Chapter 6
Stability Against Local, Non-spherical Perturbations

We have based our treatment on the assumption of strict spherical symmetry, meaning that all functions and variables (including velocities) are constant on concentric spheres. In reality there will arise small fluctuations on such a sphere, for example, simply from the thermal motion of the gas particles. Such local perturbations of the average state may be ignored if they do not grow. But in a star sometimes small perturbations may grow and give rise to macroscopic local (non-spherical) motions that are also statistically distributed over the sphere. In the basic equations the assumption of spherical symmetry can still be kept if we interpret the variables as proper average values over a concentric sphere.

However, these motions have to be considered carefully because they can have a strong influence on the stellar structure. They not only mix the stellar material but also transport energy: hot gas bubbles rise, while cooler material sinks down, i.e. energy transport is by convection, something which is known to play an important role in the earth's atmosphere.

Whether convection occurs in a certain region of a star obviously depends on the question whether the small perturbations always present will grow or stay small: a question of *stability*. We shall derive criteria which tell us whether stellar material at a certain depth is stable or not. Depending on the physical conditions one can make different simplifying assumptions which lead to different stability problems. The following dynamical problem covers most of the "normal" cases in stars.

6.1 Dynamical Instability

The kind of stability we are discussing here is based on the assumption that the moving mass elements have no time to exchange appreciable amounts of heat with the surroundings and therefore move adiabatically. This type of stability (or instability) is called *dynamical*. We will soon learn that there are other types of instability.

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First, we consider the possibility that the physical quantities (temperature, density, etc.) may not be exactly constant on the surface of a concentric sphere but rather may show certain fluctuations. In the global problem of stellar structure, one then has only to interpret the previously used functions as proper averages. For the local description, we shall simply represent a fluctuation by a mass "element" (subscript e) in which the functions have constant, but somewhat different, values than in the average "surroundings" (subscript s). For any quantity A we define the difference DA between element and surroundings[1] as

$$DA := A_\mathrm{e} - A_\mathrm{s} \ . \tag{6.1}$$

One can easily imagine an initial fluctuation of temperature, for example, a slightly hotter element with $DT > 0$. Normally one could then also expect an excess of pressure DP. However, the element will expand immediately until pressure balance with the surroundings is restored, and since this expansion occurs with the velocity of sound, it is usually much more rapid than any other motion of the element. Therefore we can assume here (and in the following) that the element always remains in pressure balance with its surroundings:

$$DP = 0 \ . \tag{6.2}$$

Consequently the assumed $DT > 0$ requires that, for a perfect gas with $\varrho \sim P/T, D\varrho < 0$, i.e. the element is lighter than the surrounding material, and the buoyancy forces will lift it upwards: temperature fluctuations are obviously accompanied by local motions of elements in a radial direction.

So, we can also take a radial shift $\Delta r > 0$ of the element as the initial perturbation for testing the stability of a layer. Consider an element that was in complete equilibrium with the surroundings at its original position r but has now been lifted to $r + \Delta r$ (cf. Fig. 6.1). In general its density will differ from that of its new surroundings by

$$D\varrho = \left[\left(\frac{d\varrho}{dr} \right)_\mathrm{e} - \left(\frac{d\varrho}{dr} \right)_\mathrm{s} \right] \Delta r \ , \tag{6.3}$$

$(d\varrho/dr)_\mathrm{e}$ determining the change of the element's density while it rises by dr; the other derivative is the spatial gradient in the surroundings.

A finite $D\varrho$ gives the radial component $K_r = -gD\varrho$ of a buoyancy force $\boldsymbol{K}$ (per unit of volume), where g again is the absolute value of the acceleration of gravity. If $D\varrho < 0$, the element is lighter and $K_r > 0$, i.e. $\boldsymbol{K}$ is directed upwards. This situation is obviously unstable, since the element is lifted further, the original perturbation being increased.

[1]Note that we use the subscript s, which is different from s used for the specific entropy in other parts of this book.

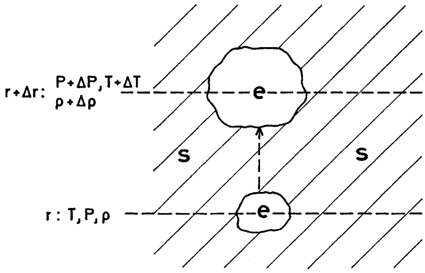

Fig. 6.1 In order to test the stability of a "surrounding" layer (s), a test "element" (e) is lifted from level r to $r + \Delta r$

If on the other hand $D\varrho > 0$, then $K_r < 0$, i.e. $\boldsymbol{K}$ is directed downwards. The element, which is heavier than its new surroundings, is drawn back to its original position, the perturbation is removed, and the layer is stable. As the *condition for stability* we obtain with $D\varrho > 0$ from (6.3) the result

$$\left(\frac{d\varrho}{dr}\right)_{\mathrm{e}} - \left(\frac{d\varrho}{dr}\right)_{\mathrm{s}} > 0\ . \tag{6.4}$$

Unfortunately this criterion is highly impractical, since it requires knowledge of density gradients that do not appear in the basic equations. It is therefore preferable to turn to temperature gradients as used in the equations of radiative and conductive transport. In order to evaluate $(d\varrho/dr)_{\mathrm{e}}$ correctly, we would have to take into account the possible energy exchange between the element and its surroundings. For simplicity let us here assume that no such exchange of energy occurs, i.e. that the element rises *adiabatically.* This is very close to reality for the deep interior of a star (see Chap. 7).

In order to transform the gradients of ϱ into those of T, we write the equation of state $\varrho = \varrho\,(P, T, \mu)$ in the following differential form:

$$\frac{d\varrho}{\varrho} = \alpha\frac{dP}{P} - \delta\frac{dT}{T} + \varphi\frac{d\mu}{\mu}\ , \tag{6.5}$$

where α and δ have already been defined in (4.2) and (4.3). But here, we have made allowance also for a possible variation of the chemical composition, which is characterized by the molecular weight μ. We therefore have

$$\alpha := \left(\frac{\partial \ln \varrho}{\partial \ln P}\right), \quad \delta := -\left(\frac{\partial \ln \varrho}{\partial \ln T}\right), \quad \varphi := \left(\frac{\partial \ln \varrho}{\partial \ln \mu}\right)\ , \tag{6.6}$$

where the three partial derivatives correspond to constant values of T, μ; P, μ; and P, T, respectively, and for a perfect gas with $\varrho \sim P\mu/T$, one has $\alpha = \delta = \varphi = 1$. In this description $d\mu$ shall represent only the change of μ due to the change of chemical composition, i.e. the variation of the concentrations of different nuclei in the deep interior. Of course, μ can also change in the outer regions for constant composition if the degree of ionization changes. This effect, however, has a well-known dependence on P and T and is supposed to be incorporated in α and δ. Thus, $d\mu = 0$ for the moving element that carries its composition along. But $d\mu \neq 0$ for the surroundings if the element passes through layers of different chemical composition.

We can immediately rewrite (6.4) with the help of (6.5) in the form

$$\left(\frac{\alpha}{P}\frac{dP}{dr}\right)_{\mathrm{e}} - \left(\frac{\delta}{T}\frac{dT}{dr}\right)_{\mathrm{e}} - \left(\frac{\alpha}{P}\frac{dP}{dr}\right)_{\mathrm{s}} + \left(\frac{\delta}{T}\frac{dT}{dr}\right)_{\mathrm{s}} - \left(\frac{\varphi}{\mu}\frac{d\mu}{dr}\right)_{\mathrm{s}} > 0 \,. \qquad (6.7)$$

The two terms containing the pressure gradient cancel each other owing to (6.2), and the other terms are usually multiplied by the so-called *scale height of pressure* H_P:

$$H_P := -\frac{dr}{d\ln P} = -P\frac{dr}{dP} \,. \qquad (6.8)$$

With (2.3), the condition for hydrostatic equilibrium, we find $H_P = P/\varrho g$, i.e. $H_P > 0$, since P decreases with increasing r. H_P has the dimension of length, being the length characteristic of the radial variation of P. In the solar photo-sphere ($g = 2.7 \times 10^4\,\mathrm{cm\,s^{-2}}$, $P = 1.0 \times 10^5\,\mathrm{dyn\,cm^{-2}}$, $\varrho = 2.6 \times 10^{-7}\,\mathrm{g\,cm^{-3}}$), one finds $H_P = 1.4 \times 10^7\,\mathrm{cm}$, while at $r = R_\odot/2$ ($g = 9.8 \times 10^4\,\mathrm{cm\,s^{-2}}$, $P = 7.3 \times 10^{14}\,\mathrm{dyn\,cm^{-2}}$, $\varrho = 1.4\,\mathrm{g\,cm^{-3}}$), H_P is much bigger, at $5.5 \times 10^9\,\mathrm{cm}$. If one approaches the stellar centre–where $g = 0$, while P remains finite–then $H_P \to \infty$.

Multiplication of (6.7) by H_P yields as a condition for stability

$$\left(\frac{d\ln T}{d\ln P}\right)_{\mathrm{s}} < \left(\frac{d\ln T}{d\ln P}\right)_{\mathrm{e}} + \frac{\varphi}{\delta}\left(\frac{d\ln\mu}{d\ln P}\right)_{\mathrm{s}} \,. \qquad (6.9)$$

Similar to the previously defined quantities ∇_{rad} and ∇_{ad}, we define three new derivatives:

$$\nabla := \left(\frac{d\ln T}{d\ln P}\right)_{\mathrm{s}}, \quad \nabla_{\mathrm{e}} := \left(\frac{d\ln T}{d\ln P}\right)_{\mathrm{e}}, \quad \nabla_\mu := \left(\frac{d\ln\mu}{d\ln P}\right)_{\mathrm{s}} \,. \qquad (6.10)$$

Here the subscripts s indicate that the derivatives are to be taken in the surrounding material. In both cases they are spatial derivatives in which the variations of T and μ with depth are considered and P is taken as a measure of depth. The quantity ∇_{e} describes the variation of T in the element during its motion, where the position of the element is measured by P. In this sense ∇_{e} and ∇_{ad} are similar, since both describe the temperature variation of a gas undergoing pressure variations; on the other hand, ∇_{rad} and ∇_μ describe the spatial variation of T and μ in the surroundings.

With the definitions (6.10) the condition (6.9) for stability becomes

$$\nabla < \nabla_{\mathrm{e}} + \frac{\varphi}{\delta}\nabla_\mu \,. \qquad (6.11)$$

In (5.27) and (5.28) we defined $\nabla_{\rm rad}$, which describes the temperature gradient for the case that the energy is transported by radiation (or conduction) only. Therefore in a layer that indeed transports all energy by radiation the actual gradient ∇ is equal to $\nabla_{\rm rad}$. Let us test such a layer for its stability and assume the elements change adiabatically: $\nabla_{\rm e} = \nabla_{\rm ad}$; the radiation layer is stable if

$$\nabla_{\rm rad} < \nabla_{\rm ad} + \frac{\varphi}{\delta}\nabla_{\mu} \ , \tag{6.12}$$

a form known as the *Ledoux criterion* (named after Paul Ledoux) for dynamical stability. In a region with homogeneous chemical composition, $\nabla_{\mu} = 0$, and one has then simply the famous *Schwarzschild criterion* for dynamical stability (named after Karl Schwarzschild):

$$\nabla_{\rm rad} < \nabla_{\rm ad} \ . \tag{6.13}$$

If in the criteria (6.12) and (6.13) the left-hand side is larger than the right, the layer is dynamically unstable. If they are equal, one speaks of marginal stability. The difference between the two criteria obviously plays a role only in regions where the chemical composition varies radially. We will see that such regions occur in the interior of evolving stars, where heavier elements are usually produced below the lighter ones, such that the molecular weight μ increases inwards (as the pressure does) and $\nabla_{\mu} > 0$. Then the last term in inequality (6.12) obviously has a stabilizing effect (φ and δ are both positive). This is plausible since the element carries its heavier material upwards into lighter surroundings and gravity will tend to draw it back to its original place.

If these criteria favour stability, then no convective motions will occur, and the whole flux will indeed be carried by radiation, i.e. the actual gradient at such a place is equal to the radiative one: $\nabla = \nabla_{\rm rad}$. If they favour instability, then small perturbations will increase to finite amplitude until the whole region boils with convective motions that carry part of the flux–and the actual gradient has to be determined in a manner described in Chap. 7. This instability can be caused either by the fact that $\nabla_{\rm rad}$ has become too high (large flux or very opaque matter), or else by a depression of $\nabla_{\rm ad}$; both cases occur in stars. And, finally, in a twilight zone, where one of the two criteria (6.12) and (6.13) says stability and the other one says instability, strange things may happen (see, for instance, Sects. 6.3 and 30.4.2).

Note that (6.12) and (6.13) are strictly local criteria, which means good and bad news. They are very practical since they can be evaluated easily for any given place by using the local values of P, T, ϱ only, without bothering about other parts of the star. And in most cases this will give satisfactory answers. In critical cases, however, this may not be sufficient. Strictly speaking, convective motions are not only dependent on the local forces (which are solely regarded by the criteria), but must be coupled (by momentum transfer, inertia, the equation of continuity) to their neighbouring layers. And in extreme cases the reaction of the whole star against a local perturbation should be taken into account. An obvious example is the precise determination of the border of a convective zone, where elements that were accelerated elsewhere "shoot over" until their motion is braked. We will come back later to such problems when they arise (see Sect. 30.4.1).

We can immediately derive a qualitative relation between the different gradients. They are best visualized in a diagram such as Fig. 6.2, where $\ln T$ is plotted against $\ln P$ (decreasing outwards) for an unstable layer violating the Schwarzschild criterion. In such a diagram, an adiabatic change follows a line with slope ∇_{ad}, the changes in a rising element are given by a line with slope ∇_{e}, while the stratifications in the surroundings and in a radiative layer are shown by lines with slopes ∇ and ∇_{rad}, respectively.

Suppose we have convection in a chemically homogeneous layer ($\nabla_\mu = 0$). The criterion (6.11) must be violated, i.e. $\nabla > \nabla_{\mathrm{e}}$. If some part of the flux is carried by convection, then the actual gradient $\nabla < \nabla_{\mathrm{rad}}$, since only a part of the total flux is left for radiative transfer. Consider a rising element that has started from a point with P_0, T_0. In Fig. 6.2 this element moves downwards to the left along the line with slope ∇_{e}. Since $\nabla > \nabla_{\mathrm{e}}$, the element (although cooling) will obviously have an increasing temperature excess over its new surroundings (the temperature of which changes with ∇). Therefore it will radiate energy into its surroundings, which means that the element cools more than adiabatically: $\nabla_{\mathrm{e}} > \nabla_{\mathrm{ad}}$. Combining these inequalities, we arrive at the relation illustrated in Fig. 6.2:

$$\nabla_{\mathrm{rad}} > \nabla > \nabla_{\mathrm{e}} > \nabla_{\mathrm{ad}} \ . \tag{6.14}$$

The fact that ∇_{e} must always be between ∇_{ad} and ∇ of the surroundings shows that the criteria (6.12) and (6.13) are also to be used in near-surface regions, where the rising elements lose much of their energy by radiation.

6.2 Oscillation of a Displaced Element

In a dynamically stable layer a displaced mass element is pushed back by buoyancy. When coming back to its original position, it has gained momentum and will overshoot and therefore start to oscillate. In the following we shall discuss this oscillation.

Consider a mass element lifted from its normal (equilibrium) position in the radial direction by an amount Δr (see Fig. 6.1). There it has an excess of density $D\varrho$ over its new surroundings given by (6.3), which for balance of pressure ($DP = 0$) and with (6.5) and the definitions (6.6), (6.8), (6.10) can easily be written as

$$D\varrho = \frac{\varrho\delta}{H_P}\left[\nabla_{\mathrm{e}} - \nabla + \frac{\varphi}{\delta}\nabla_\mu\right]\Delta r \ . \tag{6.15}$$

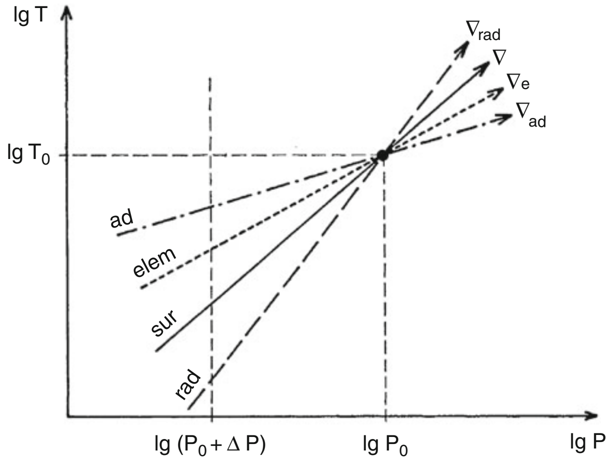

Fig. 6.2 Temperature-pressure diagram with a schematic sketch of the different gradients $\nabla(\equiv \partial \ln T / \partial \ln P)$ in a convective layer. Starting at a common point with P_0 and T_0, the different types of changes (adiabatic, in a rising element, in the surroundings, for radiative stratification) lead to different temperatures at a slightly higher point with $P_0 + \Delta P$ ($< P_0$, since P decreases outwards)

In the presence of gravity g, the resulting buoyancy force per unit volume is $K_r = -gD\varrho$, producing an acceleration of the element of

$$\frac{\partial^2(\Delta r)}{\partial t^2} = -\frac{g\delta}{H_P}\left[\nabla_{\mathrm{e}} - \nabla + \frac{\varphi}{\delta}\nabla_\mu\right]\Delta r\ . \tag{6.16}$$

Suppose now that the element, after an original displacement Δr_0, moves adiabatically ($\nabla_{\mathrm{e}} = \nabla_{\mathrm{ad}}$) through a dynamically stable layer ($D\varrho/\Delta r > 0$). The element is accelerated back towards its equilibrium position around which it then oscillates according to the solution of (6.16):

$$\Delta r = \Delta r_0\, \mathrm{e}^{\mathrm{i}\omega t}\ . \tag{6.17}$$

The frequency $\omega = \omega_{\mathrm{ad}}$ of this adiabatic oscillation is the so-called *Brunt-Väisälä frequency* given by

$$\omega_{\mathrm{ad}}^2 = \frac{g\delta}{H_P}\left(\nabla_{\mathrm{ad}} - \nabla + \frac{\varphi}{\delta}\nabla_\mu\right)\ . \tag{6.18}$$

(It plays, e.g. a role in the discussion of non-radial oscillations of a star, see Chap. 42.) The corresponding period is $\tau_{\mathrm{ad}} = 2\pi/\omega_{\mathrm{ad}}$.

We see immediately what happens in an unstable layer. If the Ledoux criterion (6.12) [or the Schwarzschild criterion (6.13) for $\nabla_\mu = 0$] is violated, then (6.18) gives $\omega_{\mathrm{ad}}^2 < 0$, such that ω_{ad} is imaginary and the time dependence of Δr is given by the factor $\exp(\sigma t)$ with a real $\sigma > 0$. Instead of oscillating, the displaced element moves away exponentially.

6.3 Vibrational Stability

In a dynamically stable layer an oscillating mass element has, in general, $DT \neq 0$. If $DT > 0$, it will lose heat to its surrounding by radiation; if $DT < 0$, it will gain heat. This means it will not move adiabatically. We consider the deviation from adiabaticity to be small, which means that the thermal adjustment time of the element is large compared to the period of the oscillation; then the temperature excess of the element can be written as

$$\begin{aligned} DT &= \left[\left(\frac{dT}{dr}\right)_{\mathrm{e}} - \left(\frac{dT}{dr}\right)_{\mathrm{s}}\right]\Delta r \\ &= -\frac{T}{H_P}(\nabla_{\mathrm{e}} - \nabla)\Delta r \ . \end{aligned} \tag{6.19}$$

Dynamical stability means that $D\varrho/\Delta r > 0$ and therefore (6.11) is fulfilled. If the layer is chemically homogeneous, then $\nabla_\mu = 0$, and (6.11) becomes $\nabla_{\mathrm{e}} - \nabla > 0$, such that (6.19) gives $DT < 0$ for $\Delta r > 0$. Above its equilibrium position the element is cooler than the surroundings and receives energy by radiation. This reduces $\nabla_{\mathrm{e}} - \nabla$, $D\varrho$, and the restoring force, such that the element is less accelerated back towards the equilibrium position. The result will be an oscillation with slowly decreasing amplitude. Formally this radiative damping shows up as a small positive imaginary part of ω in (6.17) after the exchange of heat with the surroundings is included in (6.16). The oscillatory part (real part of ω) is still very close to the adiabatic value (6.18).

If the stable layer is inhomogeneous with $\nabla\mu > 0$, it can be that with (6.11) $\nabla_{\mathrm{e}} - \nabla > 0$ also (*both* criteria for stability are fulfilled), i.e. we find again that $DT < 0$ for $\Delta r > 0$ and radiative damping as before. However, we can also imagine a situation with $\nabla_{\mathrm{e}} - \nabla < 0$ in spite of (6.11) for large enough ∇_μ. Then $DT > 0$ for $\Delta r > 0$ according to (6.19), and the lifted element, being hotter than its surroundings, will now *lose* energy by radiation. This increases $\nabla_{\mathrm{e}} - \nabla$, $D\varrho$, and the restoring force, and the element will oscillate with slowly increasing amplitude. This is an *over-stability,* or *vibrational instability.* The difficulties in this strange situation are obvious [it being the above mentioned twilight zone between the two criteria (6.12) and (6.13)]. The growing oscillation may lead to a chemical mixing of elements and surroundings and thus decrease, or eventually even destroy, the stabilizing gradient ∇_μ. But then again, it is not clear whether in such critical situations a local analysis suffices at all. The reaction of other layers of the star might provide enough damping to suppress the over-stability.

With these considerations it follows that we have to distinguish between *dynamical stability* and *vibrational stability*. The first applies to purely adiabatic behaviour of the moving mass, while the second takes heat exchange into account. A layer with a temperature gradient ∇ such that the Ledoux criterion is fulfilled but the Schwarzschild criterion is not, i.e.

$$\nabla_{\mathrm{ad}} < \nabla < \nabla_{\mathrm{ad}} + \frac{\varphi}{\delta}\nabla_\mu \ , \tag{6.20}$$

is dynamically stable but vibrationally unstable.

A dynamical instability grows on a timescale given by $(H_P/g)^{1/2}$, while in the case of a vibrational instability, the growth of amplitude is governed by the time it takes a mass element to adjust thermally to its surrounding, i.e. by the fraction of the total energy of the moving element lost by radiation per unit time. In the following we shall estimate this timescale τ_{adj}.

6.4 The Thermal Adjustment Time

Let us consider a mass element with $DT > 0$, i.e. one that will radiate into the surroundings. Superposed onto the radial energy flux $\boldsymbol{F}$, carrying energy from the stellar interior to the surface, there will be a local, non-radial flux $\boldsymbol{f}$, carrying the surplus energy of the element to its surroundings. According to (5.9) and (5.10), the absolute value f of the radiative flux from the element due to its excess temperature will be

$$f = \frac{4acT^3}{3\kappa\varrho}\left|\frac{\partial T}{\partial n}\right| \ , \tag{6.21}$$

where $\partial/\partial n$ indicates the differentiation perpendicular to the surface of the element. Suppose our element to be a roughly spherical "blob" with diameter d. We will approximate the temperature gradient in the normal direction by $\partial T/\partial n \approx 2DT/d$. The radiative loss λ per unit time from the whole surface V of the blob is then

$$\lambda = Sf = \frac{8acT^3}{3\kappa\varrho}DT\frac{S}{d} \ . \tag{6.22}$$

The quantity λ is a sort of "luminosity" of the blob, and it determines the rate by which the thermal energy of the blob of volume V changes:

$$\varrho V c_P \frac{\partial T}{\partial t} = -\lambda \ . \tag{6.23}$$

Here we can replace $\partial T/\partial t$ by $\partial(DT)/\partial t$, since the temperature of the (large) surroundings scarcely changes, owing to radiative losses of the blob. Furthermore, let $V/S \approx d/6$ (as for a sphere); then one obtains from (6.22) and (6.23) that

$$\frac{\partial(DT)}{\partial t} = -\frac{DT}{\tau_{\rm adj}} , \tag{6.24}$$

with the timescale for thermal adjustment

$$\tau_{\rm adj} = \frac{\kappa \varrho^2 c_P d^2}{16acT^3} = \frac{\varrho V c_P DT}{\lambda} . \tag{6.25}$$

The second equation follows from a comparison of (6.22)–(6.24). We see that $\tau_{\rm adj}$ is roughly the excess thermal energy divided by the luminosity, i.e. an equivalent to the Kelvin–Helmholtz timescale for a star (3.17). For sufficiently large elements that are far enough from a region of marginal stability, one has $\tau_{\rm adj} \gg 1/\omega_{\rm ad}$, which means that the radiative losses give only a small deviation from adiabatic oscillations, as discussed in Sect. 6.2.

6.5 Secular Instability

Even a small exchange of heat between a displaced mass element and its surroundings can lead to another kind of instability, which is called *thermal* or *secular instability*. We first discuss this qualitatively with an experiment which can easily be carried out with water and kitchen equipment.

In a glass jar containing cold fresh water we carefully pour over a layer of warm salty water. The salt increases the specific weight of the upper layer, but the warmth shall be enough to reduce (despite the salt content) its specific weight to below that of the underlying fresh water. If, owing to a perturbation, a blob of salty water is pushed downwards, buoyancy will push it back, i.e. the two layers are then *dynamically stable.*

But the buoyancy acts as a restoring force only as long as the element stays warm during its excursion into the cold layers. On the timescale by which it loses its excess temperature, the buoyancy diminishes and the element moves downwards because of its salt content. Indeed if one watches the two layers for some time, one can see (especially if the salty water is coloured) that small blobs of salty water slowly sink, a phenomenon called *salt-fingers.* It is an instability controlled by the heat leakage of the element. This is *secular instability.* It can not only occur in glass jars, but also in stars!

Consider a blob of stellar matter situated in surroundings of somewhat different, but homogeneous, composition, i.e. $D\mu \neq 0$, but $\nabla_\mu = 0$ (Such a situation can occur, for example, if two homogeneous layers of different compositions are above each other and a blob from one layer is displaced into the other.). The blob is supposed to be in mechanical equilibrium with its surroundings, i.e. $DP = D\varrho = 0$. This requires, however, a temperature difference according to (6.5):

$$\delta \frac{DT}{T} = \varphi \frac{D\mu}{\mu} \ . \tag{6.26}$$

For $D\mu > 0$, for example, the blob is hotter and therefore radiates towards the surroundings; the loss of energy under pressure balance ($DP = 0$) leads to an increased density and the blob sinks until again $D\varrho = 0$. Equation (6.26) is still valid and, since $D\mu$, is unchanged, $DT > 0$ as before, and so on. Obviously the blob will slowly sink (or rise for $D\mu < 0$) with a velocity v_μ such that DT always remains constant according to (6.26).

Owing to radiation, the temperature of the blob changes at the rate–DT/τ_{adj} [see (6.24)]. While sinking or rising it changes also because of the adiabatic compression (or expansion) that occurs as a result of the change of pressure, even in the absence of energy exchange. The rate of change of DT can then immediately be written as

$$\frac{1}{T}\frac{\partial}{\partial t}(DT) = \left(\nabla_{\mathrm{ad}} \frac{\partial \ln P}{\partial t} - \frac{DT}{T\tau_{\mathrm{adj}}}\right) - \nabla \frac{\partial \ln P}{\partial t} \ . \tag{6.27}$$

The rate of change of pressure is simply linked to the velocity v_μ by

$$\frac{\partial \ln P}{\partial t} = -\frac{v_\mu}{H_P} \ . \tag{6.28}$$

Using this and (6.26), together with the condition $\partial(DT)/\partial t = 0$ [which follows from (6.26), since $D\mu$ does not vary if the element moves in a chemically homogeneous region], we can solve (6.26)–(6.28) for the velocity and obtain

$$v_\mu = -\frac{H_P}{(\nabla_{\mathrm{ad}} - \nabla)\tau_{\mathrm{adj}}} \frac{\varphi}{\delta} \frac{D\mu}{\mu} \ . \tag{6.29}$$

In this case of thermal instability, therefore, the blob sinks ($v_\mu < 0$ for $D\mu > 0$) through a dynamically stable surrounding ($\nabla_{\mathrm{ad}} > 0$) with the adjustment timescale for radiative losses.

The idea of blobs finding themselves in strange surroundings ($D\mu > 0$) is not far-fetched. Secular instabilities of the kind discussed here can occur in stars, for example, of about one solar mass. After hydrogen has been transformed to helium in their cores, their central region is cooled by neutrinos, which take away energy without interacting with the stellar matter. The temperature in these stars, therefore, is highest somewhere off-centre and decreases towards the stellar surface as well as towards the centre. If, then, helium "burning" is ignited in the region of maximum temperature, the newly formed carbon is in a shell surrounding the central core (Sects. 33.4 and 33.5). This carbon-enriched shell has a higher molecular weight than the regions below: carbon "fingers" will grow and sink inwards. In later evolutionary phases, other nuclear reactions, such as neon burning, may ignite off-centre, and heavier fingers of material may sink.

6.6 The Stability of the Piston Model

Our piston model (Sects. 2.7 and 5.4) shows a stability behaviour in many respects similar to that of the blobs.

We start with the two equations that together with the equation of state describe the time dependence of the piston model. These are (2.34) and (5.39), where we assume for the sake of simplicity that $\varepsilon = \kappa = 0$. The equilibrium state is given by $T = T_s$ and $G^* = PA$.

In order to investigate the stability we denote the equilibrium values by the subscript "0" and make small perturbations of the form

$$\begin{aligned} h(t) &= h_0(1 + x\mathrm{e}^{\mathrm{i}\omega t}) \\ P(t) &= P_0(1 + p\mathrm{e}^{\mathrm{i}\omega t}) \\ T(t) &= T_0(1 + \vartheta\mathrm{e}^{\mathrm{i}\omega t}) \end{aligned} \tag{6.30}$$

with $|x|, |p|, |\vartheta| \ll 1$. We therefore neglect quadratic and higher-order expressions in these quantities.

From mass conservation ϱh = constant and from the perfect gas equation $P \sim \varrho T$, we obtain

$$p = \vartheta - x \ . \tag{6.31}$$

We now introduce (6.30) into (2.34) and obtain after linearization and using $G^* = PA$

$$M^* h_0 \omega^2 x + P_0 A p = 0 \ , \tag{6.32}$$

which with $g = P_0 A/M^*$ and with (6.31) can be replaced by

$$\left(\frac{\omega^2 h_0}{g} - 1 \right) x + \vartheta = 0 \ , \tag{6.33}$$

while the corresponding perturbation and linearization of (5.39) gives

$$\mathrm{i}\omega P_0 A h_0 x + (\mathrm{i}\omega c_v m^* T_0 + \chi T_0)\vartheta = 0 \ . \tag{6.34}$$

The two linear homogeneous equations (6.33) and (6.34) for x and ϑ can be solved if the determinant vanishes. This condition gives an algebraic equation of third order for the eigenvalue ω.

The problem becomes simple if we assume that the trapped gas changes adiabatically, i.e. if $\chi = 0$. Then (6.34), with $m^*/(Ah_0) = \varrho_0$ and with the perfect gas equation, yields

$$\frac{\Re}{\mu c_v} x + \vartheta = 0 \,, \tag{6.35}$$

and with $\Re/\mu = c_P - c_v$ (4.33) and $c_P/c_v = \gamma_{\rm ad}$ (4.37) it follows that

$$(\gamma_{\rm ad} - 1)x + \vartheta = 0 \,. \tag{6.36}$$

Setting the determinant of the equations (6.33) and (6.36) to zero gives the eigenvalue for the adiabatic motion:

$$\omega = \pm\omega_{\rm ad}, \qquad \omega_{\rm ad} = (\gamma_{\rm ad} g/h_0)^{1/2} \,. \tag{6.37}$$

Since ω is real, the adiabatic motion is an oscillation with frequency ω and constant amplitude. Therefore in the language of Sect. 6.1 our perfect gas piston model is *dynamically stable*. Note that $1/\omega_{\rm ad}$ is of the order of the hydrostatic timescale $\tau_{\rm hydr}$ defined in Sect. 2.7.

How do non-adiabatic effects change the picture? With the χ term in (6.34) we have, instead of (6.36),

$$(\gamma_{\rm ad} - 1)x + \left(1 + \frac{a}{{\rm i}\omega}\right)\vartheta = 0 \,, \tag{6.38}$$

with $a = \chi/(c_v m^*)$. Setting the determinant of (6.33) and (6.38) equal to zero now gives a *cubic equation* in ω. In general ω will be complex.

We assume χ to be small, so that the oscillation frequency must be close to the adiabatic value and we can put $\omega = \omega_{\rm ad} + \xi$, with $|\xi| \ll |\omega_{\rm ad}|$. If we neglect higher terms in ξ and χ, we find from the vanishing determinant of the system of homogeneous linear equations (6.33) and (6.38) and after some algebraic manipulation that

$${\rm i}\xi = -\frac{\gamma_{\rm ad} - 1}{2\gamma_{\rm ad}} \frac{\chi}{c_v m^*} = -\frac{\gamma_{\rm ad} - 1}{2\gamma_{\rm ad}} \frac{1}{\tau_{\rm adj}} < 0 \,, \tag{6.39}$$

where we have used (5.41). The (almost adiabatic) oscillation is therefore damped since the exponents of (6.30), ${\rm i}\omega = {\rm i}\omega_{\rm ad} + {\rm i}\xi$, have a negative real part that decreases the amplitude on a timescale $\tau_{\rm adj}$. The piston model with a leak is *vibrationally stable.*

The cubic equation for ω must have a third root, which we find easily by assuming that it describes an evolution so slow that the inertia term in (2.34) can be neglected (This has to be checked later.). Then (6.33) has to be replaced by

$$\vartheta - x = 0 \,, \tag{6.40}$$

which according to (6.31) is equivalent to $p = 0$. Indeed if the evolution is so slow that there is always hydrostatic equilibrium, the pressure is given by the (constant) weight of the piston. We then have from (6.34) and (6.40)

$$\mathrm{i}\omega = -\frac{\chi T}{P_0 A h_0 + c_v m^* T_0} = -\frac{\chi}{c_P m^*} = -\frac{1}{\gamma_{\mathrm{ad}} \tau_{\mathrm{adj}}} \,. \tag{6.41}$$

For the latter equation we have used the relation $P_0 A h_0 = \Re m^* T_0/\mu$ and (5.41). The third root gives an exponential decay in time of the initial perturbation, the timescale being comparable with τ_{adj}. If χ is sufficiently small and the evolution slow, the assumption that the inertia term is negligible is justified.

Our result (6.41) means that any deviation from thermal equilibrium $(T - T_{\mathrm{s}} \neq 0)$ vanishes within the thermal adjustment time, i.e. the thermally adjusted piston model for $\varepsilon = \kappa = 0$ is *secularly stable.* We see that it shows the same limiting cases for the stability problem (dynamical, vibrational, and secular stability) as the blobs. In Sect. 41.1 we will consider the influence on the stability of the piston model of the (here neglected) terms in (5.39) due to ε and κ.

To summarize: if the trapped gas is changing adiabatically, the piston model is dynamically stable. If there is a leak, the oscillations are damped and the gas vibrationally stable. If the thermal evolution is so slow that hydrostatic equilibrium is always achieved, it is secularly stable, if κ and ϵ are zero.