Chapter 20
Homology Relations

In physical problems it often happens that from one solution others can be obtained by simple transformations. When comparing different stellar models that are calculated under similar assumptions (concerning parameters or material functions), one therefore expects to find similarities in the solutions. It would be very helpful if we could find simple analytic expressions that transform one solution into another. It would then only be necessary to produce *one* numerical solution in order to find new ones by a transformation. There is indeed often a kind of "similarity" between different solutions, which is called *homology,* though the conditions for this are so severe that real stars will scarcely match them. There are a few cases, however, for which homology relations offer a rough, but helpful, indication for interpreting or predicting the numerical solutions. We indicate this in two examples, the main-sequence models and the homologous contraction. Except for this classical homology there is another type of homology, which applies to certain red giants (see Sect. 33.2).

20.1 Definitions and Basic Relations

When comparing different models (say of masses M and M', and radii R and R') one considers in particular *homologous points* at which the relative radii are equal: $r/R = r'/R'$. We now speak of *homologous stars* if their homologous mass shells ($m/M = m'/M'$) are situated at homologous points. To be more precise, let us consider all radii as functions of the *relative* mass values ξ, which are the same for homologous masses:

$$\xi := m/M = m'/M'. \tag{20.1}$$

We can then write the homology condition as

$$\frac{r(\xi)}{r'(\xi)} = \frac{R}{R'} \tag{20.2}$$

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for all ξ. In homologous stars the ratio of the radii r/r' for homologous mass shells is constant throughout the stars. Going from one homologous star to another, all homologous mass shells are compressed (or expanded) by the same factor R/R' (Note that therefore any two polytropic models of the same index n are homologous to each other.).

Since both models have to fulfil the stellar-structure equations, the transition has, of course, consequences for all other variables. We derive these by comparing two homologous stars of masses M and M' and of two different compositions that are supposed to be homogeneous and represented by the mean molecular weights μ and μ'. The ratio of these basic parameters will be called

$$x = M/M'; \quad y = \mu/\mu'. \tag{20.3}$$

The variables in the two models are always considered functions of the relative mass variable ξ and may be called r, P, T, l (for M, μ), and r', P', T', l' (for M', μ'), respectively. We try the following "ansatz": for homologous mass values ξ (which we omit for clarity in the following equations) the variables are supposed to have the ratios

$$\frac{r}{r'} = z = \frac{R}{R'}; \quad \frac{P}{P'} = p = \frac{P_c}{P'_c}; \quad \frac{T}{T'} = t = \frac{T_c}{T'_c}; \quad \frac{l}{l'} = s = \frac{L}{L'}, \tag{20.4}$$

where z, p, t, s *have the same values for all* ξ and where the subscript c indicates central values.

We start with homologous main-sequence models. Since they evolve within the long nuclear timescale, one can use (10.2), neglecting the inertia term, as well as the time derivatives in the energy equation (10.3). Let us assume that in these two stars in complete equilibrium (hydrostatic and thermal) the energy transport is radiative. The basic equations to be fulfilled are then (10.1), (10.2), (10.4) and (10.16) together with (10.6), where we further set ε for the total energy production rate. We write them for the first star in terms of the relative mass variable ξ as

$$\begin{aligned}
\frac{dr}{d\xi} &= c_1 \frac{M}{r^2 \varrho}, \qquad c_1 = \frac{1}{4\pi}, \\
\frac{dP}{d\xi} &= c_2 \frac{\xi M^2}{r^4}, \qquad c_2 = -\frac{G}{4\pi}, \\
\frac{dl}{d\xi} &= \varepsilon M, \\
\frac{dT}{d\xi} &= c_4 \frac{\kappa l M}{r^4 T^3}, \qquad c_4 = -\frac{3}{64\pi^2 ac}.
\end{aligned} \tag{20.5}$$

Since no time derivatives appear, the differentiations with respect to ξ are written as ordinary derivatives. In these equations we transform the variables r, P, T, l into

r', P', T', l' by use of (20.4). Noting that the z, p, t, s are independent of ξ, and that ξ contains the total mass as scaling factor, which has to be transformed by (20.3), one immediately finds the transformed equations:

$$\begin{aligned}
\frac{dr'}{d\xi} &= c_1 \frac{M'}{r'^2 \varrho'} \left[\frac{x}{z^3 d}\right], \\
\frac{dP'}{d\xi} &= c_2 \frac{\xi M'^2}{r'^4} \left[\frac{x^2}{z^4 p}\right], \\
\frac{dl'}{d\xi} &= \varepsilon' M' \left[\frac{ex}{s}\right], \\
\frac{dT'}{d\xi} &= c_4 \frac{\kappa' l' M'}{r'^4 T'^3} \left[\frac{ksx}{z^4 t^4}\right].
\end{aligned} \tag{20.6}$$

$c_1, \ldots, c_4$ are the same constants as before, and we have introduced the additional abbreviations

$$\frac{\varrho}{\varrho'} = d \; ; \quad \frac{\varepsilon}{\varepsilon'} = e \; ; \quad \frac{\kappa}{\kappa'} = k \tag{20.7}$$

for the ratios of the material functions at homologous points.

Since for the variables r', P', T', l' we could have written the same basic equations (20.5) as for r, P, T, l, a comparison of (20.6) with (20.5) shows immediately that the four factors in brackets in (20.6) must be equal to one:

$$\frac{x}{z^3 d} = 1, \quad \frac{x^2}{z^4 p} = 1, \quad \frac{ex}{s} = 1, \quad \frac{ksx}{z^4 t^4} = 1. \tag{20.8}$$

Without further specification of the material functions, we can obtain two useful relations already from the first and second of equations (20.8). They can be rewritten as

$$\frac{\varrho}{\varrho'} = \frac{M/M'}{(R/R')^3}, \quad \frac{P}{P'} = \frac{(M/M')^2}{(R/R')^4}. \tag{20.9}$$

Therefore, for all homologous points, the density changes simply as the mean density for the whole star, while P varies like $M^2 R^{-4}$.

In order to find solutions for (20.8), we represent the material functions by power laws:

$$\varrho \sim P^\alpha T^{-\delta} \mu^\varphi, \quad \varepsilon \sim \varrho^\lambda T^\nu, \quad \kappa \sim P^a T^b, \tag{20.10}$$

which from (20.7) with (20.4) give

$$d = p^\alpha t^{-\delta} y^\varphi, \quad e = p^{\lambda\alpha} t^{\nu - \lambda\delta} y^{\lambda\varphi}, \quad k = p^a t^b. \tag{20.11}$$

These can be introduced into (20.8), which are then four conditions for the powers of z, p, t, and s. We will try to represent them in terms of x and y, which, according to (20.3), describe the change of the basic parameters M and μ:

$$z = x^{z_1} y^{z_2} \ ; \quad p = x^{p_1} y^{p_2} \ ; \quad t = x^{t_1} y^{t_2} \ ; \quad s = x^{s_1} y^{s_2}. \tag{20.12}$$

Introducing these and (20.11) into (20.8), we obtain four conditions which contain only products of powers of x and y. In each condition, the exponents of x and of y must sum up to zero, since the right-hand sides of (20.8) are independent of x and y. This yields eight linear equations for the exponents $z_1, \ldots, s_2$, which are written in matrix form as

$$\begin{pmatrix} -3 & -\alpha & \delta & 0 \\ -4 & -1 & 0 & 0 \\ 0 & \lambda\alpha & (\nu - \lambda\delta) & -1 \\ -4 & a & (b-4) & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ p_1 \\ t_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \\ -1 \end{pmatrix} \tag{20.13}$$

and

$$\begin{pmatrix} -3 & -\alpha & \delta & 0 \\ -4 & -1 & 0 & 0 \\ 0 & \lambda\alpha & (\nu - \lambda\delta) & -1 \\ -4 & a & (b-4) & 1 \end{pmatrix} \begin{pmatrix} z_2 \\ p_2 \\ t_2 \\ s_2 \end{pmatrix} = \begin{pmatrix} \varphi \\ 0 \\ -\lambda\varphi \\ 0 \end{pmatrix}. \tag{20.14}$$

The solutions are

$$\begin{aligned} z_1 &= \frac{1}{2}(1 + A), \quad p_1 = -2A, \\ t_1 &= \frac{1}{2\delta}[1 + (3 - 4\alpha)A], \\ s_1 &= 1 + \frac{4-b}{2\delta} + \left[2 + 2a + \frac{3-4\alpha}{2\delta}(4-b)\right]A, \end{aligned} \tag{20.15}$$

and

$$\begin{aligned} z_2 &= \varphi B, \quad p_2 = -4\varphi B, \quad t_2 = \frac{\varphi}{\delta}[1 + (3 - 4\alpha)B], \\ s_2 &= \frac{\varphi}{\delta}(4-b) + \varphi\left[4 + 4a + \frac{3-4\alpha}{\delta}(4-b)\right]B, \end{aligned} \tag{20.16}$$

$$A = \left[\frac{4\delta(1 + a + \lambda\alpha)}{\nu + b - 4 - \lambda\delta} + 4\alpha - 3\right]^{-1}, \quad B = A\left(1 - \frac{\lambda\delta}{\nu + b - 4}\right)^{-1}. \tag{20.17}$$

20.2 Applications to Simple Material Functions

20.2.1 The Case $\delta = 0$

A special situation arises for the case that the density is independent of T, i.e. $\delta = 0$ in (20.10). The equation of state then is polytropic, the polytropic index being $n = \alpha/(1-\alpha)$, and we must recover the typical properties of polytropic stars (see Sect. 19.3). This can, in fact, be easily verified. To start with, the first two equations of system (20.13) (which represent the mechanical part) can be solved independently of the rest (the thermo-energetic part). For $\delta = 0$ we find from (20.15) and (20.17) that $A = (4\alpha - 3)^{-1}$ and $z_1 = (2\alpha - 1)/(4\alpha - 3)$. The first of (20.12) gives for homologous stars of equal composition ($y = 1$) the mass-radius relation

$$R \sim M^{z_1}. \tag{20.18}$$

For a non-relativistic degenerate electron gas, one has $\alpha = 3/5$, which gives the exponent $z_1 = -1/3$ as already obtained in Sect. 19.6.

20.2.2 The Case $\alpha = \delta = \varphi = 1, a = b = 0$

Further discussion of the above homology solutions will concentrate on the simplest case, an ideal gas ($\alpha = \delta = \varphi = 1$) with constant opacity ($a = b = 0$) [cf. (20.10)]. This extremely rough approximation to reality suffices for outlining some general properties of main-sequence stars (The assumption of homology introduces a much severer limitation on the results.).

From (20.15)–(20.17), one finds

$$\begin{aligned}
z_1 &= \frac{\nu + \lambda - 2}{\nu + 3\lambda}, & z_2 &= \frac{\nu - 4}{\nu + 3\lambda}, \\
p_1 &= 2 - 4z_1, & p_2 &= -4z_2, \\
t_1 &= 1 - z_1, & t_2 &= 1 - z_2, \\
s_1 &= 3, & s_2 &= 4.
\end{aligned} \tag{20.19}$$

The first surprising result concerns the exponents of the luminosity, s_1 and s_2. In this simple case the square brackets in the equations for s_1 and s_2 in (20.15) and (20.16) vanish, and s_1 and s_2 become simple constant numbers. In particular, *they are independent of* ν and λ, i.e. *of the special mode of energy generation.* In fact the energy equation [giving the third of (20.13)] has no influence on the luminosity, which is determined by hydrostatic equilibrium, the equations of state, and radiative energy *transfer* only. The model has to adjust so that the energy sources (ε) provide

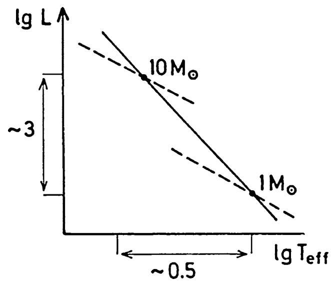

Fig. 20.1 Sketch of the Hertzsprung–Russell diagram with the locus of homologous main-sequence stars (*solid line*) of different masses for a certain constant value of ν. The *dashed lines* indicate lines of $R =$ constant

this luminosity. Introducing the exponents into (20.12), we have from (20.4) that

$$\frac{L}{L'} = \left(\frac{M}{M'}\right)^3 \left(\frac{\mu}{\mu'}\right)^4 . \tag{20.20}$$

There thus exists a mass-luminosity relation that gives a steeply increasing L with increasing M. And L varies even more strongly with the molecular weight μ (The precise values of the exponents vary for other values of a and b roughly in a range from 3 to 6, but the principle result remains.).

All other exponents depend on ν and λ. z_1 and z_2 describe the variation of the radius:

$$\frac{R}{R'} = \left(\frac{M}{M'}\right)^{z_1} \left(\frac{\mu}{\mu'}\right)^{z_2} . \tag{20.21}$$

The exponent z_1 of the $M-R$ relation is positive for all relevant combinations of λ and ν but smaller than one, i.e. R increases slightly with M. Values for typical parameters of hydrogen burning ($\lambda = 1$) via the *pp* chain ($\nu = 4\ldots5$) and the CNO cycle ($\nu \approx 15\ldots18$) are given in Table 20.1. Over this very large range of ν, z_1 varies relatively little, roughly from 0.4 to 0.8.

The $M-R$ relation together with the $M-L$ relation immediately give the locus of these stars in the Hertzsprung–Russell (HR) diagram, where lg L is plotted over $-$lg T_{eff} (see Fig. 20.1).

From (20.20) and (20.21) we have $R \sim L^{z_1/3}$ for homologous stars of identical μ. Introducing this into the definition of the effective temperature

$$\sigma T_{\text{eff}}^4 = \frac{L}{4\pi R^2}, \tag{20.22}$$

we obtain the locus as given by

$$\lg L = \frac{12}{3-2z_1} \lg T_{\text{eff}} + \text{constant}. \tag{20.23}$$

For an average value $z_1 = 0.6$, the slope is 6.67.

Table 20.1 Exponents in (20.12) for various temperature sensitivities ν of the nuclear reactions, and for $\alpha = \delta = \varphi = 1, a = b = 0, \lambda = 1$, calculated from (20.19)

ν:	4	5	15	18
z_1	0.43	0.5	0.78	0.81
z_2	0	0.13	0.61	0.67
p_1	0.29	0	−1.11	−1.24
p_2	0	−0.5	−2.44	−2.67
t_1	0.57	0.5	0.22	0.19
t_2	1.0	0.88	0.39	0.33
s_1	3	3	3	3
s_2	4	4	4	4

The exponents describe the dependence of R, P, T, L on M and μ ($R \sim M^{z_1}\mu^{z_2}$; $P \sim M^{p_1}\mu^{p_2}$; $T \sim M^{t_1}\mu^{t_2}$; $L \sim M^{s_1}\mu^{s_2}$)

Let us consider how a star of fixed M moves in the HR diagram if μ changes. From (20.20) and (20.21) we have $L \sim \mu^4, R \sim \mu^{z_2}$, which with (20.22) gives $T_{\text{eff}}^8 \sim L^{2-z_2} \approx L^{1.5}$ for $z_2 \approx 0.5$. This defines in the HR diagram a straight line of smaller slope ($\approx$ 5.3) than that of the main sequence. This line for $M =$ constant and μ increasing goes to the upper left with a slope between that of the main sequence and that of the lines $R =$ constant.

The expression for t_1 in (20.19) means that

$$T \sim M/R, \tag{20.24}$$

which simply reflects the virial theorem (thermal energy $\sim$ potential energy). Of special interest are the central values of temperature and density, T_c and ϱ_c, for which one has

$$T_c \sim M^{1-z_1}, \quad \varrho_c \sim M^{1-3z_1}. \tag{20.25}$$

The values in Table 20.1 show that for increasing M, T_c increases relatively slowly, while ϱ_c decreases. This trend is especially pronounced for CNO burning, where T_c scarcely changes at all, typical variations being $T_c \sim M^{0.2}$ and $\varrho_c \sim M^{-1.4}$ (see Fig. 20.2). The predictions of the homology relations are at least qualitatively recovered in the numerical solutions for main-sequence stars (Chap. 22).

20.2.3 The Role of the Equation of State

The procedure by which the homology solutions were obtained shows that their existence rests entirely on the fact that the right-hand sides of (20.5) contain only products of the variables, but no sums. This property is destroyed if the material functions, instead of being products of powers of P and T, contain additive terms as is in general the case with the equation of state. The simplest example is the addition of radiation pressure to an ideal gas such that $P = \Re\varrho T/\mu + aT^4/3$.

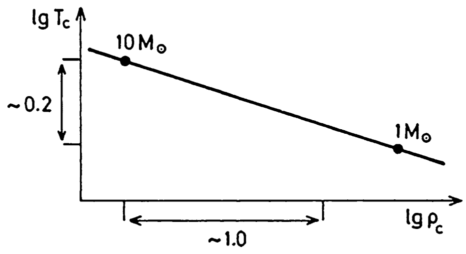

Fig. 20.2 The central values of T and ϱ (both logarithmic) for homologous main-sequence stars of various M. The slope corresponds to a temperature sensitivity ν typical for CNO burning

No strict homology relations are then possible. But one can try to make rough approximations.

One usually writes the corresponding equation of state as

$$\varrho \sim (\mu\beta)\frac{P}{T}, \quad \beta = \frac{P_{\mathrm{gas}}}{P} = \frac{1 - P_{\mathrm{rad}}}{P}. \tag{20.26}$$

The situation would be simple and homology relations would hold if β were constant throughout the model. Then a variation of β obviously has the same effect as that of μ and we would find $R \sim \beta^{z_2}, P \sim \beta^{p_2}, T \sim \beta^{t_2}, L \sim \beta^{s_2}$. In reality β is determined by P and T. For simultaneous variations of M and β, therefore

$$1 - \beta = \frac{P_{\mathrm{rad}}}{P} \sim \frac{T^4}{P} \sim \frac{M^{4t_1}}{M^{p_1}} \frac{\beta^{4t_2}}{\beta^{p_2}}, \tag{20.27}$$

which, if we simply use (20.19), gives

$$\frac{1-\beta}{\beta^4} \sim M^2. \tag{20.28}$$

Now, β is generally *not* constant inside a star [except for the polytrope $n = 3$ as treated in Sect. 19.5; compare with the identical relation (19.56)], but we can consider (20.28) as a relation between M and some kind of mean value of β. One then sees that β decreases strongly with M, i.e. the contribution of the radiation pressure to P increases with mass. Quite similarly we can write

$$L \sim M^{s_1} \beta^{s_2}. \tag{20.29}$$

Since β decreases with increasing M, (20.29) can be written as $L \sim M^{s_1 - c}$ ($c > 0$ for $s_2 > 0$) and the M–L relation becomes less steep. For $\beta \ll 1$ (large P_{rad}), relation (20.28) gives $\beta \sim M^{-1/2}$ such that $L \sim M^{s_1 - s_2/2} = M$. It is generally true that with increasing mass, the pressure in homogeneous stars is increasingly dominated by radiation pressure, and the mass-luminosity relation is less steep than for low-mass stars.

20.3 Homologous Contraction

Now we briefly consider the homologous contraction. This may apply to a chemically homogeneous star of given mass in hydrostatic equilibrium, if its radius is not fixed by an $M-R$ relation but changes in time. Let us assume that consecutive models are homologous to each other. An example in which this assumption is fulfilled is the contraction of a polytrope that does not change its polytropic index n. The solution of the Lane–Emden equation for given n yields the mass value m as a unique function of z only, where z is Emden's dimensionless radius variable, i.e. $z \sim r/R$ (see Sect. 19.2). Therefore the mass elements remain at homologous points, since their values of z do not change in time.

Homologous mass shells (ξ = constant) are here simply those which have the same value of m, since the normalizing factor M remains constant. The radius of any such shell is supposed to change by a rate $\dot{r} = \partial r/\partial t$. In two neighbouring models, separated by a time interval Δt, we have the values r and r' connected by $r' = r + \dot{r}\Delta t$. This gives

$$\frac{r'}{r} = 1 + \frac{\dot{r}}{r}\Delta t. \tag{20.30}$$

For a homologous contraction, we must require that $r'/r = R'/R =$ constant throughout the star. Then also

$$\frac{\dot{r}}{r} = \frac{\dot{R}}{R} \tag{20.31}$$

must be constant, or

$$\frac{\partial}{\partial m}\left(\frac{\partial \ln r}{\partial t}\right) = 0. \tag{20.32}$$

The relative rate of change of the other variables can then be easily expressed in terms of $\dot{r}/r$. From (20.32) we find by exchange of the two derivatives, and by using (10.1),

$$\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial r}{\partial m}\right) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi r^3 \varrho}\right) = \frac{1}{4\pi r^3 \varrho}\left(-3\frac{\dot{r}}{r} - \frac{\dot{\varrho}}{\varrho}\right) = 0, \tag{20.33}$$

which gives

$$\frac{\dot{\varrho}}{\varrho} = -3\frac{\dot{r}}{r}. \tag{20.34}$$

The pressure at a layer of mass value m is given by an integration of the hydrostatic equation as

$$P = \int_m^M \frac{Gm}{4\pi r^4} dm. \tag{20.35}$$

Differentiating this with respect to time and observing that $\dot{r}/r$ is constant throughout the model, we have

$$\dot{P} = \int_m^M \frac{\partial}{\partial t}\left(\frac{1}{r^4}\right)\frac{Gm}{4\pi} dm = -4\frac{\dot{r}}{r}\int_m^M \frac{Gm}{4\pi r^4} dm. \tag{20.36}$$

Equations (20.35) and (20.36) yield

$$\frac{\dot{P}}{P} = -4\frac{\dot{r}}{r}. \tag{20.37}$$

If we have an equation of state with $\varrho \sim p^\alpha T^{-\delta}$, then $\dot{\varrho}/\varrho = \alpha\dot{P}/P - \delta\dot{T}/T$. Solving this for $\dot{T}/T$ and replacing $\dot{\varrho}$ and $\dot{P}$ by (20.34) and (20.37), we have

$$\frac{\dot{T}}{T} = -\frac{4\alpha - 3}{\delta}\frac{\dot{r}}{r}. \tag{20.38}$$

The energy generation due to contraction is according to (4.47)

$$\varepsilon_{\rm g} = c_P T\left(\nabla_{\rm ad}\frac{\dot{P}}{P} - \frac{\dot{T}}{T}\right). \tag{20.39}$$

We introduce (20.37), (20.38) and (20.31), thus obtaining

$$\varepsilon_{\rm g} = c_P T\left(-4\nabla_{\rm ad} + \frac{4\alpha - 3}{\delta}\right)\frac{\dot{R}}{R}. \tag{20.40}$$

For an ideal monatomic gas ($\nabla_{\rm ad} = 2/5, \alpha = \delta = 1$) this becomes

$$\varepsilon_{\rm g} = -\frac{3}{5}c_P T\frac{\dot{R}}{R}. \tag{20.41}$$

Therefore $\varepsilon_{\rm g} > 0$ for contraction ($\dot{R} < 0$). We also see that $|\varepsilon_{\rm g}| \sim |\dot{R}/R|$; and since $\varepsilon_{\rm g}$ is proportional to T, it represents an energy source that is only rather moderately concentrated towards the centre.

As already mentioned, homology considerations are important for rough interpretations of numerical results, but their strict applicability is very limited. This is ultimately because homology requires a very well concerted action of all mass elements. It can hold approximately only for homogeneous stars. In Sect. 33.2 we will encounter another type of homology which considers only certain parts inside a star, and which applies to some very inhomogeneous stellar configurations.