# Chapter 19
# Polytropic Gaseous Spheres

## 19.1 Polytropic Relations

As we have seen in Sect. 10.1 the temperature does not appear explicitly in the two mechanical equations (10.1) and (10.2). Under certain circumstances this provides the possibility of separating them from the "thermo-energetic part" of the equations. For the following it is convenient to introduce once again the gravitational potential $\Phi$, as it was defined in Sect. 1.3. We here treat stars in hydrostatic equilibrium, which requires [see (1.11) and (2.3)]

$$\frac{dP}{dr} = -\frac{d\Phi}{dr}\varrho , \tag{19.1}$$

together with Poisson's equation (1.10)

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\varrho . \tag{19.2}$$

We have replaced the partial derivatives by ordinary ones since only time-independent solutions shall be considered.

In general the temperature appears in the system (19.1) and (19.2) if the density is replaced by an equation of state of the form $\varrho = \varrho(P, T)$. However, we have already encountered examples for simpler cases. If $\varrho$ does not depend on $T$, i.e. $\varrho = \varrho(P)$ only, then this relation can be introduced into (19.1) and (19.2), which become a system of two equations for $P$ and $\Phi$ and can be solved without the other structure equations. An example is the completely degenerate gas of non-relativistic electrons for which $\varrho \sim P^{3/5}$ [see (15.23)].

We shall deal here with similar cases and assume that there exists a simple relation between $P$ and $\varrho$ of the form

$$P = K\varrho^{\gamma} \equiv K\varrho^{1+\frac{1}{n}} , \tag{19.3}$$

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where $K$, $\gamma$, and $n$ are constant. A relation of the form (19.3) is called a *polytropic relation*. $K$ is the *polytropic constant* and $\gamma$ the *polytropic exponent* (which we have to distinguish from the adiabatic exponent $\gamma_{\rm ad}$). One often uses, instead of $\gamma$, the *polytropic index n,* which is defined by

$$n = \frac{1}{\gamma - 1} \; . \tag{19.4}$$

Obviously for a completely degenerate gas the equation of state in its limiting cases has the polytropic form (19.3). In the non-relativistic limit (15.23) we have $\gamma = 5/3$, $n = 3/2$, while for the relativistic limit (15.26) holds, so that $\gamma = 4/3$, $n = 3$. For such cases, where the equation of state has a polytropic form, the polytropic constant $K$ is fixed and can be calculated from natural constants.

But there are also examples for a relation of the form (19.3) where $K$ is a free parameter which is constant within a particular star but can have different values from one star to another.

Let us consider an isothermal ideal gas of temperature $T = T_0$ and mean molecular weight $\mu$. Its equation of state $\varrho = \mu P/(\Re T)$ can be written in the form (19.3), with $K = \Re T_0/\mu$, $\gamma = 1$, and $n = \infty$. Here $K$ is not fixed but depends on $T_0$ and $\mu$, and if we then use (19.3) in the stellar-structure equations, we are free to give $K$ any (positive) value for a certain star.

In a star that is completely convective the temperature gradient (except for that in a region near the surface, which we shall ignore) is given, to a very good approximation, by $\nabla = (d \ln T/d \ln P)_{\rm ad} = \nabla_{\rm ad}$ (see Sect. 7.3). If radiation pressure can be ignored and the gas is completely ionized, we have $\nabla_{\rm ad} = 2/5$ according to (13.12). This means that throughout the star $T \sim P^{2/5}$, and for an ideal gas with $\mu$ = constant, $T \sim P/\varrho$, and therefore $P \sim \varrho^{5/3}$. This again is a polytropic relation of the form (19.3) with $\gamma = 5/3$, $n = 3/2$. But now $K$ is not fixed by natural constants; it is a free parameter in the sense that it can vary from star to star.

The homogeneous gaseous sphere can also be considered a special case of the polytropic relation (19.3). Let us write (19.3) in the form

$$\varrho = K_1 P^{1/\gamma} \; ; \tag{19.5}$$

then $\gamma = \infty$ (or $n = 0$) gives $\varrho = K_1 =$ constant.

These examples have shown that we can have two reasons for a polytropic relation in a star. (1) The equation of state is of the simple form $P = K\varrho^\gamma$, with a fixed value of $K$. (2) The equation of state contains $T$ (as for an ideal gas), but there is an additional relation between $T$ and $P$ (like the adiabatic condition) that together with the equation of state yields a polytropic relation; then $K$ is a free parameter.

On the other hand, if we assume a polytropic relation for an ideal gas, this is equivalent to adopting a certain relation $T = T(P)$. This means that one fixes the temperature stratification instead of determining it by the thermo-energetic equations of stellar structure. For example, a polytrope with $n = 3$ does not necessarily have to consist of relativistic degenerate gases but can also consist of an ideal gas and have $\nabla = 1/(n+1) = 0.25$.

## 19.2 Polytropic Stellar Models

With the polytropic relation (19.3) (independent of whether $K$ is a free parameter or a constant with a fixed value), (19.1) can be written as

$$\frac{d\Phi}{dr} = -\gamma K \varrho^{\gamma-2} \frac{d\varrho}{dr} \ . \tag{19.6}$$

If $\gamma \neq 1$ (the case $\gamma = 1, n = \infty$, corresponding to the isothermal model, will be treated in Sect. 19.8), (19.6) can be integrated:

$$\varrho = \left( \frac{-\Phi}{(n+1)K} \right)^n \ , \tag{19.7}$$

where we have made use of (19.4) and chosen the integration constant to give $\Phi = 0$ at the surface ($\varrho = 0$). Note that in the interior of our model, $\Phi < 0$, giving there $\varrho > 0$. If we introduce (19.7) into the right-hand side of the Poisson equation (19.2), we obtain an ordinary differential equation for $\Phi$:

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = 4\pi G \left( \frac{-\Phi}{(n+1)K} \right)^n \ . \tag{19.8}$$

We now define dimensionless variables $z, w$ by

$$z = Ar \ , \quad A^2 = \frac{4\pi G}{(n+1)^n K^n}(-\Phi_c)^{n-1} = \frac{4\pi G}{(n+1)K}\varrho_c^{\frac{n-1}{n}} \ ,$$
$$w = \frac{\Phi}{\Phi_c} = \left( \frac{\varrho}{\varrho_c} \right)^{1/n} \ , \tag{19.9}$$

where the subscript c refers to the centre and where the relation between $\varrho$ and $\Phi$ is taken from (19.7). At the centre ($r = 0$) we have $z = 0, \Phi = \Phi_c, \varrho = \varrho_c$, and therefore $w = 1$. Then (19.8) can be written as

$$\frac{d^2w}{dz^2} + \frac{2}{z}\frac{dw}{dz} + w^n = 0 \ ,$$
$$\frac{1}{z^2}\frac{d}{dz}\left( z^2 \frac{dw}{dz} \right) + w^n = 0 \ . \tag{19.10}$$

This is the famous *Lane–Emden equation* (named after J.H. Lane and R. Emden). We are only interested in solutions that are finite at the centre, $z = 0$. Equation (19.10) shows that we then have to require $dw/dz \equiv w' = 0$. Let us assume we have a solution $w(z)$ of (19.10) that fulfils the central boundary conditions $w(0) = 1$ and $w'(0) = 0$; then according to (19.9) the radial distribution of the density is given by

$$\varrho(r) = \varrho_c w^n \ , \quad \varrho_c = \left[ \frac{-\Phi_c}{(n+1)K} \right]^n \ . \tag{19.11}$$

For the pressure we obtain from (19.3) and (19.4) that $P(r) = P_c w^{n+1}$, where $P_c = K \varrho_c^\gamma$.

Before trying to construct stellar polytropic models we shall discuss some of the mathematical properties of the solutions $w(z)$ of (19.10).

## 19.3 Properties of the Solutions

The Lane–Emden equation has a regular singularity at $z = 0$. In order to understand the behaviour of the solutions there, we expand into a power series:

$$w(z) = 1 + a_1 z + a_2 z^2 + a_3 z^3 + \dots \ , \tag{19.12}$$

with $a_1 = w'(0), 2a_2 = w''(0), \dots$. Since the gravitational acceleration $|g| = d\Phi/dr \sim dw/dz$ must vanish in the centre, we have $a_1 = 0$. Inserting (19.12) into the Emden equation (19.10), by comparing coefficients one finds

$$w(z) = 1 - \frac{1}{6} z^2 + \frac{n}{120} z^4 + \dots \ , \tag{19.13}$$

where again we have excluded the isothermal sphere $n = \infty$. Equation (19.13) shows that $w(z)$ has a maximum at $z = 0$.

Only for three values of $n$ can the solutions be given by analytic expressions. The first case is

$$n = 0: \qquad w(z) = 1 - \frac{1}{6} z^2 \ , \tag{19.14}$$

and we have already mentioned that this corresponds to the homogeneous gas sphere. Indeed $\varrho = \varrho_c w^n$ gives constant density for $n = 0$. The two other cases are

$$n = 1: \qquad w(z) = \frac{\sin z}{z} \ , \tag{19.15}$$

$$n = 5: \qquad w(z) = \frac{1}{(1 + z^2/3)^{1/2}} \ . \tag{19.16}$$

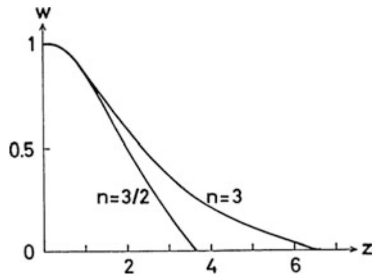


**Fig. 19.1** If $n < 5$ the solution of the Lane–Emden equation (19.10) of index $n$ starting with $w(0) = 1$ becomes zero at a finite value of $z = z_{\mathrm{n}}$. Here the solutions for $n = 3/2$ and $n = 3$ are plotted

**Table 19.1** Numerical values for polytropic models with index $n$ (after Chandrasekhar 1939)

| $n$ | $z_n$ | $\left(-z^2 \frac{dw}{dz}\right)_{z=z_n}$ | $\varrho_c/\bar{\varrho}$ |
|---|---|---|---|
| 0 | 2.4494 | 4.8988 | 1.0000 |
| 1 | 3.14159 | 3.14159 | 3.28987 |
| 1.5 | 3.65375 | 2.71406 | 5.99071 |
| 2 | 4.35287 | 2.41105 | 11.40254 |
| 3 | 6.89685 | 2.01824 | 54.1825 |
| 4 | 14.97155 | 1.79723 | 622.408 |
| 4.5 | 31.8365 | 1.73780 | 6,189.47 |
| 5 | $\infty$ | 1.73205 | $\infty$ |

The surface of the polytrope of index $n$ is defined by the value $z = z_n$, for which $\varrho = 0$ and thus $w = 0$. While for $n = 0$ and $n = 1$ the surface is obviously reached for a finite value of $z_n$, the case $n = 5$ yields a model of infinite radius. It can be shown that for $n < 5$ the radius of polytropic models is finite; for $n \geq 5$ they have infinite radius. This also holds for the limiting case $n = \infty$ (cf. Sect. 19.8).

Apart from the three cases where analytic solutions are known, the Emden equation (19.10) has to be solved numerically, beginning with the expansion (19.13) for the neighbourhood of the centre. Here the solution starts with zero tangent and $w = 1$ and decreases outwards. This can be seen from (19.13) and is illustrated in Fig. 19.1.

For a given value of $n < 5$ the integration comes to a point $z$ where $w(z)$ vanishes, i.e. $\varrho = 0$. This value of $z$, which corresponds to the surface of the polytrope, will be called $z_n$. From (19.14)–(19.16) one finds $z_0 = \sqrt{6}, z_1 = \pi, z_5 = \infty$. It is a general property of the solutions that $z_n$ grows monotonically with the polytropic index $n$. Table 19.1 gives some values of $z_n$ and the values of certain functions at $z = z_n$ which will later turn out to be useful for the construction of models.

So far, we have discussed only solutions that are regular at the centre. But solutions with a singularity at $z = 0$ can also be important if one uses them for stellar regions outside the centre. Let us, for instance, consider a star that is convective in its outer layer, while in the inner part, the energy may be transported by radiation. If the convective envelope is adiabatic, with $\nabla = \nabla_{\mathrm{ad}} = 2/5$, it is polytropic and therefore $\varrho \sim w^{3/2}$ and $P \sim w^{5/2}$. But it is unimportant whether this solution is finite at the centre, since anyway the equations do not hold in the radiative interior.

On the other hand, one may have to fit a polytropic central core to an envelope with different properties. In this case the polytropic solution has to be regular at the centre, but its behaviour for $w = \varrho = 0$ is unimportant, since it is used only up to the core surface where $\varrho$ and $P$ are non-vanishing. In the following we mainly deal with *complete polytropes*, which have a polytropic relation of the form (19.3) from surface to centre.

## 19.4 Application to Stars

We now construct polytropic models for a given index $n < 5$ and for given values of $M$ and $R$. This will turn out to be possible as long as $K$ is not fixed by the equation of state. We first derive some more relations for polytropes.

From (10.1) and (19.11) it follows that

$$m(r) = \int_0^r 4\pi \varrho r^2 dr = 4\pi \varrho_c \int_0^r w^n r^2 dr = 4\pi \varrho_c \frac{r^3}{z^3} \int_0^z w^n z^2 dz \, , \tag{19.17}$$

where we have made use of relations (19.9) and of the fact that $r^3/z^3$ is constant and can be brought in front of the integral. According to the Lane–Emden equation (19.10) the integrand $w^n z^2$ on the right is a derivative and can immediately be integrated, so that the integral becomes $-z^2 dw/dz$. We obtain

$$m(r) = 4\pi \varrho_c r^3 \left( -\frac{1}{z} \frac{dw}{dz} \right) \, , \tag{19.18}$$

where the simultaneously appearing $z$ and $r$ are related to each other by $r/z = 1/A = R/z_n$. For the special case of the surface, we have

$$M = 4\pi \varrho_c R^3 \left( -\frac{1}{z} \frac{dw}{dz} \right)_{z=z_n} \, . \tag{19.19}$$

The quantity in brackets can be derived from Table 19.1 for several values of $n$. If we introduce the mean density $\bar{\varrho} := 3M/(4\pi R^3)$, we find

$$\frac{\bar{\varrho}}{\varrho_c} = \left( -\frac{3}{z} \frac{dw}{dz} \right)_{z=z_n} \, . \tag{19.20}$$

The right-hand side of this equation depends only on $n$: for $n = 0$ it is 1–as one can see from (19.11). The higher $n$, the smaller $\bar{\varrho}/\varrho_c$, which means the higher the density concentration, as can be seen in Table 19.1.

We now have all the means at hand to construct the whole polytropic stellar model for given values of $n$, $M$, and $R$ for the case that $K$ is not fixed by the equation of state.

If $n$ is given, a numerical solution of the Lane–Emden equation (19.10) yields the functions $w(z), w'(z)$ and the values of $z_n$ and of $-z_n/(3dw/dz)_n$. If we now use $M$ and $R$ to determine the mean density $\bar{\varrho}$, (19.20) gives $\varrho_c$. On the other hand, we know the constant $A = z/r = z_n/R$ by which we adjust the dimensionless $z$ scale to the $r$ scale. We therefore know the density distribution in the model $\varrho(r) = \varrho_c w^n(z)$ from (19.11). With $\varrho_c$ and the constant $A$ we can determine $K$ from (19.9) and obtain the pressure distribution $P(r) = K\varrho^{(n+1)/n} = K\varrho_c^{(n+1)/n} w^{n+1}$. The local mass $m$ then follows from (19.18) and the (known) relation between the $z$ scale and the $r$ scale. The whole mechanical structure is now determined. It has to be emphasized that this method of constructing models for given values of $n, M$, and $R$ is only applicable if $K$ is a free parameter, otherwise the problem would be overdetermined (The case that $K$ has fixed value will be discussed in Sect. 19.6.).

As an example we try to construct a polytropic model of index 3 for the Sun ($M = 1.989 \times 10^{33}$g, $R = 6.96 \times 10^{10}$ cm). For $n = 3$ Table 19.1 gives $z_3 = 6.897$, $\varrho_c/\bar{\varrho} = 54.18$. The mean density becomes $\bar{\varrho} = 1.41\,\mathrm{g\,cm^{-3}}$; consequently the central density $\varrho_c = 76.39\,\mathrm{g\,cm^{-3}}$ and, further, $A = z_3/R = 9.91 \times 10^{-11}$. From (19.9) we find $K = 3.85 \times 10^{14}$ and consequently $P_c = 1.24 \times 10^{17}\,\mathrm{dyn/cm^2}$. For the ideal gas equation with $\mu = 0.62$ corresponding to $X \approx 0.7$, $Y \approx 0.3$ we find for the temperature $T_c = 1.2 \times 10^7$ K. A proper numerical solution of the full set of stellar-structure equations for a chemically homogeneous model of $1M_\odot$ gives $T_c = 1.5 \times 10^7$ K. We see that a polytropic estimate with $n = 3$ comes considerably closer to the honestly computed value than our crude estimate in Sect. 2.3.

## 19.5 Radiation Pressure and the Polytrope $n = 3$

We consider here only the case that $K$ is a free parameter. In the example at the end of the previous section we approximated the Sun by a polytrope of $n = 3$. This is formally equivalent to the assumption of an ideal gas ($P \sim \varrho T$) together with a constant temperature gradient $\nabla = 1/4 (T \sim P^{1/4})$. We will now show that this polytropic relation with $n = 3$ can also be obtained by a certain assumption on the radiation pressure. For an ideal gas with radiation pressure

$$P = \frac{\Re}{\mu}\varrho T + \frac{a}{3}T^4 = \frac{\Re}{\mu\beta}\varrho T, \tag{19.21}$$

we assume that the ratio $\beta = P_{\mathrm{gas}}/P$ is constant throughout the star. Now

$$1 - \beta = \frac{P_{\mathrm{rad}}}{P} = \frac{aT^4}{3P} \tag{19.22}$$

shows that $\beta =$ constant means a relation of the form $T^4 \sim P$, which we introduce into (19.21). This gives

$$P = \left(\frac{3\Re^4}{a\mu^4}\right)^{1/3} \left(\frac{1-\beta}{\beta^4}\right)^{1/3} \varrho^{4/3} , \tag{19.23}$$

which indeed is a polytropic relation with $n = 3$ for constant $\beta$. Here the polytropic constant $K$ is again a free parameter, since we can choose $\beta$ in the interval 0, 1.

In Sect. 19.10 we shall apply this to very massive stars. They are fully convective ($\nabla = \nabla_{\rm ad}$) and dominated by radiation pressure.

Relation (19.23) goes back to A.S. Eddington, who obtained it for his famous "standard model". He found that the full set of stellar-structure equations (including the thermo-energetic equations) could be solved very simply by the assumption $\kappa l/m =$ constant throughout the star. One then obtains $\beta =$ constant and therefore the polytropic relation (19.23).

## 19.6 Polytropic Stellar Models with Fixed *K*

As a typical example we have already mentioned the non-relativistic degenerate electron gas for which the equation of state (15.23) is polytropic with $n = 3/2$ and polytropic constant

$$K = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_{\rm e}} \frac{1}{(\mu_{\rm e} m_{\rm u})^{5/3}} . \tag{19.24}$$

We consider the chemical composition to be given ($\mu_{\rm e}$ fixed). Then in this expression there is no room for the choice of a free parameter as in (19.23). Although $n = 3/2$ is a particularly interesting case, we shall derive our relation for general values of the polytropic index with $n < 5$.

Let us see how to construct a model with index $n$ for a given value of $\varrho_{\rm c}$. The functions $w(z)$ and $w'(z)$ can be considered known from an integration of the Emden equation. Then $\varrho = \varrho_{\rm c} w^n$ is known as a function of $z$. According to (19.9) the relation between $r$ and $z$ is

$$\left(\frac{r}{z}\right)^2 = \frac{1}{4\pi G}(n+1) K \varrho_{\rm c}^{\frac{1-n}{n}} . \tag{19.25}$$

This can be used to derive the density also as a function of $r$, where the radius of the model is $R = z_n/A$ and the value $z_n$ is obtained from the integration. The constant $A$ depends on $\varrho_{\rm c}$, as shown by (19.25), and

$$R \sim \varrho_{\rm c}^{\frac{1-n}{2n}} . \tag{19.26}$$

As long as $n > 1$, the radius $R$ becomes smaller with increasing central density $\varrho_{\rm c}$, becoming zero for infinite $\varrho_{\rm c}$. On the other hand, the mass $M$ of the model varies with $\varrho_{\rm c}$ according to (19.19) as $M \sim \varrho_{\rm c} R^3$ or

$$M = C_1 \varrho_c^{\frac{3-n}{2n}} \; ; \quad C_1 = 4\pi \left( -\frac{w'}{z} \right)_{z_n} z_n^3 \left( \frac{n+1}{4\pi G} \right)^{3/2} K^{3/2} \, . \tag{19.27}$$

Elimination of $\varrho_c$ from (19.26) and (19.27) shows that there is a mass-radius relation of the form

$$R \sim M^{\frac{1-n}{3-n}} \, . \tag{19.28}$$

We see that for given $K$ and $n$ there is a one-dimensional manifold of models only, the parameter being *either M or R* (or $\varrho_c$), whereas there was a two-dimensional manifold *(M and R* as parameters) when $K$ was a free parameter.

Consider again the case of the non-relativistic degenerateX electron gas, which is not too bad an approximation for white dwarfs of small mass. With $n = 3/2$, (19.28) gives $R \sim M^{-1/3}$ and the surprising result that the larger the mass the smaller the radius (This is made plausible by simple considerations in Sect. 37.1.). The model will shrink with increasing mass and should finally end as a point mass for infinite $M$. But long before this, our assumed equation of state will not be valid any more, since from (19.27) we see that $\varrho_c$ is proportional to $\sim M^2$. For ever-increasing densities the electrons will become relativistic (see Sect. 16.2), and the equation of state (15.23) has to be replaced by (15.26). This means a transition from a polytrope $n = 3/2$ to one with $n = 3$ (and a different, but also given, polytropic constant $K$). In this case we shall encounter a new problem, hinted at by the exponent in (19.28).

## 19.7 Chandrasekhar's Limiting Mass

In Sect. 19.6 we have seen that a polytropic model in which the pressure is provided by a non-relativistic degenerate electron gas reaches higher central and mean densities with growing total mass $M$. But with increasing density the electrons become gradually more relativistic. This starts in the central region where the density is highest, the outer parts remaining non-relativistic. Although we know that the transition between equations of state (15.23) and (15.26) does not occur abruptly, but smoothly via the more general equation of state (15.13), one can imagine that an idealized stellar model consisting of degenerate matter can be constructed by fitting two regions smoothly together: a (relativistic) polytropic core with $n = 3$ surrounded by a (non-relativistic) polytropic envelope with $n = 3/2$. Indeed Chandrasekhar constructed his first white-dwarf model in this way.

Let us consider how this idealized model changes with growing mass $M$. At small $M$ the whole model is still non-relativistic. The relativistic core will occur for $\varrho_c \gtrsim 10^6\,\mathrm{g\,cm^{-3}}$ (Fig. 16.1) and gradually encompass larger parts of the model as $\varrho_c$ increases. One would therefore expect the model finally to approach the state where all its mass (except a small surface region) is relativistic, so that a polytrope of index $n = 3$ would describe the whole model properly; however, there is a difficulty. As one can see from (19.27) the mass does not vary with central density in the case of a polytrope of index $n = 3$ if $K$ is fixed. In this case, (19.27) gives $M = C_1$:

$$M = 4\pi \left( -\frac{w'}{z} \right)_{z_3} z_3^3 \left( \frac{K}{\pi G} \right)^{3/2} . \tag{19.29}$$

This is the only possible mass for relativistic degenerate polytropes and is called the *Chandrasekhar mass*, which after insertion of the proper numerical values yields

$$M_{\rm Ch} = \frac{5.836}{\mu_{\rm e}^2} M_\odot . \tag{19.30}$$

We therefore can expect that our series of models constructed by fitting an $n = 3/2$ envelope to an $n = 3$ core finds its end at a critical total mass $M = M_{\rm Ch}$ as given by (19.30). Or in other words our models of increasing central density tend to a finite mass and approach zero radius for $\varrho_{\rm c} \to \infty$. Of course, this final state is physically unrealistic, since the equation of state is changed by different effects at very high density (see Chaps. 16, 37 and 38).

Although we have discussed the problem only from the standpoint of polytropic models, the result for $M_{\rm Ch}$ remains numerically the same if one uses Chandrasekhar's more general equation of state (15.13) (compare the treatment in Sect. 37.1. The reason is that for extremely high density, (15.13) approaches the polytropic relation (19.3) with $\gamma = 4/3$ or $n = 3$.

It is surprising that the limiting mass not only is finite, but that it is so small that many stars exceed it. But their equation of state is not dominated by degenerate electrons, and therefore Chandrasekhar's limiting mass (19.30) has no meaning for them. White dwarfs seem to be formed of material where all the hydrogen is transformed into helium, carbon, or oxygen, such that we expect $\mu_{\rm e} = 2$ and therefore $M_{\rm Ch} = 1.46 M_\odot$. Indeed no white dwarf has been found which exceeds this mass.

In the above considerations we have approached the relativistic degenerate polytrope by way of a sequence with $\varrho_{\rm c} \to \infty$ (and consequently $R \to 0$). However, this polytrope is a particular case: we have already mentioned that according to (19.27) $M$ and $\varrho_{\rm c}$ are then no longer coupled. In other words, for $M = M_{\rm Ch}$, the central density can be arbitrary (and therefore also the radius $R$), i.e. there is a whole series of relativistic degenerate polytropes (having $\varrho_{\rm c}$ or $R$ as parameter) that all have the same mass $M_{\rm Ch}$. This is a case of neutral equilibrium (see Sect. 25.3.2).

## 19.8 Isothermal Spheres of an Ideal Gas

We now deal with the case $\gamma = 1$ or $n = \infty$, which we omitted in Sect. 19.2. Here $K = \Re T/\mu$ is a free parameter. If $\gamma = 1$, integration of (19.6) gives

$$-\frac{\Phi}{K} = \ln \varrho - \ln \varrho_{\rm c} , \tag{19.31}$$

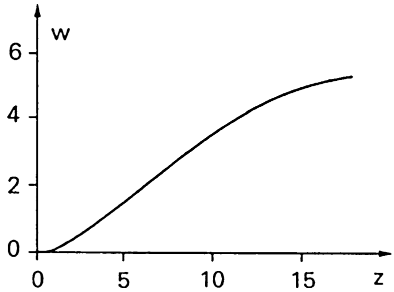


**Fig. 19.2** The solution of the Lane–Emden equation (19.35) for the case of an isothermal ideal gas ($n = \infty$)

where we have now chosen the constant of integration in such a way that the gravitational potential is zero at the centre and positive outside it. With

$$\varrho = \varrho_c e^{-\Phi/K} \tag{19.32}$$

and with the Poisson equation (19.2) we find

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = 4\pi G \varrho_c e^{-\Phi/K} \ . \tag{19.33}$$

We now introduce dimensionless variables $z, w$ by

$$z = Ar \ , \quad A^2 = \frac{4\pi G\varrho_c}{K} \ , \quad \Phi = Kw \tag{19.34}$$

and obtain the "isothermal" Lane–Emden equation

$$\frac{d^2w}{dz^2} + \frac{2}{z}\frac{dw}{dz} = e^{-w} \ , \tag{19.35}$$

which now has to be integrated with the central conditions

$$w(0) = 0 \ , \quad \left(\frac{dw}{dz}\right)_{z=0} = 0 \ . \tag{19.36}$$

Again, a power series expansion can be derived and has to be used to describe the behaviour near the centre. The solution is given in Fig. 19.2.

As already mentioned, the isothermal sphere consisting of an ideal gas has an infinite radius, like all polytropes of $n \geq 5$. It also has an infinite mass. Certainly there can be no such stars, but polytropes with $n = \infty$ can be used in order to construct models with non-degenerate isothermal cores. Such models play a role in connection with the so-called Schönberg–Chandrasekhar limit (see Sect. 30.5).

## 19.9 Gravitational and Total Energy for Polytropes

We now give a general expression for the gravitational energy $E_g$ of polytropes. We first show that quite generally

$$E_g = \frac{1}{2}\int_0^M \Phi\, dm - \frac{1}{2}\frac{GM^2}{R} \, . \tag{19.37}$$

From the definition (3.3) of $E_g$, we find

$$E_g = -G\int_0^M \frac{m}{r} dm = -\frac{1}{2}\frac{GM^2}{R} - \frac{1}{2}G\int_0^R \frac{m^2}{r^2} dr \, , \tag{19.38}$$

where the last expression has been obtained by partial integration and where we have used the fact that $m/r$ vanishes at the centre. But on the other hand

$$\frac{d\Phi}{dr} = \frac{Gm}{r^2} \tag{19.39}$$

and therefore

$$\begin{aligned} E_g &= -\frac{1}{2}\frac{GM^2}{R} - \frac{1}{2}\int_0^R \frac{d\Phi}{dr} m\, dr \\ &= -\frac{1}{2}\frac{GM^2}{R} + \frac{1}{2}\int_0^M \Phi\, dm \, , \end{aligned} \tag{19.40}$$

where again we have integrated partially and used the fact that $m\Phi$ vanishes at the centre ($m = 0$) and at the surface [$\Phi = 0$, according to our choice of the integration constant in connection with (19.7)], so we have indeed recovered (19.37). For a polytrope we can use (19.3), (19.7) and write

$$\Phi = -\frac{K\gamma}{\gamma - 1}\varrho^{\gamma-1} = -\frac{\gamma}{\gamma - 1}\frac{P}{\varrho} \tag{19.41}$$

and therefore, with (19.37),

$$E_g = -\frac{1}{2}\frac{GM^2}{R} - \frac{1}{2}\frac{\gamma}{\gamma - 1}\int_0^M \frac{P}{\varrho} dm \, . \tag{19.42}$$

According to (3.2) and (3.3) the last term on the right can be expressed by $E_g$. If we replace $\gamma$ by $n$, then

$$E_g = -\frac{1}{2}\frac{GM^2}{R} + \frac{1}{6}(n+1)E_g \tag{19.43}$$

and therefore

$$E_g = -\frac{3}{5-n}\frac{GM^2}{R} \; . \tag{19.44}$$

We now derive a similar expression for the internal energy $E_i$. In (3.8) we defined a quantity $\zeta$ by

$$\zeta := 3P/(\varrho u) \tag{19.45}$$

($u$ = internal energy per mass unit).

We saw that for an ideal gas

$$\zeta = 3(\gamma_{ad} - 1) \; . \tag{19.46}$$

This relation also holds for a more general equation of state as long as $\zeta$ is constant.

In order to show this, we take the total differentials from (19.45) and obtain

$$\zeta \, du = 3\frac{dP}{\varrho} - 3\frac{P}{\varrho^2}d\varrho \; . \tag{19.47}$$

We now assume that the differentials describe adiabatic changes. The first law of thermodynamics gives

$$du = \frac{P}{\varrho^2}d\varrho \; . \tag{19.48}$$

Then with

$$\gamma_{ad} = \frac{\varrho}{P}\frac{dP}{d\varrho} \; , \tag{19.49}$$

(19.47) yields

$$\zeta = 3\frac{\varrho}{P}\frac{dP}{d\varrho} - 3 = 3(\gamma_{ad} - 1) \; . \tag{19.50}$$

For an ideal gas with $\gamma_{ad} = 5/3$ one has $\zeta = 2$, while for an ideal gas with $\gamma_{ad} = 4/3, \zeta = 1$. In the case of a gas dominated by radiation pressure ($P = aT^4/3$ and $u = aT^4$) one finds $\zeta = 1$. Assuming $\zeta$ to be constant throughout the star and using (19.44) we find with (3.9)

$$E_i = -\frac{1}{\zeta}E_g = \frac{3}{\zeta(5-n)}\frac{GM^2}{R} \; . \tag{19.51}$$

The total energy then becomes

$$W = E_i + E_g = \frac{3}{5-n}\left(\frac{1}{\zeta} - 1\right)\frac{GM^2}{R} \; . \tag{19.52}$$

We can conclude from (19.52) that the total energy for a polytrope of finite radius vanishes when $\zeta = 1$ and in particular for the above cases of an ideal gas with $\gamma_{ad} = 4/3$ and a radiation-dominated gas.

## 19.10 Supermassive Stars

Let us consider an ideal gas with radiation pressure and assume that $\beta = P_{\text{gas}}/P =$ constant throughout the star. We have seen in (19.23) that this yields a polytrope with $n = 3$.

Relation (19.23) defines the polytropic constant $K$:

$$K = \left(\frac{3\Re^4}{a\mu^4}\right)^{1/3} \left(\frac{1-\beta}{\beta^4}\right)^{1/3} . \tag{19.53}$$

On the other hand, from (19.9) for $n = 3$ we have

$$K = \pi G \varrho_c^{2/3} \frac{R^2}{z_3^2} , \tag{19.54}$$

where we have used $A = z_3/R$. The numerical value of $z_3$ is 6.897 (Table 19.1). With (19.20) $\varrho_c$ can be expressed by $M$ and $R$:

$$\varrho_c = 54.18\bar{\varrho} = 54.18 \frac{3M}{4\pi R^3} = c_1 \frac{M}{R^3} , \tag{19.55}$$

where we have taken the numerical value from Table 19.1. From (19.53) we eliminate $K$ with (19.54) and then $\varrho_c$ with (19.55) and obtain "Eddington's quartic equation":

$$\frac{1-\beta}{\mu^4\beta^4} = \frac{a}{3\Re^4} \frac{(\pi G)^3 c_1^2}{z_3^6} M^2 = 3.02 \times 10^{-3} \left(\frac{M}{M_\odot}\right)^2 . \tag{19.56}$$

In the interval $0 \le \beta \le 1$ the left-hand side is a monotonically decreasing function of $\beta$, which therefore becomes smaller with growing $M$; this means that radiation pressure becomes the more important the larger the stellar mass.

For a pure hydrogen star of $10^6 M_\odot$ and $\mu = 0.5$, (19.56) gives $(1 - \beta)/\beta^4 = 1.9 \times 10^8$, or $\beta \approx 0.0086$.

Supermassive stars are therefore dominated by radiation pressure. One consequence is that $\nabla_{\text{ad}}$ is appreciably reduced [$\nabla_{\text{ad}} \to 1/4$, for $\beta \to 0$; see (13.12)] and the star becomes convective with $\nabla = \nabla_{\text{ad}}$. This can also be seen from an extrapolation of the main-sequence models towards large $M$ (Sect. 22.3). The adiabatic structure requires constant specific entropy $s$. For a gas dominated by radiation pressure (the density being determined by the gas, the pressure by the photons) the energy $u$ per mass unit and the pressure are given by

$$u = \frac{aT^4}{\varrho} , \quad P = \frac{a}{3} T^4 . \tag{19.57}$$

Then with the first law of thermodynamics we have

$$d_s = \frac{dq}{T} = \frac{1}{T}\left(du - \frac{P}{\varrho^2}d\varrho\right)$$
$$= \frac{4aT^2}{\varrho}dT - \frac{4aT^3}{3\varrho^2}d\varrho = d\left(\frac{4aT^3}{3\varrho}\right) \tag{19.58}$$

and

$$s = \frac{4aT^3}{3\varrho} . \tag{19.59}$$

Constant specific entropy means $\varrho \sim T^3$, which together with the pressure equation $P \sim T^4$ immediately gives $P \sim \varrho^{4/3}$. Indeed supermassive stars are polytropic with $n = 3$ as we assumed initially.

The supermassive star polytropes have a free $K$, which means that $M$ can be chosen arbitrarily (in contrast to the relativistic degenerate polytrope of the same index, where $K$ and $M$ were fixed). For each mass, $(1-\beta)/(\mu\beta)^4$ can be obtained from (19.56), and then (19.53) gives the corresponding value of $K$. But if the mass is given, there still exists an infinite number of models for different $R$. This is possible in spite of the fact that $K$ is already determined by $M$: since according to (19.55) $\varrho_c \sim \bar{\varrho} \sim M/R^3$, (19.54) shows $K$ to be independent of $R$. This is typical for the polytropic index $n = 3$.

Equation (19.59) shows that for an adiabatic change ($ds = 0$) of a given mass element $\varrho \sim T^3$, and therefore with (19.57) $P \sim \varrho^{4/3}$ or $\gamma_{\rm ad} = 4/3$. Then $\zeta = 1$ and (19.52) gives the total energy of the model $W = 0$. The supermassive configuration is in neutral equilibrium. No energy is needed to compress or expand it. In Chap. 25 we will find that $\gamma_{\rm ad} = 4/3$ corresponds to the case of marginal dynamical stability. There a simple interpretation is given for this peculiar behaviour.

## 19.11 A Collapsing Polytrope

Up to now we have only treated polytropic gaseous spheres in hydrostatic equilibrium. One can also find solutions for polytropes of $n = 3$ for which the inertia term, neglected in (19.1), is important (Goldreich and Weber 1980). Then (19.1) has to be replaced by

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{1}{\varrho}\frac{\partial P}{\partial r} + \frac{\partial \Phi}{\partial r} = 0 , \tag{19.60}$$

with $v_r = \partial r/\partial t$.

Let us consider a relativistic degenerate polytrope with $n = 3$, or $\gamma = \gamma_{\rm ad} = 4/3$. In a manner similar to that of Sect. 19.2 we define a dimensionless length-scale $z$ by

$$r = a(t)z , \quad v_r = \dot{a}z \tag{19.61}$$

such that $z$ is time independent, the whole time dependence of $r$ being contained in $a(t)$ [Note that $a$ corresponds to $1/A$ in (19.9)]. The form (19.61) describes a homologous change (compare with Sect. 20.3). If we introduce a velocity potential $\psi$ by $v_r = \partial\psi/\partial r$, we can write

$$a v_r = a\dot{a}z = a\frac{\partial\psi}{\partial r} = \frac{\partial\psi}{\partial z} , \quad \psi = \frac{1}{2}a\dot{a}z^2 , \tag{19.62}$$

where we have fixed the constant of integration in the velocity potential by $\psi = 0$ at $z = 0$. Note that the time derivative of $\psi$ in the comoving frame is

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + v_r\frac{\partial\psi}{\partial r} = \frac{\partial\psi}{\partial t} + (\dot{a}z)^2 . \tag{19.63}$$

With the new variables, Poisson's equation (19.2) can be written as

$$\frac{1}{z^2}\frac{\partial}{\partial z}\left(z^2\frac{\partial\psi}{\partial z}\right) = 4\pi G\varrho a^2 , \tag{19.64}$$

while the continuity equation (1.4) becomes with (19.62)

$$\frac{1}{\varrho}\frac{d\varrho}{dt} + \frac{1}{z^2a^2}\frac{\partial}{\partial z}\left(z^2\frac{\partial\psi}{\partial z}\right) \equiv \frac{1}{\varrho}\frac{d\varrho}{dt} + 3\frac{\dot{a}}{a} = 0 . \tag{19.65}$$

This means that $\varrho \sim a^{-3}$ (in the comoving frame), a result that is obvious from (19.61). As in (19.9) we define $w(z)$ by $\varrho = \varrho_c w^3(z)$. This $w(z)$ will turn out to be related to the Emden function of index 3, as we shall see later. Note that $\varrho_c$ is a function of time. In order to stay as close as possible to the formalism of hydrostatic equilibrium, we fix $a = r/z$ [rather as we did with $1/A$ in (19.9)] by

$$\frac{1}{a^2} = \frac{\pi G}{K}\varrho_c^{2/3} \tag{19.66}$$

such that

$$\varrho = \varrho_c w^3(z) = \left(\frac{K}{\pi G}\right)^{3/2}\frac{1}{a^3}w^3(z) . \tag{19.67}$$

We now come to the equation of motion and define

$$h := \int\frac{dP}{\varrho} = 4K\varrho^{1/3} , \tag{19.68}$$

where we have made use of (19.3) for $\gamma = 4/3$. Inserting $\psi$ and $h$ from (19.62) and (19.68) into the equation of motion (19.60) gives

$$\frac{\partial^2 \psi}{\partial r \partial t} + \frac{1}{2} \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} \right)^2 + \frac{\partial \Phi}{\partial r} + \frac{\partial h}{\partial r} = 0 , \tag{19.69}$$

which can be integrated with respect to $r$. If we set the integration constant to zero, replace $\partial\psi/\partial r$ by $\dot{a}z$, and consider (19.63), we find that

$$\frac{d\psi}{dt} = \frac{1}{2}\dot{a}^2 z^2 - \Phi - h \tag{19.70}$$

and therefore with (19.62)

$$\frac{1}{2} a\ddot{a} z^2 = -\Phi - h . \tag{19.71}$$

From (19.67) and (19.68) follows

$$h = 4K\varrho^{1/3} = 4\frac{K^{3/2}}{(\pi G)^{1/2}} \frac{1}{a} w(z) . \tag{19.72}$$

We try a similar dependence of $\Phi$ on $t$ and write

$$\Phi = 4\frac{K^{3/2}}{(\pi G)^{1/2}} \frac{1}{a} g(z) , \tag{19.73}$$

which defines the dimensionless function $g(z)$. If we insert (19.72) and (19.73) into (19.71) we find

$$\frac{1}{2} a^2 \ddot{a} = -\frac{4K^{3/2}}{(\pi G)^{1/2}} (g + w) \frac{1}{z^2} . \tag{19.74}$$

Since the left-hand side is a function of $t$ only and the right-hand side is a function of $z$ only, both sides must be constant; therefore

$$\frac{3}{4} \frac{(\pi G)^{1/2}}{K^{3/2}} a^2 \ddot{a} = -\lambda , \tag{19.75}$$

$$6\frac{g + w}{z^2} = \lambda \tag{19.76}$$

($\lambda$ = constant). The first of these equations can be integrated twice. After multiplication with $\dot{a}/a^2$, the first integration gives

$$\dot{a}^2 = \frac{8}{3}\lambda \left( \frac{K^3}{\pi G} \right)^{1/2} \frac{1}{a} , \tag{19.77}$$

where the constant of integration is set equal to zero (assuming a zero velocity when the sphere is expanded to infinity). Multiplication of (19.77) with $a$ gives

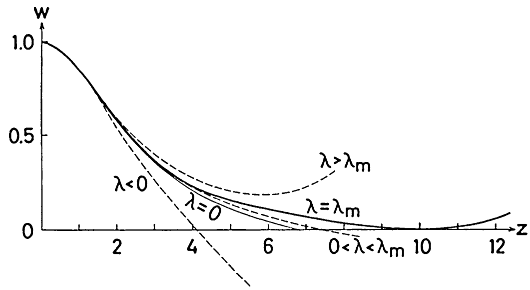


**Fig. 19.3** Solutions of (19.81) for different values of $\lambda$. In the range $0 < \lambda \leq \lambda_{\mathrm{m}}$, they describe homologously collapsing polytropes of index 3. The solution for $\lambda = \lambda_{\mathrm{m}}$ reaches the abscissa with slope zero. The *broken lines* indicate the behaviour of the solutions for different values of $\lambda$

$$a^{1/2}\dot{a} \equiv \frac{2}{3}\frac{d}{dt}(a^{3/2}) = \pm\left[\frac{8\lambda}{3}\left(\frac{K^3}{\pi G}\right)^{1/2}\right]^{1/2} \tag{19.78}$$

(the signs representing exploding or collapsing models, respectively). This can immediately be integrated, yielding for a collapse ($\dot{a} < 0$) that starts at $a_0$ for $t = 0$:

$$a^{3/2}(t) = a_0^{3/2} - \frac{3}{2}\left[\frac{8\lambda}{3}\left(\frac{K^3}{\pi G}\right)^{1/2}\right]^{1/2} t\ . \tag{19.79}$$

This expression gives the time dependence of the scaling factor $a(t)$ and therefore by way of (19.67), of the density as a function of time.

We now investigate the spatial dependence of our solution. In particular, the function $w(z)$ in (19.67) has to be determined. For this purpose we write Poisson's equation (19.2) in the dimensionless variable $z$

$$\frac{1}{z^2}\frac{\partial}{\partial z}\left(z^2\frac{\partial\Phi}{\partial z}\right) = 4\pi G\varrho a^2\ . \tag{19.80}$$

If we here replace $\Phi$ by (19.73), $g(z)$ by (19.76), and $\varrho$ by (19.67), we find

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{dw}{dz}\right) + w^3 = \lambda\ . \tag{19.81}$$

For $\lambda = 0$ this is the classical Emden equation. Solutions for $\lambda \neq 0$ deviate from hydrostatic equilibrium, the value of $\lambda$ being a measure for this deviation. From numerical integrations it follows that physically relevant solutions $w(z)$ are obtained only for very small values of $\lambda$, namely for $\lambda < \lambda_{\mathrm{m}} = 0.0065$. Otherwise the solution $w(z)$ and therefore $\varrho(r)$ do not become zero at a finite radius; they rather increase again to infinity after a minimum has been reached (see Fig. 19.3).

This figure shows also that for $\lambda < \lambda_\mathrm{m}$ the solutions deviate appreciably from the "classical" one ($\lambda = 0$) only in the outer layers, where $\lambda \ll w^3$ no longer applies.

The time-dependent solution discussed here has to be understood in the following way. Let us consider a polytrope with $n = 3$ in equilibrium; then the equilibrium is independent of radius. We have already seen that the total energy is $W = 0$ independent of the radius, see (19.52). Therefore the polytrope $n = 3$ is indifferent to radial changes. If we now assume that suddenly the pressure is slightly reduced say, because the constant $K$ is slightly diminished, then the gaseous sphere begins to contract. This contraction can be described by the two equations (19.75) and (19.76). The solution of the first gives the behaviour in time (19.79), while the second is used to derive the modification of the Lane–Emden equation due to the inertia terms. The parameter $\lambda$ is a measure of the deviation from hydrostatic equilibrium, caused by the assumed reduction of $K$.

The solutions for collapsing polytropes have been discussed by Goldreich and Weber (1980) with respect to collapsing stellar cores causing supernova outbursts (Chap. 36).