Chapter 10
The Differential Equations of Stellar Evolution

10.1 The Full Set of Equations

Collecting the basic differential equations for a spherically symmetric star in hydrostatic equilibrium derived in Chap. 1, we are then led by (1.6), (2.16), (4.47), (4.48), (7.32), and (8.4) to

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \varrho} , \tag{10.1}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} , \tag{10.2}$$

$$\frac{\partial l}{\partial m} = \varepsilon_{\mathrm{n}} - \varepsilon_\nu - c_P \frac{\partial T}{\partial t} + \frac{\delta}{\varrho}\frac{\partial P}{\partial t} , \tag{10.3}$$

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla , \tag{10.4}$$

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\varrho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right) , \quad i = 1, \ldots, I . \tag{10.5}$$

Equation (10.2) has an additional term $-\partial^2 r/\partial t^2 (4\pi r^2)^{-1}$ in case the assumption of hydrostatic equilibrium is not fulfilled. In (10.5) we have a set of I equations (one of which may be replaced by the normalization $\sum_i X_i = 1$) for the change of the mass fractions X_i of the relevant nuclei $i = 1, \ldots, I$ having masses m_i. Additional formulae regulate the mixing of the composition in convective regions, (8.27) or (8.28), or in case of diffusive processes (8.25). In (10.3), $\delta \equiv -(\partial \ln \varrho/\partial \ln T)_P$, and in (10.4), $\nabla \equiv d \ln T/d \ln P$. If the energy transport is due to radiation (and conduction), then ∇ has to be replaced by ∇_{rad}, which is given by (5.28):

R. Kippenhahn et al., *Stellar Structure and Evolution*, Astronomy and Astrophysics Library, DOI 10.1007/978-3-642-30304-3_10,

$$\nabla = \nabla_{\text{rad}} = \frac{3}{16\pi acG} \frac{\kappa l P}{m T^4} . \tag{10.6}$$

If the energy is carried by convection, then ∇ in (10.4) has to be replaced by a value obtained from a proper theory of convection; this may be ∇_{ad} in the deep interior or obtained from a solution of the cubic equation (7.26) for superadiabatic convection in the outer layers. Note that the expression on the right-hand side of (10.4) assumes hydrostatic equilibrium. This does not matter in the case of radiative transport, since the local adjustment time of the radiation field is very short, and the convection theory of Chap. 7 is valid only for stars in hydrostatic equilibrium. Otherwise another convection theory valid in rapidly changing regions would have to be used. Additional criteria such as (6.12) and (6.13) distinguish between radiative and convective transport.

In the system (10.1)–(10.5) one can distinguish certain subsystems, i.e. (10.1) and (10.2) give the mechanical part, being coupled to the thermo-energetic part only through the density ϱ–which usually also depends on T. If for some reason or other this dependence of ϱ on T is not present (or can be eliminated), then (10.1) and (10.2) can be solved regardless of the other equations to give the mechanical structure $r(m)$, $P(m)$. Equations (10.5) may be regarded as the chemical part. Under normal conditions (τ_{n} much larger than the other timescales; see Sect. 10.2) they can be decoupled from the spatial parts (10.1)–(10.4), which describe the structure of the star for a given time and given composition $X_i(m)$. This would be questionable, of course, if the chemical composition changed as rapidly as the other variables, and for changes of $X_i(m)$ more rapid than those of P, T, one would rather assume to have an "equilibrium composition" $X_i(P, T)$ at any time (see Chap. 36).

Equations (10.1)–(10.5) contain functions which describe properties of the stellar material such as $\varrho, \varepsilon_{\text{n}}, \varepsilon_\nu, \kappa, c_P, \nabla_{\text{ad}}, \delta$ and the reaction rates r_{ij}. We shall deal with these functions in Part III. Meanwhile we assume them to be known functions of P, T and the chemical composition described by the functions $X_i(m, t)$. We therefore have an equation of state

$$\varrho = \varrho(P, T, X_i) \tag{10.7}$$

and equations for the other thermodynamic properties of the stellar matter

$$c_P = c_P(P, T, X_i) , \tag{10.8}$$

$$\delta = \delta(P, T, X_i) , \tag{10.9}$$

$$\nabla_{\text{ad}} = \nabla_{\text{ad}}(P, T, X_i) , \tag{10.10}$$

as well as the Rosseland mean of the opacity (including conduction)

$$\kappa = \kappa(P, T, X_i) , \tag{10.11}$$

and the nuclear reaction rates and the energy production and energy loss via neutrinos:

$$r_{jk} = r_{jk}(P, T, X_i) \ , \tag{10.12}$$

$$\varepsilon_{\rm n} = \varepsilon_{\rm n}(P, T, X_i) \ , \tag{10.13}$$

$$\varepsilon_\nu = \varepsilon_\nu(P, T, X_i) \ . \tag{10.14}$$

In these equations, the arguments X_i stand for *all* types of nuclei ($i = 1, \ldots, I$).

It is now time to count the equations and the unknown variables. We consider the material functions on the right-hand sides of (10.1)–(10.5) to be replaced with the help of the corresponding equations (10.7)–(10.14), i.e. by functions of P, T, X_i. For I different types of nuclei being affected by reactions, (10.1)–(10.5) form a set of $4 + I$ differential equations for the $4 + I$ variables $r, P, T, l, X_1, \ldots, X_I$. We therefore have the same number of equations and unknown variables.

The independent variables are m and t. If we assume that the total mass of the star does not change in time (i.e. no gain nor loss of mass) and if we define the time at which evolution starts as $t = t_0$, then we are looking for solutions in the intervals

$$0 \le m \le M, \quad t \ge t_0 \ . \tag{10.15}$$

In the full problem we are confronted with a set of non-linear, partial differential equations. As usual, physically relevant solutions require the specification of boundary conditions (here at $m = 0, m = M$) and of initial values [e.g. $X_i(m, t_0)$]. The boundary conditions will be dealt with in Chap. 11. In order to see more clearly which initial values have to be specified we replace the two terms with time derivatives of P and T in (10.3) by one term containing the change of the entropy $s, -T\partial s/\partial t$, according to (4.47). Obviously the full problem requires specification of the functions $r(m, t_0), \dot{r}(m, t_0), s(m, t_0)$, and $X_i(m, t_0)$.

After proper initial values and boundary conditions are specified, together with the stellar mass M, the problem is to find solutions of the basic equations, i.e. the unknown variables as functions of m and t. A solution $r(m), P(m), \ldots, X_i(m)$ for a given time t in the interval [0, M] is called a *stellar model.* But before we discuss in more detail how solutions of our set of differential equations can be obtained, we first discuss simplifications of the full problem.

10.2 Timescales and Simplifications

There are three types of time derivatives in our set of equations. To each of them belongs a certain characteristic timescale. In Sect. 2.4 the term $(\partial^2 r/\partial t^2)/4\pi r^2$ in (2.16)–the dynamical version of (10.2)–was used to derive $\tau_{\rm hydr}$. From the time derivatives in (10.3) we have derived $\tau_{\rm KH}$ in Sect. 3.3. The time derivatives in (10.5) define chemical timescales τ_{Xi} which were shown to be equivalent to τ_{n} [see (4.59)] at the end of Sect. 8.2.1.

In Sect. 2.4 we showed that the inertia term in (10.2) can be neglected if the evolution is slow compared to τ_{hydr}. Therefore, if the evolution of a star is governed by thermal adjustment or by nuclear reactions ($\tau_{\mathrm{KH}} \gg \tau_{\mathrm{hydr}}$ and $\tau_{\mathrm{n}} \gg \tau_{\mathrm{hydr}}$), the equation of hydrostatic equilibrium (10.2) is appropriate. The star then evolves along a sequence of states of hydrostatic equilibrium. As initial conditions, the functions $s(m, t_0)$ and $X_i(m, t_0)$ have to be specified in this approximation.

If the star evolves on the timescale $\tau_{\mathrm{n}} \gg \tau_{\mathrm{KH}}$, then according to the discussion in Sect. 4.4, the time derivatives in the energy equation can also be neglected and (10.3) is reduced to

$$\frac{\partial l}{\partial m} = \varepsilon_{\mathrm{n}} - \varepsilon_{\nu}. \tag{10.16}$$

The star now evolves along a sequence of states in which it is not only in hydrostatic equilibrium but also thermally adjusted. We call this *complete* (mechanical and thermal) *equilibrium*. The only initial values to be given in this case are the $X_i(m, t_0)$.

In complete equilibrium the basic equations split into two parts: the "structure equations" (10.1), (10.2), (10.16) and (10.4) contain only spatial derivatives while the "chemical equations" (10.5) contain only time derivatives. Therefore, if at a certain time $t = t_0$ the $X_i(m, t_0)$ are given, the structure equations can be taken as a set of four *ordinary* differential equations describing the structure of the star at t_0.

Complete equilibrium is a good approximation for stars in many important evolutionary phases, for example, the stars on the main sequence. But even without complete equilibrium the full set of equations is usually split into two parts: the spatial part solved as a boundary value problem for a given chemical composition $X_i(m, t_0)$, and the time-dependent initial-value problem of the chemical changes. These two parts are solved in two different, alternating steps with different numerical schemes. This introduces a basic problem of inconsistency: consider the spatial problem to be solved at time t_0, with some chemical stratification given. Once the solutions for $r(m)$, $T(m)$, $P(m)$, and $l(m)$ have been found, some layers may be convective. Therefore, the chemical stratification for which the solution was determined may be altered by convective (or, more general, by any kind of) mixing, and the solution will not be consistent with the real chemical composition. The mixing is done only in the second step, after the spatial problem is solved, over a time step $\triangle t$, and after this step, the physical variables are again not a solution of the new $X_i(m, t_0 + \triangle t)$, etc. Of course one can control the severeness of this inconsistency by keeping $\triangle t$ small, but one should be aware of its fundamental nature. Another problem arises if the structure variables are kept constant over the time step $\triangle t$. In nuclear burning regions, for example, temperature and density are usually rising with time. Therefore, they would be underestimated in the nuclear reactions if kept constant over $\triangle t$, and so would be the chemical changes due to nuclear burning. This effect leads to an overestimate of main-sequence lifetimes. Again, it can be minimized by using very small values for $\triangle t$, or by a clever prediction how T and ρ (and other quantities) may change during a time step.