considered functions of m [or of $r_0 = r_0(m)$] and therefore give the variations in the *co-moving frame*. For the sake of simplicity we now assume that $\gamma_{\rm ad}$ is constant in space and time. From (40.5) and (40.6) we obtain by differentiation with respect to r_0

$$\frac{\partial x}{\partial r_0} + r_0 \frac{\partial^2 x}{\partial r_0^2} = -3\frac{\partial x}{\partial r_0} - \frac{1}{\gamma_{\rm ad}}\frac{\partial p}{\partial r_0} . \tag{40.7}$$

Eliminating $\partial_p/\partial r_0$, p, and d from (40.4)–(40.7) gives

$$x'' + \left(\frac{4}{r_0} - \frac{\varrho_0 g_0}{P_0}\right) x' + \frac{\varrho_0}{\gamma_{\rm ad} P_0}\left[\omega^2 + (4 - 3\gamma_{\rm ad})\frac{g_0}{r_0}\right] x = 0 , \tag{40.8}$$

where a prime denotes a derivative with respect to r_0.

This second-order differential equation describes the relative amplitude $x(r_0)$ as function of depth for an adiabatic oscillation of frequency ω. In addition one has to fulfil boundary conditions, one at the centre and one at the surface. At the centre the coefficient of x' in (40.8) is singular, while the coefficient of x remains regular since $g_0 \sim m/r_0^2 \sim r_0$. Because one has to demand that x is regular there, this gives the central boundary condition $x' = 0$.

With a simple expansion into powers of r_0 of the form $x = a_0 + a_1 r_0 + a_2 r_0^2 + \ldots$, one finds that the regular solution starts from the centre outwards with $a_1 = 0$ and

$$a_2 = -\frac{1}{10}\frac{\varrho_c}{\gamma_{\mathrm{ad}} P_c}\left[\omega^2 + (4 - 3\gamma_{\mathrm{ad}})\frac{4\pi}{3} G\varrho_c\right] a_0 \ , \tag{40.9}$$

where the subscript c indicates central values of the unperturbed solution.

For the surface the simple condition $P_1 \equiv p\ P_0 = 0$ is often used. However, one can find a slightly more realistic boundary condition. We simplify the atmosphere by assuming its mass m_{a} to be comprised in a thin layer at $r = R(t)$, which follows the changing R during the oscillations and provides the outer boundary condition at each moment by its weight. We neglect, however, its inertia. Then at the bottom of the "atmosphere" we have

$$4\pi R^2 P - \frac{G m_{\mathrm{a}} M}{R^2} = 0 \ , \tag{40.10}$$

and in the equilibrium state we have

$$4\pi R_0^2 P_0 = \frac{G m_{\mathrm{a}} M}{R_0^2} \ . \tag{40.11}$$

Using this and (40.1), we find from (40.10) that after linearization

$$p + 4x = 0 \ . \tag{40.12}$$

We can rewrite this condition in terms of x and x'. If we replace p in (40.12) by (40.6) and then d by (40.5), the outer boundary condition at $r_0 = R_0$ becomes

$$\gamma_{\mathrm{ad}} R_0 x' - (4 - 3\gamma_{\mathrm{ad}}) x = 0. \tag{40.13}$$

The interior boundary condition at $r_0 = 0$ was

$$x' = 0 \tag{40.14}$$

If we multiply the differential equation (40.8) by $r_0^4 P_0$, we can write it in the form

$$(r_0^4 P_0 x')' + \frac{r_0^4 \varrho_0}{\gamma_{\mathrm{ad}}}\left[\omega^2 + (4 - 3\gamma_{\mathrm{ad}})\frac{g_0}{r_0}\right] x = 0 \ . \tag{40.15}$$

Together with the (linear, homogeneous) boundary conditions (40.13) and (40.14) this defines a classical *Sturm–Liouville problem* with all its consequences.

From the theory of eigenvalue problems of the Sturm–Liouville type, a series of theorems immediately follows that we shall here list without proofs (which can be found in standard textbooks):

1. There is an infinite number of eigenvalues ω_n^2.
2. The ω_n^2 are real and can be placed in the order $\omega_0^2 < \omega_1^2 < \ldots$, with $\omega_n^2 \to \infty$ for $n \to \infty$.
3. The eigenfunction x_0 of the lowest eigenvalue ω_0 has no node in the interval $0 < r_0 < R_0$("fundamental"). For $n > 0$, the eigenfunction x_n has n nodes in the above interval ("nth overtone").
4. The normalized eigenfunctions x_n are complete and obey the orthogonality relation
$$\int_0^{R_0} r_0^4 \varrho_0 \, x_m \, x_n \, d r_0 = \delta_{mn} \ , \tag{40.16}$$

where δ_{mn} is the Kronecker symbol.

The eigenfunctions permit the investigation of the evolution in time of any arbitrary initial perturbation described by $x_m = x_m(r_0), \dot{x}_m = \dot{x}_m(r_0)$ at $t = 0$. Indeed if one writes down the expansion of the initial perturbations in terms of the eigenfunctions,

$$x_m(r_0) = \sum_{n=0}^{\infty} c_n x_n(r_0) \ , \quad \dot{x}_\mathrm{m}(r_0) = \sum_{n=0}^{\infty} d_n x_n(r_0) \ , \tag{40.17}$$

where the c_n, d_n are real, then

$$\begin{aligned} x(r_0, t) &= \mathrm{Re} \left[\sum_{n=0}^{\infty} (a_n \mathrm{e}^{\mathrm{i} w_n t} + b_n \mathrm{e}^{-\mathrm{i} w_n t}) x_n(r_0) \right] , \\ \dot{x}(r_0, t) &= \mathrm{Re} \left[\sum_{n=0}^{\infty} \mathrm{i} w_n (a_n \mathrm{e}^{\mathrm{i} w_n t} - b_n \mathrm{e}^{-\mathrm{i} w_n t}) x_n(r_0) \right] \end{aligned} \tag{40.18}$$

with complex coefficients a_n, b_n, fulfil the time-dependent equation of motion (40.15) with the initial conditions (40.17) at $t = 0$ if a_n, b_n satisfy

$$a_n + b_n = c_n \ , \quad \mathrm{Re}[\mathrm{i} w_n (a_n - b_n)] = d_n . \tag{40.19}$$

Now we come to the question of stability. Since the perturbations are assumed to be adiabatic, it is dynamical stability we are asking for. We have seen that ω_n^2 is real, so that if $\omega_n^2 > 0$, then $\pm\omega_n$ is real, and the perturbations according to (40.1) are purely oscillatory (with constant amplitude): the equilibrium is dynamically stable. If $\omega_n^2 < 0$ then $\pm\omega_n$ is purely imaginary, say $\pm\omega_n = \pm i\chi$ with real χ. The general time-dependent solution for this model is a sum of expressions of the form

$$A x_n \mathrm{e}^{-\chi t} + B x_n \mathrm{e}^{\chi t} \ , \tag{40.20}$$

where A, B are complex constants. Hence at least one of the two terms describes an amplitude growing exponentially in time. This term will necessarily show up in the expansion (40.18) of an arbitrary perturbation and dominate after sufficient time: the equilibrium is dynamically unstable.

The two regimes are separated by the case of marginal stability with $\omega_0^2 = 0$, which according to earlier considerations (Sect. 25.3.2) is expected to occur for $\gamma_{ad} = 4/3$. We now show that this in fact follows from the rather general formalism used here. For simplicity let us assume that $P_0 \to 0$ at the outer boundary.

Integration of (40.15) over the whole star for the fundamental mode ($n = 0$) gives

$$\left[r_0^4 P_0 x_0'\right]_0^{R_0} + \frac{\omega_0^2}{\gamma_{ad}} \int_0^{R_0} r_0^4 \varrho_0 x_0 dr_0 + \frac{4 - 3\gamma_{ad}}{\gamma_{ad}} \int_0^{R_0} r_0^3 \varrho_0 g_0 x_0 dr_0 = 0 \,. \tag{40.21}$$

The boundary term on the left vanishes and we find

$$\omega_0^2 = (3\gamma_{ad} - 4) \frac{\int_0^{R_0} r_0^3 \varrho_0 g_0 x_0 dr_0}{\int_0^{R_0} r_0^4 \varrho_0 g_0 x_0 dr_0} . \tag{40.22}$$

Since x_0, as eigenfunction of the fundamental, does not change sign in the interval, we have sign $\omega_0^2 = \text{sign}(3\gamma_{ad} - 4)$. Therefore $\gamma_{ad} > 4/3$ gives $\omega_0^2 > 0$, and the equilibrium is dynamically stable, because all $\omega_n^2 > \omega_2^0$ for $n > 0$ (see above). If $\gamma_{ad} < 4/3$, then for the fundamental (and possibly for a finite number of overtones), $\omega_n^2 < 0$, and the equilibrium is dynamically unstable.

Here we have assumed that γ_{ad} is constant throughout the stellar model, though the main result is unchanged if γ_{ad} varies; in order to guarantee dynamical stability, then, a mean value of γ_{ad} has to be $> 4/3$.

Of course, we could have carried through the whole procedure using m as independent variable instead of r_0. Then (40.4) and (40.5) would have had to be replaced by the equivalent equations (25.19) and (25.20).

40.2 The Homogeneous Sphere

To illustrate the procedure of Sect. 40.1 we apply it to the simplest, but very instructive, case of a gaseous sphere of constant density, where we have an easy analytical access to the eigenvalues and eigenfunctions.

If ϱ is constant in space, then

$$r_0 = \left(\frac{3m}{4\pi\varrho_0}\right)^{1/3} , \quad g_0 = \frac{Gm}{r_0^2} = \frac{4\pi}{3}Gr_0\varrho_0 , \tag{40.23}$$

and from integration of the equation of hydrostatic equilibrium (2.3) we find

$$P_0(r_0) = \frac{2\pi}{3}G\varrho_0^2\left(R_0^2 - r_0^2\right) , \tag{40.24}$$

where R_0 is the surface radius in hydrostatic equilibrium.

If we introduce the dimensionless variable $\xi = r_0/R_0$ and define

$$\tilde{A} := \frac{3\omega^2}{2\pi G\varrho_0\gamma_{\mathrm{ad}}} + \frac{2(4-3\gamma_{\mathrm{ad}})}{\gamma_{\mathrm{ad}}} , \tag{40.25}$$

then instead of (40.8) we can write

$$\frac{d^2x}{d\xi^2} + \left(\frac{4}{\xi} - \frac{2\xi}{1-\xi^2}\right)\frac{dx}{d\xi} + \frac{\tilde{A}}{1-\xi^2}x = 0 . \tag{40.26}$$

This differential equation has singularities at the centre and at the surface and we look for solutions which are regular at both ends.

The simplest such solution of (40.26) is obvious: $x = x_0 =$ constant is an eigenfunction for $\tilde{A} = 0$. The corresponding eigenfrequency follows from (40.25):

$$\omega_0^2 = \frac{4\pi}{3}G\varrho_0(3\gamma_{\mathrm{ad}} - 4) . \tag{40.27}$$

This represents the fundamental, since the eigenfunction $x =$ constant has no node. The expression (40.27) for the eigenvalue follows immediately from (40.22) for $x_0 =$ constant, $\varrho_0 =$ constant. Note that (40.27) shows the famous period–density relation for pulsating stars: $\omega_0^2/\varrho_0 =$ constant.

For the overtones we try polynomials in r_0. Indeed if for the first overtone we take $x = 1 + b\xi^2$ with constant b, then (40.26) can be solved with $b = -7/5$ and $\tilde{A} = 14$. The corresponding eigenvalue is obtained from (40.25), (40.27) and we have

$$\omega_1^2 = \omega_0^2\left(1 + \frac{7\gamma_{\mathrm{ad}}}{3\gamma_{\mathrm{ad}} - 4}\right) ; \quad x_1 = 1 - \frac{7}{5}\xi^2 . \tag{40.28}$$

The eigenfunction has one node at $\xi = (5/7)^{1/2}$, i.e. at $r_0 = 0.845R_0$. For $\gamma_{\mathrm{ad}} = 5/3$ the ratio of the frequencies of first overtone and fundamental is $\omega_1/\omega_0 = 3.56$.

One can now try higher polynomials with free coefficients in order to find the higher overtones. But we leave this to the reader, the first three eigenfunctions being illustrated in Fig. 40.1.

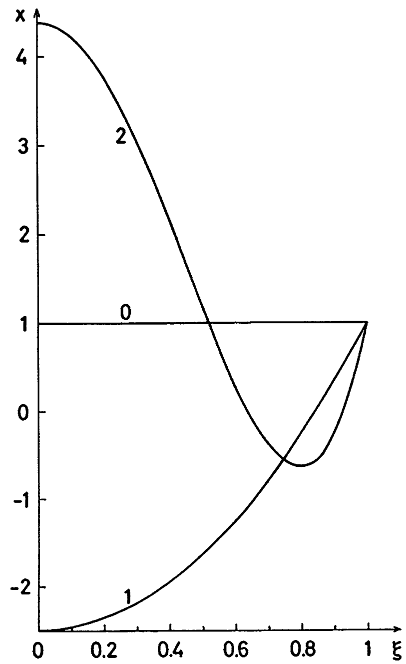

Fig. 40.1 The first three eigenfunctions for radial adiabatic pulsations of the homogeneous sphere

40.3 Pulsating Polytropes

Let us now investigate the (spherically symmetric) radial oscillations of polytropic models of index n as discussed in Chap. 19. We therefore express the quantities of the unperturbed model which appear in the coefficients of (40.8),

$$r_0 \ , \quad \varrho_0 g_0 / P_0 \ , \quad \varrho_0 / P_0 \ , \quad \varrho_0 g_0 / (P_0 r_0) \ ,$$

by the Lane–Emden function $w(z)$ and by its dimensionless argument z. From (19.9) we have

$$g_0 = \frac{\partial \Phi_0}{\partial r_0} = A\Phi_c \frac{dw}{dz} \ ; \quad A^2 = \frac{4\pi G}{[(n+1)K]^n}(-\Phi_c)^{n-1} \ , \tag{40.29}$$

while (19.7) yields

$$\varrho_0 = \left[\frac{-\Phi_c w}{(n+1)K}\right]^n , \tag{40.30}$$

the subscript c denoting central values in the unperturbed model. If we use the polytropic relation (19.3), we find

$$\frac{\varrho_0}{P_0} = \frac{1}{K}\varrho^{-1/n} = -\frac{n+1}{\Phi_c w} \ , \tag{40.31}$$

and we then have

$$\frac{g_0 \varrho_0}{P_0} = -A \frac{n+1}{w} \frac{dw}{dz} \tag{40.32}$$

and

$$\frac{g_0}{r_0} = \frac{\Phi_c A^2}{z} \frac{dw}{dz} \,. \tag{40.33}$$

If we replace r_0 by $z = Ar_0$, the oscillation equation (40.8) becomes

$$\frac{d^2x}{dz^2} + \left(\frac{4}{z} + \frac{n+1}{w} \frac{dw}{dz} \right) \frac{dx}{dz} + \left[\Omega^2 - \frac{(4 - 3\gamma_{ad})(n+1)}{\gamma_{ad}} \frac{1}{z} \frac{dw}{dz} \right] \frac{x}{w} = 0 \,. \tag{40.34}$$

Equation (40.34) is singular at the centre ($z = 0$) and at the surface ($w = 0$). Ω is a dimensionless frequency:

$$\Omega^2 = \frac{n+1}{\gamma_{ad}(-\Phi_c)A^2} \omega^2 \tag{40.35}$$

In (40.34) only γ_{ad}, the polytropic index n, and the Lane–Emden function for this index appear. Therefore the dimensionless eigenvalue Ω^2 obtained from (40.34) depends only on n and γ_{ad}, but not on other properties of the polytropic model, say M or R. The relation (40.35) between Ω and ω can be expressed differently. Using (40.30) for the centre ($w = 1$) and (40.29) we have

$$\omega^2 = \frac{\gamma_{ad}(-\Phi_c)A^2}{n+1} \Omega^2 = \frac{4\pi G \gamma_{ad} \varrho_c}{n+1} \Omega^2 \,. \tag{40.36}$$

Since for a given n the central density ϱ_c and the mean density $\bar{\varrho}$ of the whole unperturbed model differ only by a constant factor, one finds from (40.36) $w^2 = \text{constant} \cdot \bar{\varrho}$, or with the period $\Pi = 2\pi/\omega$

$$\Pi \sqrt{\bar{\varrho}} = \left[\frac{(n+1)\pi}{\gamma_{ad} G \Omega^2} \left(\frac{\bar{\varrho}}{\varrho_c} \right)_n \right]^{1/2} \,. \tag{40.37}$$

For a given mode, say the fundamental, the right-hand side depends only on the polytropic index n and on γ_{ad}. This is the famous *period–density relation*. It is also approximately fulfilled for more realistic stellar models.

If one assumes for a δ Cephei star that $M = 7M_\odot$ and $R = 80R_\odot$, its mean density is $\approx 2 \times 10^{-5}\,\mathrm{g\,cm^{-3}}$. If the period is 11^{d}, then $\Pi(\bar{\varrho})^{1/2} \approx 0.049$ (Π in days, $\bar{\varrho}$ in $\mathrm{g\,cm^{-3}}$). This constant gives a period of about 220 days for a supergiant with $\bar{\varrho} = 5 \times 10^{-8}\,\mathrm{g\,cm^{-3}}$, while for a white dwarf (with $\bar{\varrho} \approx 10^6\,\mathrm{g\,cm^{-3}}$), it gives a period of 4 s. Indeed the supergiant period is of the order of those observed for Mira stars, while very short periods are observed for white dwarfs.

The dimensionless equation (40.34) depends on n and γ_{ad}, where the polytropic index n is a measure of the density concentration, say of $\varrho_c/\bar{\varrho}$, while γ_{ad} is a measure of the stiffness of the configuration. If $\gamma_{ad} = 4/3$, then $\Omega = 0$ is an eigenvalue and x = constant the corresponding eigenfunction, as can be seen from (40.34); the model is then marginally stable and after compression does not go back to its original size. The larger the γ_{ad}, the better the stability, since the compressed model will expand more violently after being released. This can be understood with the help of the considerations in Sect. 25.3.2.

Numerical solutions of the eigenvalue problem show how variations in n and γ_{ad} modify the solutions. Because of the singularities of (40.34) at both ends of the interval $0 < z < z_n$ (z_n is the value of z for which the Lane–Emden function of index n vanishes) the numerical solution is not straightforward. The simplest way is to choose a trial value $\Omega = \Omega^*$ and to start two integrations with power series regular at $z = 0$ and at $z = z_n$. The outward and inward integrations are continued to a common point somewhere, say at $z^* = z_n/2$. There the two solutions will have neither the same value $x(z^*)$ nor the same derivative $(dx/dz)^*$. Since the differential equation is linear and homogeneous, we can multiply one of the solutions by a constant factor such that both get the same value at z^*. But then they probably still disagree in $(dx/dz)^*$. Agreement in the derivatives can be achieved by gradually improving Ω, carrying out new integrations, and so on. By such iterations a solution for the whole interval can be obtained.

Whether by such a procedure one arrives at the fundamental or at an overtone depends in general on the trial Ω^*. If it is near the fundamental, we will end up with the fundamental eigenvalue and eigenfunction. In any case the number of nodes will reveal which mode has been found.

Since (40.34) is linear and homogeneous, the solution may be multiplied by an arbitrary constant factor, in which way we can normalize the solution such that at the surface $x(z_n) = 1$. For the polytrope $n = 3$ the eigenfunctions of different modes for $\gamma_{ad} = 5/3$ are shown in Fig. 40.2 and the eigenfunction of the fundamental for different values of γ_{ad} is displayed in Fig. 40.3.

The variation of γ_{ad} is indeed important. To see this, we assume an ideal monatomic gas with radiation pressure as discussed in Chap. 13. From (13.7), (13.12) and (13.15) we find after some algebra that

$$\gamma_{ad} = \frac{1}{\alpha - \delta\nabla_{ad}} = \frac{32 - 24\beta - 3\beta^2}{24 - 21\beta} \ . \tag{40.38}$$

For the limit cases $\beta = 1$ ($P_{rad} = 0$) and $\beta = 0$ ($P_{gas} = 0$) the adiabatic exponent γ_{ad} takes the values 5/3 and 4/3, respectively. We see that our assumption γ_{ad} = constant throughout the model holds only as long as β = constant. Fortunately this is the case for the polytrope $n = 3$, since $1 - \beta \sim T^4/P$ and $T \sim \omega$, $P \sim \omega^{n+1}$. In (40.34) the radiation pressure only appears in the quantity

$$\varphi := -\frac{4 - 3\gamma_{ad}}{\gamma_{ad}} = 3 - \frac{4}{\gamma_{ad}} \ . \tag{40.39}$$

For vanishing and dominating radiation pressure, φ takes the values 0.6 and 0, respectively.

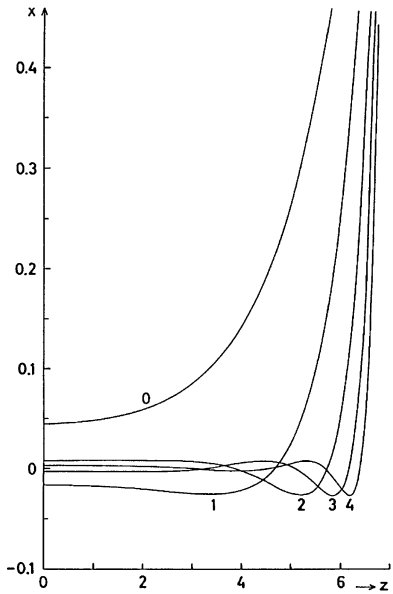

Fig. 40.2 Eigenfunctions for radial adiabatic pulsations of the polytrope $n = 3$ for $\varphi = 0.6$ (After Schwarzschild 1941)

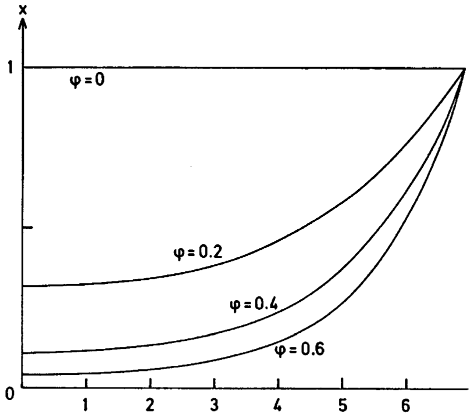

Fig. 40.3 The fundamental eigenfunction for radial adiabatic pulsations of the polytrope $n = 3$ for different values of φ. Radiation pressure diminishes the ratio of the amplitude at the surface to that of the centre. If the radiation pressure dominates the gas pressure completely ($\varphi = 0$) the relative amplitude x is constant

Fundamental and overtone solutions of (40.34) for $n = 3$ and for different values of φ have been found numerically by Schwarzschild (1941). For $\varphi = 0.6$ ($\gamma_{ad} = 5/3$) the (dimensionless) eigenfrequency for the fundamental and the first overtones are $\Omega_0^2 = 0.1367$, $\Omega_1^2 = 0.2509$, $\Omega_2^2 = 0.4209$, $\Omega_3^2 = 0.6420$, $\Omega_4^2 = 0.9117$. The corresponding eigenfunctions are shown in Fig. 40.2.

The influence of β on the fundamental eigenfunction can be seen in Fig. 40.3. With increasing radiation pressure (φ decreasing) the relative amplitude x drops less and less steeply from the surface to the centre. The ratio $x_{\text{surface}}/x_{\text{centre}}$ is 22.4 for $\varphi = 0.6$ and 9.1 for $\varphi = 0.4$. In the limit $\varphi \to 0$ (pure radiation pressure) x even becomes constant. Indeed, for $\gamma_{ad} = 4/3$ and for the eigenvalue $\Omega = 0$, $x =$ constant is a solution as we know already.