Chapter 21
Simple Models in the U–V Plane

There are stars in which the nuclear energy generation proceeding close to the centre creates such a high energy flux that the whole central region is convective. These stars can be described by models with a convective core and a radiative envelope. In later stages of stellar evolution the nuclear fuel in the central region of the star is exhausted and nuclear burning takes place only at the surface of a burned-out core. Under certain circumstances these models with shell burning can be described by a core that is isothermal, since no energy has to be transported there, and that is surrounded by a radiative envelope. In both cases a core solution of one type has to be fitted to an envelope solution of another type. In the following we shall deal with a classical fitting procedure which in the past was often used to construct models for such stars (see Schwarzschild 1958; Wrubel 1958) and which gives valuable insight into some of their general properties. Moreover, procedures like this can be helpful in certain special cases where the usual, iterative numerical methods are not practicable.

21.1 The U–V Plane

We define two dimensionless quantities using (1.2) and (2.4):

$$U := \frac{d\ln m}{d\ln r} = \frac{4\pi r^3 \varrho}{m}, \quad V := -\frac{d\ln P}{d\ln r} = \frac{\varrho}{P}\frac{Gm}{r}. \tag{21.1}$$

A solution which is regular in the stellar centre has the central values $U = 3$, $V = 0$, as can easily be seen: a small sphere around the centre has the mass $m = 4\pi r^3 \varrho_c/3$, so that there $U \to 3$ and $V \sim r^2 \to 0$. Near the surface the numerical value of U becomes very small (as ϱ does), as well as P/ϱ ($\sim T$ for the ideal gas or $\sim \varrho^{\gamma-1}$ for polytropes). Therefore V becomes very large.

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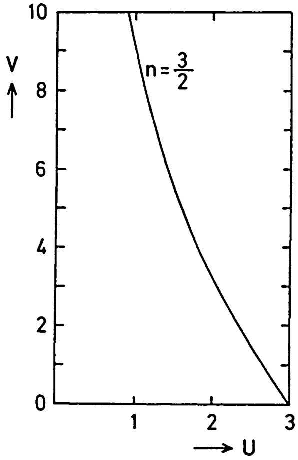

Fig. 21.1 The polytrope $n = 3/2$ in the U–V plane. The stellar centre is in the lower-right corner ($U = 3$, $V = 0$)

Compare two homologous models. Then U as well as V have the same value in homologous mass shells. Indeed with $r/r' = R/R', m/m' = M/M'$, and (20.9) it follows that

$$U = \frac{4\pi r^3 \varrho}{m} = \frac{4\pi r'^3 \varrho'}{m'} = U' \quad \text{and correspondingly } V = V'. \tag{21.2}$$

U and V are therefore also called *homology invariants*.

We now determine the quantities U and V for polytropes. From (19.11) and (19.18), we find

$$U = -w^n \left(\frac{1}{z}\frac{dw}{dz}\right)^{-1}. \tag{21.3}$$

With the expansion (19.12) one can see that indeed $U \to 3$ for $z \to 0$, independent of the value of n. We furthermore find–with $\varrho = \varrho_c w^n$, $P = P_c(\varrho/\varrho_c)^{1+1/n} = P_c w^{n+1}$, and (19.18)–from (21.1) that

$$V = \frac{4\pi G \varrho_c^2 r^2}{P_c}\left(-\frac{1}{z}\frac{dw}{dz}\right)\frac{1}{w}, \tag{21.4}$$

and with (19.3) and (19.9)

$$V = -(n+1)\frac{z}{w}\frac{dw}{dz}, \tag{21.5}$$

which indeed vanishes at the centre and becomes large near the surface where $w \to 0$. Note that the functions $U(z)$ and $V(z)$ depend only on n: they are independent of any other parameter of the model. This is the property which makes a discussion of the U–V plane worthwhile. The function $V = V(U)$ for $n = 3/2$ is plotted in Fig. 21.1.

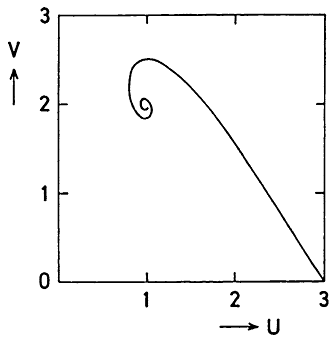

Fig. 21.2 The isothermal sphere for an ideal gas in the U–V plane. The centre ($r = 0$) is in the lower-right comer ($U = 3$, $V = 0$), while for the surface ($r \to R = \infty$) the curve spirals into the point $U = 1, V = 2$

The above polytropic relations hold for finite n only. The isothermal polytrope for an ideal gas ($n = \infty$) again is an exceptional case. Instead of (21.3) and (21.5) one finds from (21.1) and the relations of Sect. 19.8

$$U = e^{-w} \left(\frac{1}{z} \frac{dw}{dz} \right)^{-1}, \quad V = z \frac{dw}{dz}, \tag{21.6}$$

where w now is the solution of (19.35). This case is shown in Fig. 21.2: although the corresponding polytropic model has an infinite radius, its image curve in the U–V plane spirals into the point $U = 1$, $V = 2$, which represents the surface ($z = \infty$). The spiral of the isothermal gaseous sphere unwinds and reaches higher and higher values of V if degeneracy becomes important. In the limit case of complete non-relativistic degeneracy, the image curve approaches that of the polytrope $n = 3/2$ of Fig. 21.1.

The U–V plane has often been used to construct simple stellar models by fitting core and envelope solutions. Clearly this is most profitable when the core is polytropic with given index n and therefore all possible cores are represented by a single, known curve in the plane. This is the case for stars with convective cores (polytropic with $n = 3/2$) or with non-degenerate isothermal cores ($n = \infty$).

The fitting requires continuity of r, P, T, l at the interface. If μ is continuous, then also ϱ–and according to (21.1)–U and V have to be continuous at the fitting point: core and envelope curves intersect (compare Figs. 21.3 and 21.4). If μ is discontinuous at the interface having there the values μ_1, μ_2, then the continuity of P and T for an ideal gas requires $\varrho_1/\varrho_2 = \mu_1/\mu_2$, and (21.1) shows that

$$\frac{U_1}{U_2} = \frac{V_1}{V_2} = \frac{\varrho_1}{\varrho_2} = \frac{\mu_1}{\mu_2}, \tag{21.7}$$

where subscripts 1 and 2 refer to core and envelope solutions at the interface respectively. This means that the points (U_1, V_1) and (U_2, V_2) lie on a straight line through the origin.

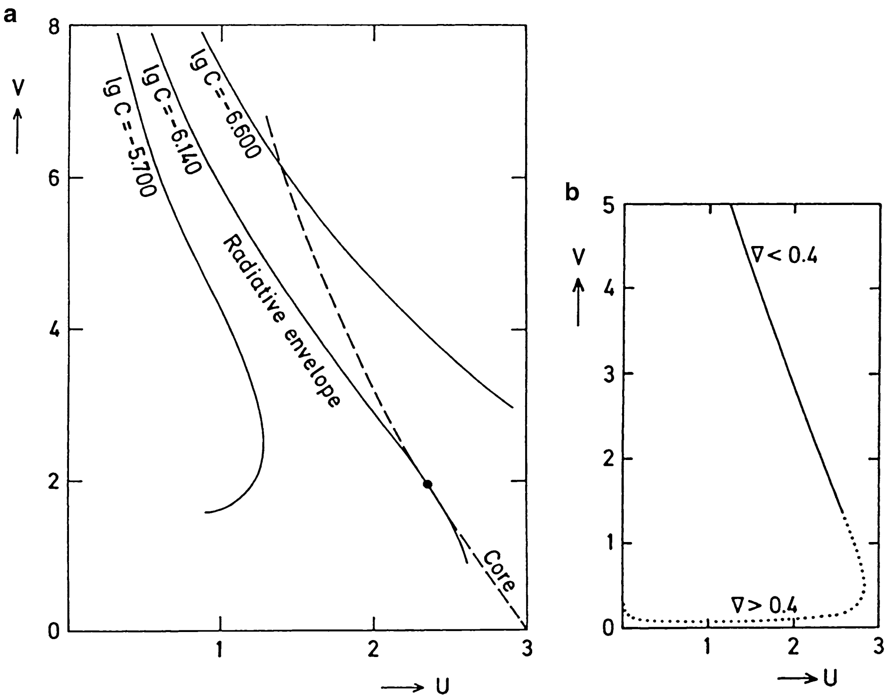

Fig. 21.3 Fitting a radiative-envelope solution with a convective core in the U–V plane. (**a**) Three envelope solutions with different values of the parameter C come from the upper left downwards (*solid lines*). One of them fits to the convective-core solution (*dashed line*), which is given by the polytrope of $n = 3/2$ and starts in the centre at $U = 3$, $V = 0$. At the fitting point, both curves have the same gradient $\nabla = \nabla_{\mathrm{ad}} = 0.4$ and the same tangent. (**b**) A radiative-envelope solution in the U–V plane. The solution is shown by a *solid line* as far as $\nabla < 0.4$, and by a *dotted line* where $\nabla > 0.4$ such that the assumption of radiative transport breaks down (After Schwarzschild 1958)

21.2 Radiative Envelope Solutions

We first consider solutions for the envelope where $\varepsilon = 0$ and therefore $l = \text{constant} = L$. The gas is supposed to be ideal, and the opacity is approximated by a power law

$$\kappa = \kappa_0 \varrho^a T^{-b}, \tag{21.8}$$

where $\kappa_0 = \text{constant}$ (Note that here a representation in ϱ and T is used which gives a different exponent b than a representation in P and T.).

We want to obtain many different solutions from a given one by simple scaling. For this aim we replace P, T, m, r by the dimensionless Schwarzschild variables y, t, q, x (Schwarzschild 1946):

$$P = \frac{GM^2}{4\pi R^4} y \,, \quad T = \frac{\mu}{\Re} \frac{GM}{R} t \,, \quad m = qM, \quad r = xR. \tag{21.9}$$

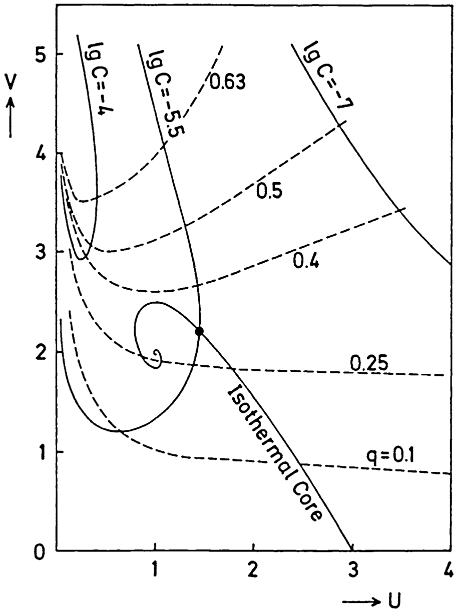

Fig. 21.4 Three envelope solutions with different parameters C and the curve of the non-degenerate isothermal core in the U–V plane. *The dashed lines* combine those points of the envelope solutions where $q = m/M$ reaches certain values. Since, in the case of a homogeneous model, envelope and core solution must be fitted continuously in the U–V plane, one can see that no complete models are possible for isothermal cores with more than about 0.38 M (This limit is even lower if the core has a higher molecular weight than the envelope.). A possible fit for $q \approx 0.3$ between the envelope curve for $\lg C = -5.5$ and the isothermal-core curve is indicated by a *heavy dot*

The equation of state gives the density as

$$\varrho = \frac{M}{4\pi R^3}\frac{y}{t}. \tag{21.10}$$

One can easily see that then the homology variables become $U = x^3 y/(qt)$ and $V = q/(tx)$. The stellar-structure equations (10.1) and (10.2) give

$$\frac{dx}{dq} = \frac{t}{x^2 y}, \quad \frac{dy}{dq} = -\frac{q}{x^4}, \tag{21.11}$$

while the equation for energy transport (10.4) with expression (10.6) gives

$$\frac{dt}{dq} = -C\frac{y^a}{t^{a+b+3}x^4}, \tag{21.12}$$

with

$$C = \frac{3\kappa_0}{4ac(4\pi)^{a+2}} \left(\frac{\Re}{\mu G}\right)^{b+4} L R^{b-3a} M^{a-b-3}. \tag{21.13}$$

At the surface $q = 1$, and the solutions have to fulfil the boundary conditions

$$y = 0\,, \quad x = 1\,, \quad y/t = 0, \tag{21.14}$$

the last of which guarantees that according to (21.10) the density vanishes there.

The singularity of the system (21.11) and (21.12) at the surface can be overcome by an approximation. If one puts q = constant = 1 for the whole near-surface region, one finds from (21.11) and (21.12) that

$$\frac{dy}{dt} = \frac{1}{C}\frac{t^{a+b+3}}{y^a}\,, \quad \frac{dt}{dx} = -\frac{a+1}{a+b+4}\frac{1}{x^2}. \tag{21.15}$$

The first equation has been integrated (the integration constant being chosen in such a way that $y = t = 0$ at the surface). This is used for eliminating y from (21.11) and (21.12), which then give the second equation (21.15).

The two ordinary differential equations (21.15) are integrated by separation of the variables. The solutions can be used near the surface down to a safe distance from the singularity. From there on the normal equations (21.11) and (21.12) can be numerically integrated inwards.

Obviously one obtains a one-parameter set of solutions, the parameter being C. Three such envelope solutions in the U–V plane are shown in Fig. 21.3a. All of them come from the upper left and miss the central boundary condition ($U = 3$, $V = 0$), since they have a singularity there. This does not matter, since anyway we have to fit them to a core solution (compare also with Sect. 12.1). From (21.11) and (21.12) it results that

$$\nabla \equiv \frac{d\ln T}{d\ln P} = \frac{y}{t}\frac{dt}{dy} = C\frac{y^{a+1}}{t^{a+b+4}q}, \tag{21.16}$$

from which one can see that owing to the factor q^{-1} the value of ∇ tends to infinity near the centre. In fact ∇ is small near the surface and increases inwards until it reaches the critical value ∇_{ad} (see Fig. 21.3b). Further inwards the Schwarzschild criterion (6.13) requires convection and the radiative-envelope solutions are no longer valid.

21.3 Fitting of a Convective Core

In order to obtain a model with a convective core inside a radiative envelope we have to fit the solutions of Sect. 21.2 with a polytropic solution of $n = 3/2$ starting at the centre ($U = 3$, $V = 0$). The fit has to be done at the point where the envelope solution reaches $\nabla = \nabla_{ad}$. Joining all these points on the different envelope solutions (different C) gives a line $\nabla = \nabla_{ad}$ in the U–V plane, which intersects the core polytrope at the fitting point U^*, V^*. The envelope solution through this point has the value $C = C^*$. Because of the condition that the gradient ∇ is also continuous there, the solutions for core and envelope are tangential to each other, as can be seen in Fig. 21.3a. At the fitting point the variables of the envelope solution may be q^*, y^*, x^*, t^*, while the core polytrope has the variables z^*, w^*.

Let us assume a certain value for the mean molecular weight μ in the envelope. The fit has fixed $C = C^*$, which according to (21.13) gives a relation between L, R, and M. But L is determined by the energy generation in the core, for which we assume a rate of

$$\varepsilon = \varepsilon_0 \varrho T^\nu. \tag{21.17}$$

In the convective core we can connect the Emden variable z with r by $r = zr^*/z^*$, where $r^* = x^* R$ from the outer solution. Then $r^* dl/dr = z^* dl/dz$, and with $\varrho = \varrho_c w^{3/2}$, $T = T_c w$, we have the energy equation with $\lambda = l/L$

$$\frac{d\lambda}{dz} = Bz^2 w^{\nu+3}, \quad B = \frac{4\pi\varepsilon_0}{L}\left(\frac{x^* R}{z^*}\right)^3 \varrho_c^2 T_c^\nu. \tag{21.18}$$

Continuity of ϱ and T in core and envelope solutions requires

$$\varrho^* = \varrho_c w^{*3/2} = \frac{M}{4\pi R^3}\frac{y^*}{t^*}, \tag{21.19}$$

$$T^* = T_c w^* = \frac{\mu}{\Re}\frac{GM}{R} t^*. \tag{21.20}$$

With these two equations we can express ϱ_c, T_c as functions of w^*, y^*, t^* (all known from the integrations) and of M and R. The expressions inserted into (21.18) give

$$B = B_0 \varepsilon_0 \left(\frac{\mu G}{\Re}\right)^\nu \frac{M^{\nu+2}}{LR^{\nu+3}}, \tag{21.21}$$

where B_0 is known from the numerical integrations to the fitting point. Since L is to be generated in the core, $\lambda = l/L = 1$ at the fitting point. Therefore integration of (21.18) gives

$$1 = \int_0^{z^*} \frac{d\lambda}{dz} dz = B \int_0^{z^*} z^2 w^{\nu+3} dz. \tag{21.22}$$

This fixes the value $B = B^*$, since z^* is known, and the integral follows from a simple quadrature.

The fitting procedure now has yielded two numerical values C^*, B^*. Therefore for a given value of M one obtains L and R from (21.13) and (21.21). Of course, one has to check afterwards that (21.17) only gives negligible contributions to L in the envelope solution (where $l =$ constant was assumed).

Models of this type were first constructed by Cowling (1935). They have the advantage that l appears in the structure equations only for the envelope where it is constant ($= L$).

21.4 Fitting of an Isothermal Core

In stellar evolution we shall have to discuss models with an isothermal helium core surrounded by a hydrogen-rich envelope. The luminosity is generated in a thin shell at the interface. This will be idealized by assuming a discontinuity of l (from 0 to L) at the interface.

Let us discuss here a model in which μ is continuous at the interface so that the image curve in the U–V plane is continuous at the fit.

In Fig. 21.4 we have plotted envelope solutions together with the isothermal-core solution for an ideal gas. Along each envelope curve the value of q decreases inwards. We have also plotted some lines $q =$ constant. As one can see from the figure there are no fits possible with $q > q_{\max} \approx 0.38$, i.e. when more than 38 % of the total mass lies within the isothermal core. For given $q < q_{\max}$ a fit is possible. An example for a fit at $q \approx 0.3$ is shown in Fig. 21.4. One can show that such a fit determines a model completely for given M. Physically more realistic is a model in which μ is higher in the core than in the envelope, which we idealize by a jump of μ at the interface. Then the curve in the $U - V$ plane is discontinuous, fulfilling the conditions (21.7) at the interface ($\mu_1 > \mu_2$). If one tries to fit core and envelope with this condition, and say $\mu_1/\mu_2 = 1.333/0.62$, one finds that $q_{\max}$ is considerably smaller: no fits are possible at $q > q_{\max} \approx 0.1$. This gives the Schönberg–Chandrasekhar limit for isothermal cores consisting of an ideal gas (see Sect. 30.5) enclosed by the stellar envelope.