Chapter 11
Boundary Conditions

As usual in mathematical physics, the boundary conditions constitute an important part of the whole problem, and their influence on the solutions is not easy to foresee. This is connected with the fact that the boundary conditions for the problem of stellar structure cannot be imposed at one end of the interval $[0, M]$ only but rather are split into some that are given at the centre and some near the surface of the star. The central conditions are simple, whereas the surface conditions implicate observable quantities and a completely different, much more complicated transport equation. It is therefore advisable to get some feeling about their influence on the stellar structure. We discuss these problems for the case of complete equilibrium.

11.1 Central Conditions

Two boundary conditions can be immediately written down for the centre, defined by $m = 0$. Since the density ϱ must go to a reasonable, finite, and non-vanishing value (there can be no singularity and no cavity in the centre), we must have $r = 0$. And since the energy sources also remain finite (positive or negative), l must vanish at the centre as well:

$$m = 0: \quad r = 0, \quad l = 0. \tag{11.1}$$

This was the simple part. Unfortunately nothing is a priori known about the central values of pressure P_c and temperature T_c, so the conditions (11.1) still allow a two-parameter set of solutions, obtained by outward integrations starting with arbitrary P_c, T_c, and $r = l = 0$.

It is useful to know the behaviour of the four functions r, l, P, T in the vicinity of the centre, $m \to 0$, for a given time $t = t_0$. The equation of continuity (10.1) may be written as

$$d(r^3) = \frac{3}{4\pi\varrho} dm, \tag{11.2}$$

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which can be integrated for constant $\varrho = \varrho_c$, i.e. for small enough values of m and r, giving

$$r = \left(\frac{3}{4\pi\varrho_c}\right)^{1/3} m^{1/3}. \tag{11.3}$$

This can be considered the first term in a series expansion of r around $m = 0$. A corresponding integration of the energy equation (10.3) yields

$$l = (\varepsilon_n - \varepsilon_\nu + \varepsilon_g)_c\, m. \tag{11.4}$$

In both cases we have used the proper boundary conditions (11.1) by taking the integration constants to be zero.

Eliminating r for small values of m by (11.3), we obtain from the hydrostatic equation (10.2)

$$\frac{dP}{dm} = -\frac{G}{4\pi}\left(\frac{4\pi\varrho_c}{3}\right)^{4/3} m^{-1/3}, \tag{11.5}$$

which can be integrated to yield

$$P - P_c = -\frac{3G}{8\pi}\left(\frac{4\pi}{3}\varrho_c\right)^{4/3} m^{2/3}. \tag{11.6}$$

The pressure gradient must, of course, vanish at the centre, which can be seen by writing the hydrostatic equation (2.4) in the form

$$\frac{dP}{dr} \sim \frac{m}{r^2} \sim \frac{r^3}{r^2} \to 0 \tag{11.7}$$

for $r \to 0$.

The variation of temperature will first be considered in the radiative case, for which (5.12) requires that

$$\frac{dT}{dm} = -\frac{3}{64\pi^2 ac}\,\frac{\kappa l}{r^4 T^3}. \tag{11.8}$$

With $P \to P_c, T \to T_c$, κ tends to some well-defined value κ_c. Replacing $l(\sim m)$ by (11.4) and $r(\sim m^{1/3})$ by (11.3) now, we can integrate (11.8) for small values of m and obtain the first equation (11.9). In the case of (adiabatic) convection we start from (7.32) with $\nabla = \nabla_{ad}$ and replace r by (11.3). An integration for small values of m then gives the second equation (11.9):

$$T^4 - T_c^4 = -\frac{1}{2ac}\left(\frac{3}{4\pi}\right)^{2/3} \kappa_c(\varepsilon_n - \varepsilon_\nu + \varepsilon_g)_c \varrho_c^{4/3} m^{2/3} \text{ (radiative)},$$

$$\ln T - \ln T_c = -\left(\frac{\pi}{6}\right)^{1/3} G\,\frac{\nabla_{ad,c}\varrho_c^{4/3}}{P_c} m^{2/3} \text{ (convective)}. \tag{11.9}$$

11.2 Surface Conditions

The strict surface conditions are rather complicated and unwieldy. For rough estimates one might therefore prefer to use a crude approximation, provided that it is simple.

An extreme step in this direction would be to take the naïve "zero conditions"

$$m \to M : \quad P \to 0, \quad T \to 0. \tag{11.10}$$

These at least reflect correctly the fact that, in the outermost region of the star, P and T go to very small values compared to those in the interior. But, of course, in reality, there is a gradual and rather extended transition to the finite values of P, T of the diffuse interstellar medium.

The next step is to find a sphere that we can reasonably call the "surface" of the star and that defines the total stellar radius $r = R$. The theory of stellar atmospheres suggests the use of the *photosphere*, from where the bulk of the radiation is emitted into space, and which is found where the optical depth τ of the overlying layers,

$$\tau := \int_R^\infty \kappa \varrho \, dr = \bar{\kappa} \int_R^\infty \varrho \, dr, \tag{11.11}$$

is equal to 2/3. Here we have defined a mean opacity $\bar{\kappa}$, averaged over the stellar atmosphere. In hydrostatic equilibrium the pressure at this level is given by the weight of the matter above. We can well approximate the gravitational acceleration by the constant value $g_0 = GM/R^2$, since the bulk of the matter in these layers is anyway very close to the photosphere. Then

$$P_{r=R} = \int_R^\infty g \varrho \, dr = g_0 \int_R^\infty \varrho \, dr, \tag{11.12}$$

and if we eliminate here the integral over ϱ by that in the second equation (11.11), we find with $\tau = 2/3$ that

$$P_{r=R} = \frac{GM}{R^2} \frac{2}{3} \frac{1}{\bar{\kappa}}. \tag{11.13}$$

The temperature at the photosphere is equal to the *effective temperature* $T_{r=R} = T_{\text{eff}}$ of the star defined by

$$L = 4\pi R^2 \sigma \, T_{\text{eff}}^4. \tag{11.14}$$

Here $\sigma = ac/4$ is the Stefan-Boltzmann constant of radiation. T_{eff} is thus the temperature of that black body which yields the same surface flux of energy as the star.

In (11.11) we have replaced κ by an average value. As we usually have detailed knowledge about the opacity, an obvious improvement is to take into account the pressure (or density) and temperature dependency of κ. This requires knowledge about the temperature stratification in the atmosphere. One common approach is to use the *Eddington approximation*, which is

$$T^4(\tau) = \frac{3}{4}\left(L/4\pi R^2\sigma\right)\left(\tau + \frac{2}{3}\right), \tag{11.15}$$

which obviously results in $T = T_{\rm eff}$ for $\tau = 2/3$.

Equation (11.11) can be transformed into a differential equation for the radius, $dr/d\tau = -1/(\kappa\varrho)$, and with $dP/dr = -g\varrho$ we obtain

$$\frac{dP}{d\tau} = \frac{Gm}{r^2\kappa} \tag{11.16}$$

which is to be integrated from $\tau = 0$ to $\tau = 2/3$ with the boundary condition $P(\tau = 0) = 0$. The Eddington approximation (11.15) is used to determine $\kappa(P, T)$. For the gravitational acceleration on the right-hand side of (11.16) we can savely use GM/R^2. We thus obtain an improved value for $P_{r=R}$. This approach is called the *Eddington grey atmosphere* because of the use of the Eddington approximation and the Rosseland mean for the opacity, and it is indeed used in stellar evolution codes, which solve the stellar structure equations (10.1)–(10.5) from $m = 0$ to $m = M$.

The *photospheric conditions* (11.13) and (11.14) or (11.16) and (11.14) represent two relations between the surface values ($m \to M$) of the functions P, T, r, l. They are certainly a better approximation for the surface conditions than (11.10). Their severest defect is that they refer to a level where the assumption made for deriving the transport equation (5.12) (small mean free path of the photons) breaks down. At this level, one should use the more complicated transport equation for stellar atmospheres. Indeed such attempts have been made, and full stellar atmosphere models are connected to those of the stellar interior at a suitable optical depth. Examples are the work by Schlattl et al. (1997) for the solar case and VandenBerg et al. (2008) for low-mass stars.

Quite generally, the correct surface conditions can be formulated as follows: the interior solution should fit smoothly to a solution of the stellar-atmosphere problem. Let us put this into a more mathematical form.

The transition between interior and outer (atmospheric) solutions is made at a certain mass value $m_{\rm F}$, the "fitting mass", which should be far enough in to ensure that the interior equations are still valid there. On the other hand, $m_{\rm F}$ should still be close enough to M that, for simplicity, we can always use thermally adjusted outer solutions with constant $l = L$. The smaller $M - m_{\rm F}$, the less energy can be stored or released in these outer layers.

For the stellar-interior problem, we consider the mass M and the chemical composition to be given. The theory of stellar atmospheres tells us that for given M and $X_i(M)$, there is a two-parameter set of possible atmospheric solutions, the parameters being, for example, R and $T_{\rm eff}$, or R and L [which are connected by (11.14)]. Any one of these possible atmospheric solutions can be extended by integration downwards to $m_{\rm F}$ and may yield there the four "exterior" values $r = r_{\rm F}^{\rm ex}, P = P_{\rm F}^{\rm ex}, T = T_{\rm F}^{\rm ex}, l = l_{\rm F}^{\rm ex} = L$.

The outer boundary conditions now require for $m = m_{\mathrm{F}}$ that one quartet $r_{\mathrm{F}}^{\mathrm{ex}}, \ldots, l_{\mathrm{F}}^{\mathrm{ex}}$ obtained from an outer solution has to match the corresponding values $r_{\mathrm{F}}^{\mathrm{in}}, \ldots, l_{\mathrm{F}}^{\mathrm{in}}$ of the interior solution, which extends from the centre to m_{F}:

$$r_{\mathrm{F}}^{\mathrm{ex}} = r_{\mathrm{F}}^{\mathrm{in}}, \quad P_{\mathrm{F}}^{\mathrm{ex}} = P_{\mathrm{F}}^{\mathrm{in}}, \quad T_{\mathrm{F}}^{\mathrm{ex}} = T_{\mathrm{F}}^{\mathrm{in}}, \quad l_{\mathrm{F}}^{\mathrm{ex}} = l_{\mathrm{F}}^{\mathrm{in}}. \tag{11.17}$$

These four simultaneous fits are in principle possible, since the solutions have enough degrees of freedom: the interior solution has two (we can vary the central values P_{c} and T_{c}), and the outer solution also has two (variation of R and L). The fact that both solutions have two degrees of freedom is reflected in the following alternative representation, which is often used in numerical computations. Imagine that many outer integrations are carried out for many pairs of parameters R and L. At $m = m_{\mathrm{F}}$, they yield the four functions $r_{\mathrm{F}}^{\mathrm{ex}}(R, L)$, $P_{\mathrm{F}}^{\mathrm{ex}}(R, L)$, $T_{\mathrm{F}}^{\mathrm{ex}}(R, L)$, $l_{\mathrm{F}}^{\mathrm{ex}}(R, L)$. The last one is very simple, namely $l_{\mathrm{F}}^{\mathrm{ex}} = L$. The first one is certainly well behaved, and we can invert it without complications, obtaining $R = R(r_{\mathrm{F}}^{\mathrm{ex}}, L)$. This is now used to replace the argument R in the functions $P_{\mathrm{F}}^{\mathrm{ex}}$ and $T_{\mathrm{F}}^{\mathrm{ex}}$, which can then be considered known functions π and θ of $r_{\mathrm{F}}^{\mathrm{ex}}$ and $l_{\mathrm{F}}^{\mathrm{ex}} = L$:

$$\begin{aligned} P_{\mathrm{F}}^{\mathrm{ex}}\big(R(r_{\mathrm{F}}^{\mathrm{ex}}, L), L\big) &:= \pi\big(r_{\mathrm{F}}^{\mathrm{ex}}, L\big), \\ T_{\mathrm{F}}^{\mathrm{ex}}\big(R\big(r_{\mathrm{F}}^{\mathrm{ex}}, L\big), L\big) &:= \theta\big(r_{\mathrm{F}}^{\mathrm{ex}}, L\big). \end{aligned} \tag{11.18}$$

For any given pair $r_{\mathrm{F}}^{\mathrm{ex}}, L$, the π and θ give the corresponding values of pressure and temperature for one outer solution. We now replace the variables $P_{\mathrm{F}}^{\mathrm{ex}}, \ldots, l_{\mathrm{F}}^{\mathrm{ex}} = L$ in (11.18) by $P_{\mathrm{F}}^{\mathrm{in}}, \ldots, l_{\mathrm{F}}^{\mathrm{in}}$, using the fit conditions (11.17):

$$P_{\mathrm{F}}^{\mathrm{in}} = \pi\big(r_{\mathrm{F}}^{\mathrm{in}}, L\big), \quad T_{\mathrm{F}}^{\mathrm{in}} = \theta\big(r_{\mathrm{F}}^{\mathrm{in}}, L\big). \tag{11.19}$$

These are the outer boundary conditions for the interior solution. Obviously, if these are fulfilled, there is always an outer solution that continuously matches the interior solution. We can now drop the distinction between the variables of the exterior and interior solutions at $m = m_{\mathrm{F}}$ expressed in the superscripts "ex" and "in".

The fulfilment of the boundary conditions is illustrated in Fig. 11.1, where the functions π and θ (obtained from outer solutions) are sketched over the r_{F}-L plane. We have also indicated the surfaces $\tilde{\pi}(r_{\mathrm{F}}, L)$ and $\tilde{\theta}(r_{\mathrm{F}}, L)$, which give the corresponding functions of the *interior* solutions obtained by varying P_{c} and T_{c}. The intersection of the surfaces ($\pi = \tilde{\pi}$ and $\theta = \tilde{\theta}$) gives the matches of P_{F} or of T_{F}, respectively. We project the intersections into the r_{F}-L plane (dot-dashed lines), and where these projections intersect, we have the desired match of all four variables.

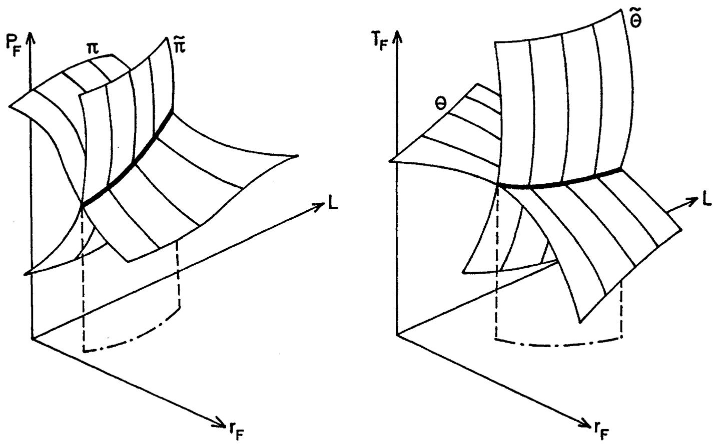

Fig. 11.1 The function values P_F (or T_F) at the fitting mass $m = M_F$ are plotted over r_F and L. The surface π (or θ) contains the values obtained by all possible integrations downwards from the photosphere. The surface $\tilde{\pi}$ (or $\tilde{\theta}$) contains the corresponding values obtained from all possible integrations outwards from the centre. The *heavy line* shows the intersection of π and $\tilde{\pi}$ (or θ and $\tilde{\theta}$), the *dot-dashed line* the projection of this intersection into the r_F-L plane (All surfaces are freely invented sketches)

11.3 Influence of the Surface Conditions and Properties of Envelope Solutions

We confine ourselves here to "normal" stars in complete (mechanical and thermal) equilibrium. For the outer envelope of such a star, it is characteristic that l and m vary very little over wide ranges of r (This is because ε is negligible and ϱ is very small; for example, only about 10 % of the solar mass lies outside $r = R_\odot/2$.). This allows the derivation of approximate solutions that demonstrate the influence of the outer layers on the interior solution.

11.3.1 Radiative Envelopes

Since m varies so little in the envelope, it seems advisable to take another independent variable, for which we may choose the pressure P, since it varies monotonically with m. The equation of radiative transport is derived from (5.12) and (2.5) as

$$\frac{\partial T}{\partial P} = \frac{3}{64\pi\sigma G}\frac{\kappa l}{T^3 m} \tag{11.20}$$

($\sigma = ac/4$). Let us approximate the dependence of κ on P and T by a power law of the form

$$\kappa = \kappa_0 P^a T^b, \tag{11.21}$$

with κ_0 = constant and exponents typically $a > 0, b < 0$. By proper choice of κ_0, a, and b we can represent reasonably (though, of course, not correctly) the run of κ over wide ranges of the envelope. Introducing (11.21) into (11.20) results in

$$\frac{T^{3-b}}{P^a}\frac{\partial T}{\partial P} = \frac{3\kappa_0}{64\pi\sigma G}\frac{l}{m}, \tag{11.22}$$

and now we take $l \approx L$ and $m \approx M$ (this, together with the approximation of κ, determines how far inwards we are allowed to extend our solution). Then the right-hand side is constant and (11.22) can be integrated by separation of the variables:

$$T^{4-b} = B(P^{1+a} + C), \tag{11.23}$$

where C is a constant of integration, while the positive constant B is given by

$$B = \frac{4-b}{1+a}\frac{3\kappa_0}{64\pi\sigma G}\frac{L}{M}. \tag{11.24}$$

For an illustrative example we now fix the exponents: $a = 1, b = -4.5$, which corresponds to the famous Kramers opacity for bound–free and free–free absorption in stellar material (see Chap. 17), and which is a good approximation for envelopes of moderate temperatures. Then (11.23) becomes

$$T^{8.5} = B(P^2 + C), \tag{11.25}$$

a solution for the envelope that will now be discussed. It is illustrated in Fig. 11.2, which gives $\lg T$ against $\lg P$, so that the slope of a solution is equal to the value of $\nabla \equiv d \ln T / d \ln P$. Differentiation of (11.25) gives the slope

$$\nabla = 0.235\frac{BP^2}{T^{8.5}}. \tag{11.26}$$

The multitude of possible solutions differ by their value of the integration constant C.

$\underline{C = 0}$: The solution (11.25) now gives

$$\frac{T^{8.5}}{BP^2} = 1, \tag{11.27}$$

for which (11.26) yields the slope $\nabla = 2/8.5 \approx 0.235$. This is smaller than the usual value of $\nabla_{ad} = 2/5$ (see Chap. 14), and therefore the solution is consistent with (the assumed) radiative transport, shown in Fig. 11.2 as the straight solid line $\lg T = (2 \lg P + \lg B)/8.5$. Obviously $T \to 0$ for $P \to 0$, and this solution would reach the zero boundary condition if we were to extend it outwards over the photosphere.

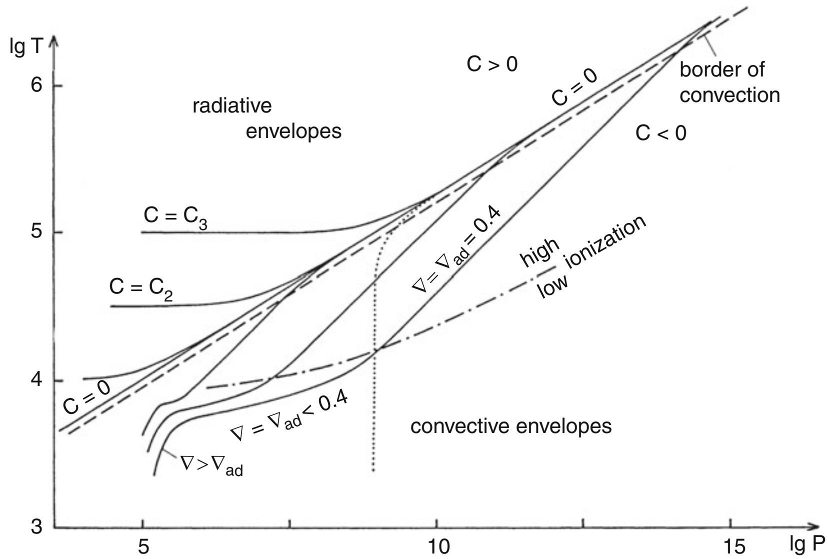

Fig. 11.2 A lg$T$–lg$P$ diagram for illustrating typical properties of envelope solutions as discussed in the text (see there for details)

$\underline{C > 0}$: Since $B > 0$, (11.25) yields

$$\frac{T^{8.5}}{BP^2} > 1. \tag{11.28}$$

Comparing this with (11.26) and (11.27), we see that in Fig. 11.2, the solutions with $C > 0$ lie above that with $C = 0$ and that they have a smaller slope, $\nabla < 2/8.5$. The layers are therefore all the more radiative. For $P^2 \ll C$ equation (11.25) becomes $T^{8.5} \approx BC$ = constant. This shows that towards the surface these solutions tend to a constant (and rather high) T. Three of them (for 3 different values $C_1 < C_2 < C_3$ of C) are illustrated by solid lines on the left of Fig. 11.2. On each line, one point corresponds to the photosphere with $T = T_{\text{eff}}$. Obviously we will find such radiative-envelope solutions below the photospheres with T_{eff} larger than some critical value (close to 10^4 K). Towards the interior, P will finally increase so far that $P^2 \gg C$ in (11.25) and the solution approximates closely that for $C = 0$. Since all solutions with $C > 0$ asymptotically approach the solution $C = 0$, the precise starting values at the surface do not greatly influence the solution in the deep interior.

$\underline{C < 0}$: Equation (11.25) now gives

$$\frac{T^{8.5}}{BP^2} < 1, \tag{11.29}$$

which with (11.26) and (11.27) shows that these solutions lie below the curve for $C = 0$ and that their slope is larger, $\nabla > 2/8.5$. A discussion quite analogous to that for $C > 0$ shows immediately that these solutions have the structure indicated in Fig. 11.2 by the dotted line. They bend downwards from the line $C = 0$, become gradually steeper, and tend vertically to a finite P for $T \to 0$ (With a proper scaling of the coordinates the curves $C > 0$ and $C < 0$ are simply symmetric with respect to the line $C = 0$.). However, the assumption of radiative transport breaks down when convection sets in, which is the case for $\nabla = \nabla_{ad}$ (see Sect. 6.1). This is close to 0.4 in the interior of not too massive stars, while ionization effects near the surface can make it considerably smaller (see Chap. 14). This limit is derived by equating the right-hand side of (11.26) with ∇_{ad}:

$$T^{8.5} = \frac{0.235}{\nabla_{ad}} BP^2. \tag{11.30}$$

For constant ∇_{ad} this corresponds to a straight line given by $\lg T = (2 \lg P + \lg B + \lg(0.235/\nabla_{ad}))/8.5$. For $\nabla_{ad} = 0.4$ this lower border for radiative solutions is plotted in Fig. 11.2 (dashed line). Near the surface, ionization effects decrease ∇_{ad} considerably below 0.4, and therefore the border line should be curved *upwards* in its lowest part.

11.3.2 Convective Envelopes

The radiative solutions with $C < 0$ extending from the interior have to be terminated at the broken line in Fig. 11.2 given by (11.30), where convection sets in, and have to be replaced in the outer regions by solutions valid for convective transport. Three such convective solutions are shown as solid lines in the lower part of Fig. 11.2. In order to construct them we have to consider their slope $d \lg T/d \lg P$ $(= \nabla)$. As long as the solutions stay in regions of high enough density, convection is very effective (cf. Sect. 7.3) and the slope is equal to the adiabatic gradient ∇_{ad}.

We can start the convective solutions near the border of convection with a slope given by $\nabla = \nabla_{ad} = 0.4$. With decreasing temperature the curves come into regions where the most abundant elements (hydrogen and helium) are no longer completely ionized (see Chap. 14). For hydrogen this occurs around $T = 10^4$ K, depending somewhat on P (cf. the dependence of the Saha equation on the electron density). Partial ionization depresses ∇_{ad} appreciably below 0.4 such that the curves with a slope $\nabla = \nabla_{ad}$ are less steep and closely approach one another.

Finally the curves come into regions of such low density that convection is ineffective and the stratification is over-adiabatic, $\nabla > \nabla_{ad}$ (Chap. 7). Correspondingly the curves in Fig. 11.2 become rather steep until they reach the photospheric point. Unfortunately the precise slope ∇ in the over-adiabatic part can only be calculated from a convection theory, with all its uncertainties. Anyway, convective envelopes start at cool photospheres, and with decreasing T_{eff}, the convection gradually reaches deeper into the interior. Small variations (due to numerical or physical uncertainties) of T_{eff} or of the over-adiabatic part lead to curves that are widely separated in the interior.

11.3.3 Summary

Making a few simplifying assumptions, we have been able to derive convenient solutions for the temperature-pressure stratification of stellar envelopes, i.e. for the layers below the photosphere. In the case of radiative envelopes, the assumptions concerned κ, m, and l. An opacity law like (11.21) is certainly a poor approximation if one takes the same values of a, b, κ_0 for too wide a range, or for very different envelopes. The discussions can, however, be easily repeated for different values of a, b, κ_0 [e.g. $a = 0, b = 0, \kappa_0 = 0.2(1 + X_{\text{H}})$, as in the case of electron scattering, Sect. 17.1] giving essentially similar results. The assumption $l =$ constant certainly holds for $T < 10^6$ K, where nuclear burning is negligible, though the assumption $m =$ constant $= M$ breaks down much earlier. But, even if we stress these assumptions somewhat by extending the solutions too far inwards, we will still obtain the correct qualitative behaviour.

Radiative envelopes are found below all hot photospheres ($T > 9,000$ K). Towards the deep interior these solutions converge rapidly to the solution with $C = 0$. The interior is therefore relatively insensitive to details of the outer boundary conditions, in particular to the photospheric details.

Below cool atmospheres there are convective envelopes, which extend farther downwards the smaller T_{eff} is. This suggests that a minimum value of T_{eff} might exist where the whole star has become convective (cf. the Hayashi line, Chap. 24). The inward extension of the convective part depends rather sensitively on the precise position of the photosphere and the details of the over-adiabatic layer. Small changes in even the outer solution, which are otherwise rather unimportant, can exert a remarkable influence on the interior, and the same is true for the uncertainties in the treatment of superadiabatic convection.

11.3.4 The $T-r$ Stratification

Sometimes it is useful to know how $T = T(r)$ increases below the photosphere. From the definition of $\nabla \equiv d \ln T / d \ln P$ we have $dT = T \nabla dP/P$, where we replace dP by using the hydrostatic equation in the form

$$dP = -\frac{Gm}{r^2} \varrho \, dr = Gm\varrho \, d\left(\frac{1}{r}\right) \tag{11.31}$$

and eliminate $T\varrho/P = \mu/\Re$ by means of the equation of state for a perfect gas. We then have

$$dT = \nabla \frac{G\mu}{\Re} m \, d\left(\frac{1}{r}\right). \tag{11.32}$$

For the outer envelope with low density we may approximate m by the surface value M, so that if ∇ is constant between points 1 and 2, we can integrate (11.32) to obtain

$$T_1 - T_2 = \nabla \frac{GM\mu}{\Re}\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \tag{11.33}$$

Let the subscript 2 indicate the photosphere, i.e. $T_2 = T_{\text{eff}}$ and $r_2 = R$. Now at any point $r = r_1$ in the envelope we have

$$T - T_{\text{eff}} = f\left(\frac{R}{r} - 1\right), \quad f = \nabla \frac{g\mu}{\Re} \frac{M}{R}. \tag{11.34}$$

As a simple example we take $M = M_\odot, R = R_\odot$ and a solution with $C = 0$ (see Sect. 11.3.1), for which we found that $\nabla = 0.235$. With $\mu = 1$ we find that $f = 5.4 \times 10^6$ K. This large value of f provides for a very rapid increase of T below the photosphere. Within only 2 % of the radius, T has reached 10^5 K. And at $r \approx 0.8R$ (where $m \approx 0.99M$ still) the temperature exceeds 10^6 K, which also shows that the "average" T for all mass elements of the star is well above 10^6 K.