# Chapter 25
# Stability Considerations

Even the most beautiful stellar model is not worth anything if one does not know whether it is stable or not. Stability is discussed again and again throughout this book. Here we review the different types of stability considerations necessary for stars. We intend to make the basic mechanisms and concepts plausible rather than present the full formalism; the reader will find this, for example, in the review article by Ledoux (1958).

## 25.1 General Remarks

It is not easy to give a very general concept of stability that is applicable to all possible cases. Different definitions are discussed in La Salle and Lefschetz (1961). We may use, for example, the following: let the solution of a system of (time-dependent) differential equations be a set of functions $y_1(t)$, $y_2(t), \ldots$ which we comprise in the symbol $y(t)$. We define a "distance" between two such solutions $y^a(t)$, $y^b(t)$ by

$$||y^a(t) - y^b(t)|| := \sum_i \left[ \left( y_i^a(t) - y_i^b(t) \right)^2 \right]. \tag{25.1}$$

We then call the solution $y^a(t)$ stable at $t = t_0$ if for any $t_1 > t_0$ and for any small positive number $\delta$ there exists a small positive number $\delta$ such that any other solution $y^b(t)$ having the distance $||y^a(t_0) - y^b(t_0)|| < \delta$ at $t = t_0$ will keep a distance $||y^a(t_1) - y^b(t_1)|| < \varepsilon$.

This definition in plain words says that a solution is stable at a given point $t_0$ if all solutions that at $t = t_0$ are in its neighbourhood remain neighbouring solutions. The problems we are interested in can be reduced to first-order systems in time. Therefore the above definition of neighbouring solutions also guarantees neighbouring derivatives.

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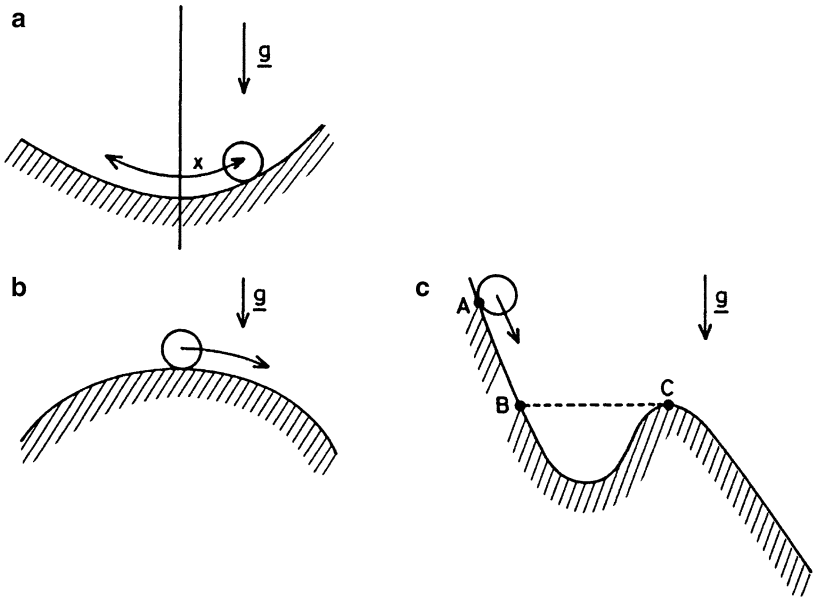


**Fig. 25.1** An example of stability in mechanics. A *ball* on a surface under the influence of gravity (**a**) in stable and (**b**) in unstable equilibrium. In (**c**) the motion starting at point $A$ is stable, but, starting with zero velocity at point $B$, the motion is unstable

One normally is familiar with stability problems in mechanics. We recall a few simple examples, the first being the freely rolling ball on a curved surface which is concave in the direction opposite to gravity (see Fig. 25.1a). One solution is that of equilibrium, where the ball rests in the lowest position. The initially neighbouring position is obtained by a small perturbation, say, by a slight horizontal displacement. The ball will then move about the equilibrium position, but it will never increase its distance above its initial value: the equilibrium position is stable and friction would merely restore the ball to its equilibrium position. In the case of a convex surface (see Fig. 25.1b) the equilibrium is unstable, since after a small displacement the ball will move further and further from the equilibrium position. While these examples deal with the stability of an equilibrium in which the solution is time independent, our general definition also concerns time-dependent solutions. The motion of a ball rolling on the surface in Fig. 25.1c can be stable or unstable. The motion is stable if it starts with zero velocity at a point $A$ *above* $B$ (non-periodic motion), or *below* $B$ (periodic motion). But a motion starting exactly at $B$ with zero velocity and ending at rest at $C$ is unstable: a slight perturbation of the initial conditions can either produce a periodic motion (the ball never overcomes the summit $C$) or cause the ball to roll beyond $C$ and never come back.

When considering the influence of friction, one may naïvely expect that it stabilizes an otherwise unstable motion, since it uses up energy. But the following example will show that friction can also produce instability.

We again consider the ball in the spherical bowl (Fig. 25.1a). But now we assume that the bowl is rotating with an angular velocity $\omega$ around a vertical axis through the minimum. Without friction no angular momentum can be transferred to the ball which therefore does not know anything of rotation and behaves as in the non-rotating case: the lowest position is stable. If there is friction, however, and the ball is "kicked" out of its lowest (equilibrium) position, it will take up angular momentum from the rotating bowl. For sufficiently large $\omega$ the ball goes to a new equilibrium position outside the axis around which it rotates with $\omega$ and where the tangential components of centrifugal and gravity forces balance each other. The lowest position has obviously become unstable by the inclusion of friction.

## 25.2 Stability of the Piston Model

Closer to stars than the above mechanical examples is the piston model introduced in Sect. 2.7, since it also incorporates thermal effects. We consider the stability of an equilibrium solution with a certain constant height $h$. Will a solution originating from a small displacement of the piston remain in its neighbourhood? This stability problem has already been discussed in Sect. 6.6, where we made approximations appropriate for the illustration of the stability of convective blobs. We now improve the model by adding some complications typical of stars.

### *25.2.1 Dynamical Stability*

In this case one assumes that there is no heat leakage, no nuclear energy generation, and no absorption, i.e. $\varepsilon = \kappa = \chi = 0$ in (5.39). Therefore the entropy of the gas remains constant during the displacement of the piston. In Sect. 6.6, we investigated the resulting (adiabatic) oscillations of the model around the equilibrium position, though with constant weight $G^*$ only. We now allow $G^*$ to vary with height $[G^* = G^*(h)]$ as we did in Sect. 3.2. This can be achieved, for instance, by putting the piston model into an inhomogeneous gravitational field. Then the equation of motion (2.34)

$$M^* \frac{d^2h}{dt^2} = -G^* + PA \tag{25.2}$$

with the perturbations (6.30) gives after linearization, instead of (6.32),

$$M^* h_0 \omega^2 x + P_0 A_p - G_h^* G_0^* x = 0. \tag{25.3}$$

Here $G_h^* := d \ln G^* / d \ln h (< 0)$, while $G_0^* = P_0 A = g_0 M^*$ is the equilibrium value of $G^*$ and $g_0$ is that of $g$. With the perturbed perfect gas equation (6.31) we find

$$\left[\frac{\omega^2 h_0}{g_0} - G_h^* - 1\right] x + \vartheta = 0. \tag{25.4}$$

This together with the adiabatic equation (6.36),

$$(\gamma_{\rm ad} - 1)x + \vartheta = 0, \tag{25.5}$$

gives for the eigenvalues of adiabatic oscillations $\omega = +\omega_{\rm ad}$ and $\omega = -\omega_{\rm ad}$ with

$$\omega_{\rm ad} = \left[(\gamma_{\rm ad} + G_h^*)\frac{g_0}{h_0}\right]^{1/2}, \tag{25.6}$$

which replaces (6.37). Recall that the perturbation changes with time as $e^{i\omega t}$. We see that $\omega_{\rm ad}$ is a real number only as long as $\gamma_{\rm ad} > -G_h^*$. In this case the small perturbation is followed by a periodic oscillation which remains small for all times. It is therefore stable in the sense of our definition of stability at the beginning of this paragraph. But if $\gamma_{\rm ad} < -G_h^*$, then $\omega_{\rm ad}$ is imaginary and one of the eigenvalues $\omega$ gives an amplitude growing exponentially in time: the equilibrium solution is unstable (We will see in Sect. 25.3.2 that for stars the analogue of $\gamma_{\rm ad} > -G_h^*$ is $\gamma_{\rm ad} > 4/3$.).

### *25.2.2 Inclusion of Non-adiabatic Effects*

We now drop the assumption of strict adiabaticity. Non-adiabatic changes were previously included in Sect. 5.4 (refer also to the last part of Sect. 6.6). The energy equation of the piston model (5.39) includes the non-adiabatic terms for nuclear generation $\varepsilon$, absorption $\kappa$, and heat leakage $\chi$. We consider $\varepsilon$ and $\kappa$ as functions of $P$ and $T$, while $\chi$ shall be constant. Let $F$ be the radiative flux through the gas. In the case of thermal equilibrium (vanishing time derivatives) we have [see (5.37)]

$$\varepsilon_0 m^* + \kappa_0 m^* F = \chi(T_0 - T_s), \tag{25.7}$$

where subscript 0 indicates the equilibrium and subscript s the surroundings. If we perturb this equilibrium according to (6.30), we find for the perturbations after linearization

$$\begin{aligned} &i\omega(c_v m^* T_0 \vartheta + P_0 A h_0 x) \\ &\quad = \varepsilon_0 m^*(p\,\varepsilon_P + \vartheta\varepsilon_T) + \kappa_0 m^* F(p\kappa_P + \vartheta\kappa_T) - \chi T_0 \vartheta, \end{aligned} \tag{25.8}$$

where the derivatives

$$\varepsilon_P = \left(\frac{\partial \ln \varepsilon}{\partial \ln P}\right)_T, \quad \varepsilon_T = \left(\frac{\partial \ln \varepsilon}{\partial \ln T}\right)_P,$$
$$\kappa_P = \left(\frac{\partial \ln \kappa}{\partial \ln P}\right)_T, \quad \kappa_T = \left(\frac{\partial \ln \kappa}{\partial \ln T}\right)_P \tag{25.9}$$

are taken at the values $P_0, T_0$.

The equation of motion (25.2) yielded (25.4) for which we now assume constant weight of the piston ($G_h^* = 0$, giving dynamical stability):

$$\left(\frac{\omega^2 h_0}{g_0} - 1\right) x + \vartheta = 0. \tag{25.10}$$

Since $\varrho \sim h^{-1}$, the equation of state for an ideal (or "perfect") gas gives (6.31)

$$p = \vartheta - x. \tag{25.11}$$

System (25.8), (25.10) and (25.11) comprises three linear homogeneous algebraic equations for the perturbations $p, \vartheta, x$. To find a solution it is necessary that the determinant of the coefficients vanishes:

$$\frac{h_0}{g_0} \mathrm{i} u_0 \omega^3 - \frac{h_0}{g_0}(e_P + e_T)\omega^2 - \frac{5}{3} u_0 \mathrm{i}\omega + e_T = 0 \tag{25.12}$$

with

$$e_P = \varepsilon_0 \varepsilon_P + \kappa_0 F \kappa_P, \quad e_T = \varepsilon_0 \varepsilon_T + \kappa_0 F \kappa_T - \frac{\chi T_0}{m^*}, \quad u_0 = c_v T_0, \tag{25.13}$$

where for the last relation we have assumed the gas to be ideal and monatomic (Note that $P_0 A h_0 / m^* = P_0/\varrho_0 = 2u_0/3$.). Equation (25.12) becomes one with *real* coefficients if instead of $\omega$ we use the eigenvalue $\sigma := \mathrm{i}\omega$,

$$\frac{h_0}{g_0} u_0 \sigma^3 - \frac{h_0}{g_0}(e_P + e_T)\sigma^2 + \frac{5}{3} u_0 \sigma - e_T = 0. \tag{25.14}$$

This is a third-order equation for the eigenvalue $\sigma$ (or $\omega$). While in the adiabatic case ($e_P = e_T = 0$) we obtained two solutions $\sigma = \pm\sigma_{\mathrm{ad}} = \pm i\omega_{\mathrm{ad}}$ (where $\omega_{\mathrm{ad}}$ was real), we now have *three* eigenvalues. If the non-adiabatic terms $e_P, e_T$ are small, we can expect that two (conjugate complex) eigenvalues lie near the adiabatic ones:

$$\sigma = \sigma_{\mathrm{r}} \pm \mathrm{i}\omega_{\mathrm{ad}}, \quad \omega_{\mathrm{ad}} = \left(\gamma_{\mathrm{ad}} \frac{g_0}{h_0}\right)^{1/2}, \tag{25.15}$$

where $\sigma_{\mathrm{r}}$ is real and $|\sigma_{\mathrm{r}}| \ll \omega_{\mathrm{ad}}$. While in the adiabatic case the oscillation was strictly periodic, the real part $\sigma_{\mathrm{r}}$ causes the amplitude of the oscillation to grow or decrease in time, depending on the sign of $\sigma_r$. Because of $|\sigma_r| \ll \omega_{ad}$ these changes take place over a time much longer than the oscillation period, actually on a scale corresponding to $\tau_{adj}$ in (5.41). This type of stability behaviour is called the *vibrational stability* (compare Sect. 6.6). If the oscillation grows in time, the solution leaves the neighbourhood of equilibrium, which therefore is unstable.

We now turn to the third root of (25.12) or (25.14), which occurs necessarily with the dissipative terms $e_P$, $e_T$. Instead of solving the third-order equation (25.14), we will follow some heuristic arguments. The addition of non-adiabatic terms has changed the rapid oscillations only to the extent that their amplitude varies on long timescales (of the order of $\sigma_r^{-1}$). We now look for the existence of a third solution changing with this long timescale only. Then the inertia terms can be neglected and, consequently, the terms with $\sigma^3$ and $\sigma^2$ disappear in (25.14). The solution of (25.14) for this so-called *secular stability* problem is

$$\sigma = \sigma_{sec} = \mathrm{i}\omega_{sec} = \frac{3}{5}\frac{e_T}{u_0}. \tag{25.16}$$

For sufficiently small non-adiabaticity $e_T$, we can achieve $|\sigma_{sec}| \ll \omega_{ad}$, and neglecting the $\sigma^2$ and $\sigma^3$ terms in (25.14) was justified. If $\sigma_{sec} < 0$, any perturbation will decay within a kind of thermal adjustment time $\tau_{adj} \approx \sigma_{sec}^{-1}$ and the equilibrium is secularly stable. But if $\sigma_{sec} > 0$, then it will grow on that timescale (independently of vibrational stability): The equilibrium is secularly unstable.

We have now found the three well-known types of stability behaviour: dynamical, vibrational, and secular stability . This classification is possible since $|\omega_{ad}| \gg |\omega_{sec}|$, which is equivalent to saying that $\tau_{hydr} \ll \tau_{adj}$. From one type of stability one cannot draw any conclusions about the behaviour of another type, for example, a dynamically stable model can still be vibrationally or secularly unstable. If the model were dynamically unstable, the other instabilities would be of no interest since the model would move out of equilibrium long before any other instability can develop.

We will find more or less the same behaviour in stars where also $\tau_{hydr} \ll \tau_{adj} \approx \tau_{KH}$. However, there we cannot solve the eigenvalue problem analytically any more. This is the reason why we dwelt in such length on the stability of the piston model.

## 25.3 Stellar Stability

For the problem of *stellar* stability a very general definition, like that given at the beginning of Sect. 25.1, has to be taken with care. For example, a star may be stable in one phase (e.g. on the main sequence) and later on become unstable (e.g. in the Cepheid phase). At any stage of evolution the solution (the stellar model) is obtained for certain parameters, for instance, a certain chemical composition or a certain distribution of entropy. It is reasonable to ask whether this solution is stable in the following sense: Does a small perturbation decay rapidly compared to the change of the parameters of the model (e.g. its chemical composition)? Then we would call the model stable. Therefore, the question of the Cepheid stability is irrelevant for the stability of its main-sequence progenitor since the chemical composition is different. The solution for a certain phase of evolution, in general, is obtained by solving approximate equations. For example, complete equilibrium may be assumed in the case of the main sequence, while only the inertia terms are dropped for the evolution through the Cepheid phase. If such approximate models approach an instability in the run of their evolution, the neglected time derivatives become important and have to be taken into account. In general, then, the solution obtained from better approximations tells us in which direction the evolution really goes.

### *25.3.1 Perturbation Equations*

We want to investigate the stability of a stellar model in complete equilibrium for given input parameters $M$ and chemical composition. Let the model be described by $r_0(m)$, $P_0(m)$, $T_0(m)$, $l_0(m)$, which solve the time-independent stellar structure equations. We test its stability by investigating how a neighbouring (perturbed) solution evolves in time. We here restrict ourselves to spherically symmetric perturbations which depend on $m$ and $t$ in such a way that the perturbed variables become

$$\begin{aligned} r(m,t) &= r_0(m)\left[1 + x(m)e^{\mathrm{i}\omega t}\right], \\ P(m,t) &= P_0(m)\left[1 + p(m)e^{\mathrm{i}\omega t}\right], \\ T(m,t) &= T_0(m)\left[1 + \vartheta(m)e^{\mathrm{i}\omega t}\right], \\ l(m,t) &= l_0(m)\left[1 + \lambda(m)e^{\mathrm{i}\omega t}\right], \end{aligned} \tag{25.17}$$

where the absolute values of $x$, $p$, $\vartheta$, and $\lambda$ are $\ll 1$. These variables have to fulfill the time-dependent equations (10.1)–(10.4). As an example let us introduce (25.17) into the equation of motion (10.2). If we linearize with respect to $p$ and $x$, this becomes

$$\begin{aligned} &P_0'\left(1 + pe^{\mathrm{i}\omega t}\right) + P_0 p' e^{\mathrm{i}\omega t} \\ &\quad = -\frac{Gm}{4\pi r_0^4}\left(1 - 4x\mathrm{e}^{\mathrm{i}\omega t}\right) + \frac{\omega^2}{4\pi r_0} x\mathrm{e}^{\mathrm{i}\omega t}, \end{aligned} \tag{25.18}$$

where primes indicate derivatives with respect to $m$. Since $P_0, r_0$ obey (10.2), we have $P_0' = -Gm/(4\pi r_0^4)$: The time-independent terms in (25.18) cancel each other, the exponentials drop out, and we are left with (25.19). By a similar procedure, we find for the case of a radiative layer and an equation of state of the form $\varrho \sim P^\alpha T^{-\delta}$

from (10.1), (10.3), (10.4) the equations (25.20)–(25.22):

$$p' = -\frac{P_0'}{P_0}\left[p + \left(4 + \frac{r_0^3}{Gm}\omega^2\right)x\right], \tag{25.19}$$

$$x' = -\frac{1}{4\pi r_0^3 \varrho_0}(3x + \alpha p - \delta\vartheta). \tag{25.20}$$

$$\lambda' = -\frac{\varepsilon_0}{l_0}(\lambda - \varepsilon_P p - \varepsilon_T \vartheta) - \mathrm{i}\omega\frac{P_0\delta}{l_0\varrho_0}\left(\frac{\vartheta}{\nabla_{\mathrm{ad}}} - p\right), \tag{25.21}$$

$$\vartheta' = \frac{P_0'}{P_0}\nabla_{\mathrm{rad}}[\kappa_P p + (\kappa_T - 4)\vartheta + \lambda - 4x]. \tag{25.22}$$

Equations (25.19)–(25.22) are four linear homogeneous differential equations of first order for the variables $p$, $\vartheta$, $x$, $\lambda$ which have to obey certain boundary conditions corresponding to those of the unperturbed solutions. They have to be regular in the centre and to be fitted to an atmosphere. We will deal with the boundary conditions in Chaps. 40 and 41, where they are shown to be equivalent to four linear homogeneous equations. Therefore, solutions exist only for certain *eigenvalues* of $\omega$, which have to be found numerically. There exists an infinite number of eigenvalues for which the system can be solved. For each eigenvalue $\omega^*$ one obtains a set of *eigenfunctions* $p^*(m)$, $\vartheta^*(m)$, $x^*(m)$, $\lambda^*(m)$.

The term with $\omega^2(\sim \ddot{r})$ in (25.19) comes from the inertial terms in the equation of motion, while in (25.21), the term with $\mathrm{i}\omega(\sim \dot{P}, \dot{T})$ is due to the time derivatives in the energy equation. The two corresponding timescales are $\tau_{\mathrm{hydr}}$ and $\tau_{\mathrm{adj}} = \tau_{\mathrm{KH}}$. Since $\tau_{\mathrm{hydr}} \ll \tau_{\mathrm{KH}}$, we have a situation similar to that described for the piston model in Sect. 25.2. Correspondingly, in general, we can speak of dynamical, vibrational, and secular stability.

There are, however, more complicated cases where this classification of stability behaviour is not possible. For example, the relevant thermal timescale may not be that of the whole star but a much shorter one for a small subregion. If the characteristic wavelength of a thermal perturbation is short enough, the corresponding adjustment time can become comparable or shorter than $\tau_{\mathrm{hydr}}$ (of the whole star). Another example is the case of a dynamically stable model which evolves in such a way that it approaches marginal stability ($\omega_{\mathrm{ad}} \to 0$). Then the oscillations become so slow that they certainly will not be adiabatic anymore: $1/\omega_{\mathrm{ad}} \gg \tau_{\mathrm{KH}}$ (although $\tau_{\mathrm{hydr}} \ll \tau_{\mathrm{KH}}$ still).

### *25.3.2 Dynamical Stability*

Since in Chap. 40 we will treat this problem thoroughly, we merely present some general results here. Instead of solving all four equations (25.19)–(25.22), one can consider oscillations taking place on the timescale $\tau_{\mathrm{hydr}}$. Since $\tau_{\mathrm{hydr}} \ll \tau_{\mathrm{adj}}$, the temperature of the matter changes almost adiabatically. Instead of solving (25.21) and (25.22) one just replaces $\vartheta$ by $p\nabla_{\rm ad}$ in (25.20). Therefore (25.19) and (25.20) present two equations for $p$ and $x$ with the eigenvalue $\omega^2$. As we will see in Chap. 40 the eigenvalue problem is self-adjoint. Then there exists an infinite series of eigenvalues $\omega_n^2$ which are real. ($\omega_n$ is either real or purely imaginary). Therefore, they either correspond to periodic oscillations ($\omega_n^2 > 0$) or exponentially decreasing/increasing solutions ($\omega_n^2 < 0$). The same behaviour was found for the adiabatic case of the piston model. But now, with an infinite number of eigenvalues, stability demands that for *all* eigenvalues $\omega_n^2 > 0$, while even a *single* eigenvalue with $\omega_n^2 < 0$ is sufficient for instability.

How a star behaves after it is adiabatically compressed or expanded depends on the numerical value of $\gamma_{\rm ad}$. This can be most easily seen in the case of homologous changes. Let us consider a concentric sphere $r = r(m)$ in a star of hydrostatic equilibrium.

The pressure there is equal to the weight of the layers above a unit area of the sphere, as shown by integrating the hydrostatic equation:

$$P = \int_m^M \frac{Gm}{4\pi r^4} dm. \tag{25.23}$$

We now compress the star artificially and assume the compression to be adiabatic and homologous. In general, after this procedure, the star will no longer be in hydrostatic equilibrium.

If a prime indicates values after the compression, then homology demands that the right-hand side of (25.23) varies like $(R'/R)^{-4}$ [cf. (20.37)] where $R$ is the stellar radius, while adiabaticity *and* homology demand that the left-hand side varies as

$$(\varrho'/\varrho)^{\gamma_{\rm ad}} = (R'/R)^{-3\gamma_{\rm ad}} \tag{25.24}$$

according to (20.9). Therefore, if $\gamma_{\rm ad} = 4/3$, the pressure on the left-hand side of (25.23) increases stronger with the contraction than the weight on the right: The resulting force is directed outwards, and the star will move back towards equilibrium: it is dynamically stable.

For $\gamma_{\rm ad} < 4/3$ the weight increases stronger than the pressure and the star would collapse after the initial compression (dynamical instability). For $\gamma_{\rm ad} = 4/3$, the compression leads again to hydrostatic equilibrium: One has neutral equilibrium. The condition $\gamma_{\rm ad} > 4/3$ corresponds to the dynamical stability condition $\gamma_{\rm ad} > -G_h^*$ for the piston model (Sect. 25.2.1).

In Chap. 40 we will see that $\gamma_{\rm ad} = 4/3$ is also a critical value for non-homologous perturbations. If $\gamma_{\rm ad}$ is not constant within a star, for instance, because of ionization, then marginal stability occurs if a certain mean value of $\gamma_{\rm ad}$ over the star reaches the critical value 4/3.

It should be noted that radiation pressure can bring $\gamma_{\rm ad}$ near the critical value 4/3 (see Sect. 13.2). This is the reason why supermassive stars are in indifferent equilibrium, i.e. they are marginally stable (see Sect. 19.10).

The critical value 4/3 depends strongly on spherical symmetry and Newtonian gravitation. The 4 in the numerator comes from the fact that the weight of the envelope in Newtonian mechanics varies as $\sim r^{-2}$ and has to be distributed over the surface of our sphere, giving another $r^{-2}$. The 3 in the denominator comes from the $r^3$ in the formula for the volume of a sphere. Therefore, effects of general relativity change the critical value (see Sect. 38.2) of $\gamma_{ad}$ and make the models less stable. Since we have assumed spherical symmetry in deriving the critical value of $\gamma_{ad}$, rotation changes it, too. It can decrease the critical value of $\gamma_{ad}$ and make the models more stable.

### 25.3.3 Non-adiabatic Effects

The inclusion of non-adiabatic effects in a dynamically stable model brings us to the question of its vibrational and secular stability (A dynamical instability makes a perturbation grow so rapidly that any other possible instability of vibrational or secular type is irrelevant because of their much longer timescales.). Vibrational stability means an oscillation with nearly adiabatic frequency but with slowly decreasing (stability) or increasing amplitude (instability). Such oscillations describe the behaviour of pulsating stars and therefore are treated in detail in Chap. 41.

Secular (or thermal) stability is governed by thermal relaxation processes. In general these proceed on timescales long compared to $\tau_{hydr}$ and, therefore, the inertia terms in the equation of motion can be dropped. This means that the term $\sim \omega^2$ in (25.19) can be omitted. Equations (25.19)–(25.22) together with proper boundary conditions can then be solved, yielding an infinite number of secular eigenvalues $\omega_{sec}$. Normally they are purely imaginary (as in the case of the piston model). This is what one expects from a thermal relaxation process, such as in the problem of diffusion of heat. It is therefore all the more surprising that in certain cases a few complex eigenvalues occur (Aizenman and Perdang 1971). The oscillatory behaviour here comes from heat flowing back and forth between different regions in the star (Obviously this could not occur in the single layer of the piston model.). If instead of $\omega$ we again use $\sigma := \mathrm{i}\omega$, the system (25.19)–(25.22) has real coefficients. Therefore the eigenvalues $\sigma$, if complex, appear in conjugate complex pairs. Again, the sign of the real part of $\sigma$ (the imaginary part of $\omega$) distinguishes between secular stability or instability.

The most important application of the secular problem to stellar evolution concerns the question whether a nuclear burning is stable or not. Secular instability in degenerate regions leads to the flash phenomenon, while in thin (nondegenerate) shell sources, it results in quasiperiodic thermal pulses.

In order to make the secular stability of a central burning plausible, we treat a simple model of the central region, assuming homologous changes of the rest of the star. Other secular instabilities which occur in burning shells or which are due to nonspherical perturbations will be discussed later (Sects. 33.5 and 34.2).

### 25.3.4 The Gravothermal Specific Heat

Let us consider a small sphere of radius $r_s$ and mass $m_s$ around the centre of a star in hydrostatic equilibrium. If the sphere is sufficiently small, then $P$ at $r_s$ and the mean density in the sphere are good approximations for the central values $P_c$, $\varrho_c$. Suppose that, as a reaction to the addition of a small amount of heat to the central sphere, the whole star is slightly expanding and let the expansion be homologous. Then any mass shell of radius $r$ after expansion has the radius $r + dr = r(1 + x)$, where $x$ is constant for all mass shells. If after the expansion the pressure in the sphere is $P_c + dP_c$, then, similarly to (20.34) and (20.37), the resulting changes of $\varrho_c$ and $P_c$ are

$$\frac{d\varrho_c}{\varrho_c} = -3x, \quad p_c := \frac{dP_c}{P_c} = -4x. \tag{25.25}$$

We now write the equation of state in differential form,

$$\frac{d\varrho_c}{\varrho_c} = \alpha p_c - \delta\vartheta_c, \tag{25.26}$$

($\vartheta$c $:= dT_c/T_c$) as in (6.5) but here with constant chemical composition. Elimination of $d\varrho_c/\varrho_c$ and of $x$ from (25.25) and (25.26) gives

$$p_c = \frac{4\delta}{4\alpha - 3}\vartheta_c. \tag{25.27}$$

According to the first law of thermodynamics the heat $dq$ per mass unit added to the central sphere is

$$dq = du + P\,dv = c_P T_c(\vartheta_c - \nabla_{ad} p_c) := c^* T_c \vartheta_c, \tag{25.28}$$

where we have used (4.18), (4.21) and where according to (25.27)

$$c^* = c_P\left(1 - \nabla_{ad}\frac{4\delta}{4\alpha - 3}\right). \tag{25.29}$$

This quantity has the dimension of a specific heat per mass unit. Indeed, $dT = dq/c^*$ gives the temperature variation in the central sphere if the heat $dq$ is added. In thermodynamics we are used to defining specific heats with some mechanical boundary conditions, for example, $c_P$ and $c_v$. For $c^*$ the mechanical condition is that the gas pressure is kept in equilibrium with the weight of all the layers with $r > r_s$. This $c^*$ is called the *gravothermal specific heat.*

For an ideal monatomic gas ($a = \delta = 1$, $\nabla_{ad} = 2/5$), as we have approximately in the central region of the Sun, one finds from (25.29) that $c^* < 0$. This is fortunate, since if in the Sun the nuclear energy generation is accidentally enhanced for a moment ($dq > 0$), then $dT < 0$, the region cools, thereby reducing the overproduction of energy immediately. Therefore the negative specific heat acts as a stabilizer. At first glance it seems as if the decrease of temperature after an injection of heat contradicts energy conservation. But one has also to take into account the $Pdv$ work done by the central sphere. Indeed, while the centre cools ($\vartheta_c < 0$), the whole star expands, since elimination of $p_c$ and $d\varrho_c/\varrho_c$ from (25.25) and (25.26) gives $x = -\delta\vartheta_c/(4\alpha - 3)$, which in the case $\alpha = \delta = 1$ yields $x > 0$. It turns out that, if heat is added to the central sphere, more energy is used up by the expansion, and therefore some must be taken from the internal energy. This behaviour is essentially connected with the virial theorem (see Sect. 3.1). A corresponding property can be found for the piston model by assuming a variable weight $G^*$ of the piston as in Sect. 3.2.

For a nonrelativistic degenerate gas ($\delta \to 0, \alpha \to 3/5$) equation (25.29) gives $c^* > 0$: the addition of energy to the central sphere heats up the matter, which can lead to thermal runaway.

### 25.3.5 Secular Stability Behaviour of Nuclear Burning

Having derived a handy expression for $dq$, we shall now use it in the energy balance of the central sphere considered in Sect. 25.3.4. Energy is released in the sphere by nuclear reactions and transported out of it by radiation (we assume here that the central region is not convective). In the steady state gains and losses compensate each other. Let $\varepsilon$ be the mean energy generation rate, and $l_s$ the energy per unit time which leaves the sphere; then $\varepsilon m_s - l_s = 0$. Now the equilibrium is supposed to be perturbed on a timescale $\tau$, such that $\tau$ is much larger than $\tau_{hydr}$ but short compared to the thermal adjustment time of the sphere. Then, while hydrostatic equilibrium is maintained, the thermal balance is perturbed.

For the perturbed state the energy balance is

$$m_s d\varepsilon - dl_s = m_s \frac{dq}{dt} \equiv m_s c^* \frac{dT_c}{dt}. \tag{25.30}$$

Here, $dq$ is the heat gained per mass unit, which is expressed by $c^* dT_c$ according to (25.28).

If we now perturb the equation for radiative heat transfer (5.12),

$$l \sim \frac{T^3 r^4}{\kappa} \frac{dT}{dm}, \tag{25.31}$$

we obtain for $l_s$

$$\frac{dl_s}{l_s} = 4\vartheta_c + 4x - \kappa_P p_c - \kappa_T \vartheta_c. \tag{25.32}$$

For the perturbation of $dT/dm$ we have made use of the fact that for homology $\vartheta = dT/T =$ constant and therefore $d(dT/dm) = d(T\vartheta)/dm = \vartheta dT/dm$.

From (25.25), (25.27) and (25.32) it follows that

$$\frac{dl_s}{l_s} = \left[4 - \kappa_T - \frac{4\delta}{4\alpha - 3}(1 + \kappa_P)\right]\vartheta_c. \qquad (25.33)$$

This, introduced into (25.30), gives

$$\begin{aligned}\frac{m_s}{l_s}\frac{dq}{dt} &= (m_s d\varepsilon - dl_s)l_s = \varepsilon_T \vartheta_c + \varepsilon_P p_c - \frac{dl_s}{l_s} \\ &= \left[(\varepsilon_T + \kappa_T - 4) + \frac{4\delta}{4\alpha - 3}(\varepsilon_P + \kappa_P + 1)\right]\vartheta_c, \end{aligned} \qquad (25.34)$$

where we have made use of $l_s = \varepsilon m_s$ and of (25.27). Then with (25.30) we find

$$\frac{m_s c^* T_c}{l_s}\frac{d\vartheta_c}{dt} = \left[(\varepsilon_T + \kappa_T - 4) + \frac{4\delta}{4\alpha - 3}(1 + \varepsilon_P + \kappa_P)\right]\vartheta_c. \qquad (25.35)$$

The sign of the bracket tells us whether for $dT_c > 0$ the additional energy production exceeds the additional energy loss of the sphere ($[\ldots] > 0$). The sign of $c^*$ tells us whether in this case the sphere heats up ($c^* > 0$) or cools ($c^* < 0$). Normally $\varepsilon_T$ is the leading term in the bracket, so that indeed $[\ldots] > 0$. We first assume an ideal gas ($\alpha = \delta = 1, c^* < 0$) and obtain

$$\frac{m_s c^* T_c}{l_s}\frac{d\vartheta_c}{dt} = [\varepsilon_T + \kappa_T + 4(\varepsilon_P + \kappa_P)]\vartheta_c. \qquad (25.36)$$

Since $c^* < 0$, one finds from (25.36) that $(d\vartheta_c/dt)/\vartheta_c < 0$, meaning that the perturbation $dT_c$ decays and the equilibrium is stable if

$$\varepsilon_T + \kappa_T + 4(\varepsilon_P + \kappa_P) > 0. \qquad (25.37)$$

This criterion is normally fulfilled. The only "dangerous" term is $\kappa_T$, which can be as low as to $-4.5$ for Kramers opacity. But then, even $\varepsilon_T = 5$ for the *pp* chain suffices to fulfill (25.37), since the other terms are positive.

Any temperature increase $dT_c > 0$ would cause a large additional energy overproduction $\varepsilon_0 \varepsilon_T dT_c/T_c$. But since the gravothermal heat capacity $c^* < 0$, the sphere reacts with $dT_c < 0$, and this cooling brings energy production back to normal. We then can say that the burning in a sphere of ideal gas proceeds in a stable manner, the negative gravothermal specific heat acts like a thermostat. This, for example, is the case in the Sun.

We go back to (25.35) for the general equation of state. Since normally $\varepsilon_T$ dominates the other terms in the square bracket (in some case $\varepsilon_T > 20$), we neglect them for simplicity. Then (25.35) can be written

$$\frac{d\vartheta_c}{dt} = \frac{l_s \varepsilon_T}{m_s T_c c^*} \vartheta_c := \frac{1}{D} \vartheta_c. \tag{25.38}$$

Obviously $D < 0$ indicates stability, $D > 0$ instability. Since $\varepsilon_T > 0$ and, for an ideal gas, $c^* < 0$, the quantity $D$ is negative: The nuclear burning is stable.

For a nonrelativistic degenerate gas we have $\delta = 0$, $\alpha = 3/5$. Therefore, $c^* > 0$ and $D > 0$: Any nuclear burning with a sufficiently strong temperature dependence will then be unstable. This is the reason, for instance, why in the central regions of a white dwarf there can be no strong nuclear energy source [as first shown by Mestel (1952)]; the star would be destroyed by thermal runaway, or at least heat up until it was not degenerate and then expand. Of course, then it would no longer be a white dwarf. The same instability is also responsible for the phenomenon of the so-called flash (compare Sect. 33.4) which occurs if a new nuclear burning starts in a degenerate region. Note that the appearance of $4\alpha - 3$ in the denominator in several equations, including (25.29) for $c^*$ and (25.35), does not become serious even if $\alpha \to 3/4$ for partial nonrelativistic degeneracy, since the singularity can be removed from the equation which one obtains if $c^*$ is inserted in (19.35) by multiplication with $4\alpha - 3$.

From (25.38) one can draw another conclusion. Let us assume that in the central region of a star there is no nuclear burning but that energy losses by neutrinos (Sect. 18.7) are important. The nuclear energy production in the star may take place in a concentric shell of finite radius. Part of this energy flows outwards, providing the star's luminosity, while part of it flows from the shell inwards towards the centre where it goes into neutrinos. The maximum temperature then is in the shell and not in the stellar centre. In Sect. 33.5 we shall see that this really can be the case in models of evolved stars. If we now again look at (25.38), we have to be aware that $l_s < 0$. If $\varepsilon_T > 0$, as it is for neutrino losses (see Sect. 18.7), all the above conclusions are contradicted because of the different sign of $l_s$: The equilibrium is stable if $c^* > 0$, i.e. for degeneracy, but unstable if $c^* < 0$, which is the case for an ideal gas.

All our discussions here were based on the assumption of homologous changes in the stellar model. Although stars clearly never change precisely in such a simple way, it turns out that the above conclusions describe qualitatively correctly the secular stability behaviour of stars. Deviations from homology only influence the factors [e.g. in the bracket in (25.36)], thus modifying the exact position of the border between secular stability and instability.