Chapter 16
The Equation of State of Stellar Matter

In Chap. 15 we dealt with degeneracy of arbitrary degree for the electron gas. We now discuss the combined effect of *all* components of stellar matter, starting with the ion gas.

16.1 The Ion Gas

In the non-degenerate case, electron pressure $P_e = n_e kT$ and ion pressure $P_{ion} = n_{ion} kT$ are of the same order of magnitude; they are even equal in the case of ionized hydrogen with $n_e = n_{ion}$. For sufficiently low temperature or sufficiently high density the ions can become degenerate, too. If they are Fermi particles such as protons, they will behave in phase space like the electrons, so that, for P_{ion} and n_{ion}, relations such as (15.29)–(15.31) hold if the mass of the ions m_{ion} is used instead of m_e, and ψ is now the degeneracy parameter for the ions. Again the transition between perfect-gas behaviour and degeneracy is roughly at $\psi = 0$. We write (15.39) in the form

$$\frac{n_j}{T^{3/2}} = \text{constant}\,(m_j)^{3/2} F_{1/2}(\psi), \tag{16.1}$$

where n_j and m_j refer to either electrons or ions. Suppose that the electron gas has a certain value of $\psi = \psi^*$ for $n_e = n_e^*$. An ion gas of the same temperature has the same degeneracy parameter $\psi = \psi^*$ for $n_{ion} = (m_{ion}/m_e)^{3/2} n_e^* \approx 8 \times 10^4 n_e^*$. Therefore the ions require much higher densities to become degenerate. For the interior of normal stars one can assume that even if the electrons are degenerate the ions still obey Boltzmann statistics; thus, because of the Pauli principle, the degenerate electrons have much higher momentum than the non-degenerate ions, and the electron pressure is much larger than the pressure of the ions: $P = P_{ion} + P_e \approx P_e$.

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Even when the ion gas does not contribute noticeably to the pressure, it provides the main contribution to the mass density ϱ. This has already been taken into account by relating n_e to $\varrho = n_\mathrm{e}\mu_\mathrm{e}m_\mathrm{u}$, for example in (15.39). Furthermore, the ions can influence the thermodynamic properties of the plasma considerably.

One should be aware that, for certain types of stars, the treatment of the ions is not as simple as described here, since they can be subject to rather complicated interactions, for example, those indicated in Sects. 16.4 and 16.5.

16.2 The Equation of State

For normal stellar matter, the equation of state is then given by

$$P = P_\mathrm{ion} + P_\mathrm{e} + P_\mathrm{rad} = \frac{\Re}{\mu_0}\varrho T + \frac{8\pi}{3h^3}\int_0^\infty p^3 v(p) \frac{dp}{\mathrm{e}^{E/kT-\psi}+1} + \frac{a}{3}T^4 \ , \tag{16.2}$$

$$\varrho = \frac{4\pi}{h^3}(2m_\mathrm{e})^{3/2} m_\mathrm{u}\mu_\mathrm{e} \int_0^\infty \frac{E^{1/2}dE}{\mathrm{e}^{E/kT-\psi}+1} \ , \tag{16.3}$$

where $v(p) = \partial E/\partial p$ according to (15.7) and where E is given by (15.8). If the electron gas is highly degenerate, then also $P_\mathrm{rad} \ll P_\mathrm{e}$ and $P \approx P_\mathrm{e}$.

For given ϱ and T and chemical composition (μ_0), (16.3) can be used to determine ψ. Then ϱ, ψ, and T determine P via (16.2). The equation of state $P = P(\varrho, T)$ for all degrees of degeneracy, including relativistic effects, is therefore given here in implicit form.

An expression similar to (16.2) can be obtained for the internal energy u per unit mass:

$$u = \frac{U_\mathrm{ion} + U_\mathrm{e} + U_\mathrm{rad}}{\varrho} = \frac{3}{2}\frac{\Re}{\mu_0}T + \frac{8\pi}{h^3\varrho}\int_0^\infty \frac{p^2 E(p)dp}{\mathrm{e}^{E/kT-\psi}+1} + \frac{aT^4}{\varrho} \ , \tag{16.4}$$

where the U are the energies per unit volume, and the first term on the right corresponds to the (perfect monatomic) ion gas.

Figure 16.1 shows the $\lg\varrho$–$\lg T$ plane for the ranges relevant for the interiors of most stars. In different regions, different effects dominate the total pressure, for example, in some places the electron degeneracy and in others the radiation pressure. We will derive rough borders between these different regimes.

Let us first consider the lines ψ = constant for given μ_e in this diagram. In the non-relativistic regime, (15.39) shows that ψ is constant for $T \sim \varrho^{2/3}$, i.e. on straight lines of slope 2/3 in the $\lg\varrho$–$\lg T$ plane. In the relativistic regime ψ = constant for $T \sim \varrho^{1/3}$ according to (15.45), i.e. on straight lines with slope 1/3.

We have already seen that the perfect-gas approximation $P_\mathrm{gas} = \Re\varrho T/\mu$ becomes valid for large negative values of ψ. For large positive values of ψ complete degeneracy is a good approximation for the electron gas, and $P \approx P_\mathrm{e}$ for

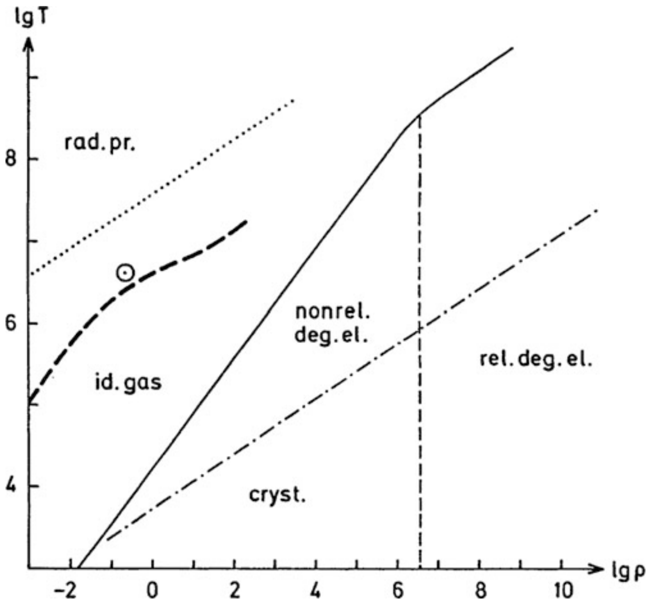

Fig. 16.1 Rough sketch of regions in the $\lg \varrho$–$\lg T$ plane (ϱ in $\mathrm{g\,cm^{-3}}$, T in K), in which the equation of state is dominated by radiation pressure (above the *dotted line* given here by $P_{\mathrm{rad}} = P_{\mathrm{gas}}$ for $\mu = 0.5$), and by the degenerate electron gas (below the *solid line* given here by (16.6) and (16.8) for $\mu_{\mathrm{e}} = 2$), which can be relativistic (right of the *vertical broken line* given by (16.7) for $\mu_{\mathrm{e}} = 2$) or non-relativistic (left of the *vertical broken line*). The *dot-dashed line* indicates the melting temperature as given by (16.26) for $\mu_0 = 4$. By comparing with (14.45) one can see that the Saha formula is valid almost nowhere in the plotted domain. The *heavy dashed curve* on the left corresponds to a model of the present Sun

the non-relativistic case is given by (15.23). We can define the border between the two regimes by the condition that both approximations yield the same value for the pressure:

$$\frac{\Re}{\mu}\varrho T = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_{\mathrm{e}}}\left(\frac{\varrho}{\mu_{\mathrm{e}} m_{\mathrm{u}}}\right)^{5/3} . \tag{16.5}$$

Equation (16.5) is equivalent to

$$\frac{T}{\varrho^{2/3}} = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_{\mathrm{e}}\Re m_{\mathrm{u}}^{5/3}}\frac{\mu}{\mu_{\mathrm{e}}^{5/3}} = 1.207\times 10^5\,\frac{\mu}{\mu_{\mathrm{e}}^{5/3}} , \tag{16.6}$$

where the numerical constant is in cgs units. Equation (16.6) gives a straight line with slope 2/3 in Fig. 16.1 (lower left part of the solid line), which is obviously a line of ψ = constant for given μ, μ_{e}. To the left of it the electrons behave almost like a perfect gas; to the right they are degenerate and dominate the pressure.

We now ask where relativistic effects become important. The transition between the non-relativistic and relativistic cases occurs around $x \approx 1$, where the relativity parameter x is given by (15.11). Then (15.4) together with $\varrho = \mu_e m_u n_e$ gives

$$\varrho = \frac{8\pi m_u m_e^3 c^3}{3h^3}\mu_e = 9.74 \times 10^5 \mu_e \text{ (cgs)} . \tag{16.7}$$

In the plane of Fig. 16.1, (16.7) defines a vertical border line between relativistic (at larger ϱ) and non-relativistic degeneracy (at smaller ϱ). The same procedure which yielded (16.6) can be used with (15.26) in order to define the border between relativistic degeneracy and non-degeneracy:

$$\frac{T}{\varrho^{1/3}} = \left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{8\Re} \frac{1}{m_u^{4/3}} \frac{\mu}{\mu_e^{4/3}} = 1.496 \times 10^7 \frac{\mu}{\mu_e^{4/3}} , \tag{16.8}$$

where the numerical constant is in cgs. The corresponding straight line of slope 1/3 is the upper-right part of the solid line in Fig. 16.1, again being a line of $\psi =$ constant for given μ, μ_e.

In a similar way we can determine a border between the regime of perfect gas pressure and that of dominating radiation pressure . From

$$\frac{\Re}{\mu}\varrho T = \frac{a}{3}T^4 \tag{16.9}$$

we find

$$\frac{T}{\varrho^{1/3}} = \left(\frac{3\Re}{a\mu}\right)^{1/3} = \frac{3.2 \times 10^7}{\mu^{1/3}} , \tag{16.10}$$

where the constant is in cgs. This line of slope 1/3 is dotted in Fig. 16.1.

In Fig. 16.1 it is indicated how T grows with increasing density in the Sun. As one can see, the interior regions of the Sun avoid the area in the diagram where radiation pressure is important, as well as that of degeneracy. However, we will have to deal with other cases in which the equation of state is more complicated. This concerns highly evolved stars, but also unevolved stars of very low mass (For a review see Van Horn 1986.).

16.3 Thermodynamic Quantities

With the implicit form (16.2) and (16.3) and with the expression (16.4) for the internal energy we are in principle able to determine δ, c_P, and ∇_{ad}. Since in general no analytic methods are known one can try to determine the thermodynamic quantities numerically. Here we just give them for some limiting cases for which analytic expressions can be derived. For the sake of simplicity we neglect the effects of radiation and we suppose the ions to be a perfect gas.

In the cases of complete degeneracy of a non-relativistic or an extremely relativistic electron gas, it is obvious from equations (15.23) and (15.26) that the quantities α, δ as defined in (4.2) and (4.3) are $\alpha = 3/5, \delta = 0$, or $\alpha = 3/4, \delta = 0$ respectively.

We define the ratio η of ion pressure to total pressure

$$\eta := \frac{P_{\rm ion}}{P_{\rm ion} + P_{\rm e}} \ . \tag{16.11}$$

For strong non-relativistic degeneracy (15.39), (15.41), and (15.43) for $\psi \gg 1$, imply that

$$\begin{aligned} P_{\rm e} &\approx \frac{4}{15} B_1 (\psi k T)^{5/2} \ , \quad B_1 = \frac{4\pi}{h^3}(2m_{\rm e})^{3/2} \ , \\ \varrho &\approx \frac{2}{3}\mu_{\rm e} m_{\rm u} B_1 (\psi k T)^{3/2} \ , \end{aligned} \tag{16.12}$$

which together with $P_{\rm ion} = \Re\varrho T/\mu_0 = k\varrho T/(m_{\rm u}\mu_0)$ and (16.11) result in

$$\eta \approx \frac{5}{2}\frac{\mu_{\rm e}}{\mu_0}\frac{1}{\psi} \ . \tag{16.13}$$

The larger ψ (the stronger the degeneracy), the smaller η, and therefore the smaller the contribution of the ion gas to the total pressure.

The value of δ can be obtained from the relation

$$\delta = -\left(\frac{\partial \ln \varrho}{\partial \ln T}\right)_P = -\left(\frac{\partial \ln \varrho}{\partial \ln T}\right)_\psi + \frac{\left(\frac{\partial \ln \varrho}{\partial \ln \psi}\right)_T \left(\frac{\partial \ln P}{\partial \ln T}\right)_\psi}{\left(\frac{\partial \ln P}{\partial \ln \psi}\right)_T} \ , \tag{16.14}$$

which follows from the total differentials of the functions $\varrho = \varrho(\psi, T), P = P(\psi, T)$. For $P = P_{\rm e}$ the partial derivatives can be taken from (16.12), and (16.14) gives $\delta = 0$. For a small but non-vanishing contribution $P_{\rm ion}$ we write according to (16.11) the total pressure $P = P_{\rm e}/(1-\eta) \approx (1+\eta)P_{\rm e}$. If we then use the expressions (16.12) and (16.13), we obtain for the non-relativistic case

$$\delta \approx \frac{3}{5}\eta \approx \frac{3}{2}\frac{\mu_{\rm e}}{\mu_0}\frac{1}{\psi} \ . \tag{16.15}$$

For the extremely relativistic electron gas we find from (15.45) and (15.46), with the lowest terms of the expansion (15.43), that

$$\begin{aligned} P_{\rm e} &= \frac{B_2}{4}(\psi k T)^4 \ , \quad B_2 = \frac{8\pi}{3c^3h^3} \ , \\ \varrho &= \mu_{\rm e} m_{\rm u} B_2 (\psi k T)^3, \end{aligned} \tag{16.16}$$

and in the same way we obtained (16.13) and (16.15) we now get

$$\eta \approx 4\frac{\mu_e}{\mu_0}\frac{1}{\psi}\,, \quad \delta = \frac{3}{4}\eta = \frac{3\mu_e}{\mu_0}\frac{1}{\psi}\,. \tag{16.17}$$

In order to derive c_P we need the internal energy u. Let us again neglect the radiation field here; then u contains a component u_e of the (degenerate) electron gas and a component u_{ion} of the (perfect) ion gas: $u = u_e + u_{ion}$. In the non-relativistic case, (15.42) gave $U_e = 3P_e/2$ for the internal energy U_e per unit volume of the electron gas, independent of ψ. A corresponding relation $U_{ion} = 3P_{ion}/2$ holds for the non-degenerate ions, and therefore

$$u = \frac{U}{\varrho} = \frac{3}{2}\frac{P_{ion} + P_e}{\varrho} = \frac{3}{2}\frac{P}{\varrho}\,. \tag{16.18}$$

This gives the derivative

$$\left(\frac{\partial u}{\partial T}\right)_P = -\frac{3}{2}\frac{P}{\varrho T}\left(\frac{\partial \ln \varrho}{\partial \ln T}\right)_P = \frac{3}{2}\frac{P\delta}{\varrho T}\,, \tag{16.19}$$

which is used in the definition (4.4) of c_P:

$$c_P = \left(\frac{\partial u}{\partial T}\right)_P - \frac{P}{\varrho^2}\left(\frac{\partial \varrho}{\partial T}\right)_P = \left(\frac{\partial u}{\partial T}\right)_P + \frac{P\delta}{\varrho T} = \frac{5}{2}\frac{P\delta}{\varrho T}\,. \tag{16.20}$$

Then (4.21) gives $\nabla_{ad} = 2/5$, the same value we obtained for the perfect gas with $\beta = 1$ [see (13.12)]. Since we have derived it without making use of the degree of degeneracy, the numerical value 2/5 for ∇_{ad} is independent of ψ, but holds only for non-relativistic degeneracy.

In the extreme relativistic case, (15.27) shows that $U_e = 3P_e$, while again $U_{ion} = 3P_{ion}/2$ for the non-degenerate ions. The total energy density is then

$$u = u_e + u_{ion} = 3\frac{P_e}{\varrho} + \frac{3}{2}\frac{P_{ion}}{\varrho} = 3\frac{P}{\varrho} - \frac{3}{2}\frac{P_{ion}}{\varrho} = 3\frac{P}{\varrho} - \frac{3}{2}\frac{\Re}{\mu_0}T; \tag{16.21}$$

the specific heat is

$$c_P = -\frac{4P}{\varrho^2}\left(\frac{\partial \varrho}{\partial T}\right)_P - \frac{3}{2}\frac{\Re}{\mu_0} = \frac{4P}{\varrho T}\delta - \frac{3}{2}\frac{\Re}{\mu_0}\,, \tag{16.22}$$

so that we can now determine ∇_{ad}:

$$\nabla_{ad} = \frac{P\delta}{\varrho T c_P} = \frac{1}{4 - \frac{3}{2}\frac{\Re}{\mu_0}\frac{\varrho T}{P\delta}}\,. \tag{16.23}$$

From (16.16) and (16.17) we find that

$$P \approx P_e = \frac{B_2}{4}(\psi kT)^4 , \quad \varrho = B_2 \mu_e m_u (\psi kT)^3 , \quad \delta = 3\frac{\mu_e}{\mu_0}\frac{1}{\psi} , \tag{16.24}$$

and therefore $3\Re\varrho T/\mu_0 = 4P\delta$, which with (16.23) gives $\nabla_{ad} = 1/2$. This is the value for the fully degenerate, extreme relativistic case.

16.4 Crystallization

Up to now we have treated the ions as a perfect gas, which means we have neglected their interaction. However, this no longer suffices for high densities and particularly low temperatures, in which case the Coulomb interaction of the ions must be considered: instead of moving freely, the ions tend to form a rigid lattice, which minimizes their total energy. This occurs when the thermal energy $3kT/2$ becomes comparable with the Coulomb energy per ion of charge $-Ze$. If we define a volume V_{ion} per ion by $n_{ion}V_{ion} = 1$ (where n_{ion} is the number density of ions) and a mean separation r_{ion} between the ions, we have $V_{ion} = 4\pi r_{ion}^3/3$. Then the ratio

$$\Gamma_C := \frac{(Ze)^2}{r_{ion}kT} = 2.7 \times 10^{-3}\frac{Z^2 n_{ion}^{1/3}}{T} \tag{16.25}$$

is a measure for the importance of this effect, the numerical constant having units of cgs. $\Gamma_C \ll 1$ would mean that the electrostatic energy plays a minor role and the ions have a Boltzmann distribution, while $\Gamma_C \gg 1$ indicates that the kinetic energy of the ions is negligible and that they try to form a conglomerate that has a lower energy, i.e. they form a crystal.

More detailed considerations (see, for instance, Shapiro and Teukolsky 1983) indicate that $\Gamma_C \approx 170$ is a critical value for the transition between the two types of behaviour of the ion gas. With this value for Γ_C and using the relation $\varrho = \mu_0 m_u n_{ion}$ we obtain the critical temperature T_m (melting temperature):

$$T_m \approx \frac{Z^2 e^2}{\Gamma_c k}\left(\frac{4\pi\varrho}{3\mu_0 m_u}\right)^{1/3} = 1.3 \times 10^3 Z^2 \mu_0^{-1/3}\varrho^{1/3} , \tag{16.26}$$

where the numerical constant is in cgs units. The corresponding straight line is plotted (dot-dashed) in Fig. 16.1.

In the interior of evolved stars we have high densities, but the temperature is well above the melting temperature. The situation is different in cooling white dwarfs, where the temperature becomes smaller with time, while the density remains virtually unchanged. We will come back to this in Chap. 37, which deals with white dwarfs.

16.5 Neutronization

If in a plasma the electrons have sufficient energy, they can combine with the protons to form neutrons. If m_n and m_p are the masses of neutron and proton, then the electron must have the total energy $E_{tot} > E^* = c^2(m_n - m_p)$. At low densities the neutron will decay within 11 min back into a proton–electron pair, where the electron has the total energy E^* and a kinetic energy $E^*_{kin} = E^* - m_c c^2$; however, the situation can be different if the gas is completely degenerate and the phase space is filled up to the (kinetic) Fermi energy E_F. If the Fermi energy E_F exceeds E^*_{kin}, the electrons released do not have enough energy to find an empty cell in phase space, and the neutrons cannot decay, i.e. the Fermi sea of electrons has stabilized the neutrons.

In order to estimate under which conditions this occurs we write the relation (15.6) between E and p in the form

$$p = \frac{1}{c}(E^2 - m_e^2 c^4)^{1/2} . \tag{16.27}$$

If we put $E = E_{kin} + m_e c^2 = E_F + m_e c^2 = c^2(m_n - m_p) = 1.294 \times 10^6$ eV, we can determine the corresponding Fermi momentum p_F from (16.27) and obtain $x = p_F/(m_e c) \approx 2.2$. Then, according to (15.15) and taking $\varrho = \mu_e m_u n_e$ with $\mu_e = 2$, we find $\varrho \approx 2.4 \times 10^7\,\mathrm{g\,cm^{-3}}$. Therefore, if a proton–electron gas is compressed to a density above this value, then the gas undergoes a transition into a neutron gas (“neutronization”).

For stellar matter the situation is more complicated, since at sufficiently high densities the plasma contains heavier nuclei, and not just protons. The nuclei capture electrons (inverse β decay) and become neutron-rich isotopes. This requires much higher electron energies than those just estimated, since the neutrons in the nucleus are degenerate and the new ones have to be raised above the Fermi energy. Correspondingly higher plasma densities are required to provide the electrons with the necessary energy. If the nuclei become too neutron rich they start to break up, releasing free neutrons. The density at which this “neutron drip” starts is of the order of several $10^{11}\,\mathrm{g\,cm^{-3}}$, but the exact value depends on the nuclear model one is using in detailed calculations. Hillebrandt (1991) gives $\varrho_{drip} \approx 3 \times 10^{11}\,\mathrm{g\,cm^{-3}}$, Pethick and Ravenhall (1991) estimate $\varrho_{drip} \approx 3.5 \times 10^{11}\,\mathrm{g\,cm^{-3}}$.

Let us briefly consider the effect on the equation of state. Up to ϱ_{drip} the total pressure $P \approx P_e$ is provided by relativistic electrons. With further increases of ϱ, the number density n_e increases by less than an amount proportional to ϱ, owing to the capture of some electrons. Therefore the pressure rises by less than $\varrho^{4/3}$. Consequently $\gamma_{ad} \equiv (d \ln P/d \ln \varrho)_{ad}$ is reduced below 4/3, which can be seen in Fig. 16.2, where the slope of the curve $P = P(\varrho)$ is suddenly reduced for $\log \varrho \gtrsim 11.7$. At still higher ϱ the increasing number of free neutrons contribute gradually more to P.

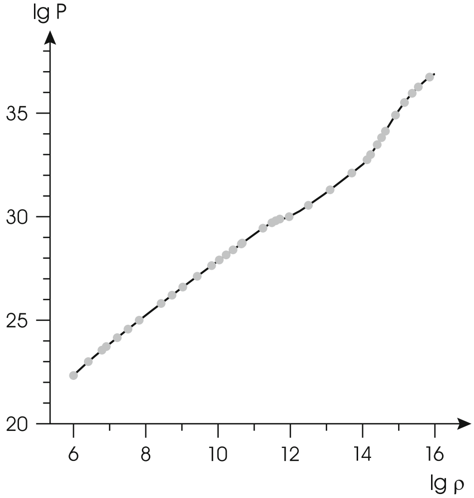

Fig. 16.2 The equation of state for very high densities. On logarithmic scales the pressure P_e (in dyn cm^{-2}) is plotted against the density ϱ (in g cm^{-3}). The *grey* symbols refer to experimental or theoretical data points from various sources. See Haensel et al. (2007), p. 15, for more details. Figure adapted from their Fig. 1.3

With increasing ϱ the neutrons become increasingly degenerate–as a perfect Fermi gas they would give the slope 5/3. But then interaction between neutrons becomes important, and the details of the equation of state are very uncertain, for example, depending on rather badly known properties of the particles. For more details see Sects. 37.2 and 38.1 and Shapiro and Teukolsky (1983).

16.6 Real Gas Effects

Although the stellar plasma can to a great extent be treated as a perfect gas, the assumptions for a perfect gas are not truly fulfilled: there are interaction forces, such as the Coulomb force, acting between the constituents, and atoms and ions cannot always be considered to be pointlike. Therefore an accurate equation of state has to include such effects.

We already encountered *pressure ionization* in Sect. 14.6, which is a consequence of the spatially overlapping energy levels, which leads to interacting ionic potentials. We noted that there is no good theory to treat this in a simple way, but that one has to modify the Saha equation somehow to avoid its wrong behaviour at high pressure. A good theory for pressure ionization has to work with quantum-mechanical atomic models. The effect of pressure ionization is increasingly important for cool, dense stars of low mass ($M < 1\,M_\odot$) and gas planets. Saumon et al. (1995) have developed an equation for state for dense gases, which, due to the complexity of the problem, is limited to hydrogen-helium mixtures. This and other modern equations of state are provided in tabular form, for example, as tables of $P(\rho, T)$ and $u(\rho, T)$ for various chemical mixtures. The thermodynamic quantities, which are or use derivatives of P and u, are either computed from the tables, or are provided as tables, too.

Another interaction becoming important at low temperatures, when molecules are able to form, are the classical *van der Waals* forces, which are attractive forces of electrically neutral, but polarized particles. Their consideration leads in the simplest approximation to the equation of state for a real gas

$$(P + n^2 a)(1 - nb) = nkT. \tag{16.28}$$

The meaning of the additional terms $n^2 a$ and nb is easy to understand: nb is the effect of volume reduction due to the finite size of the particles, which leads therefore at given temperature to a pressure increase. The second term, $n^2 a$, is the effect of the attractive forces, which result in a reduction of the gas pressure P. a and b are parameters depending on the microscopic properties of the gas particles. An equation of state of this type results in phase transitions, which indeed were found in the equation of state of Saumon et al. (1995) (For a derivation of (16.28) see Weiss et al. 2004.).

In Sect. 16.4, we discussed crystallization, which is due to the electrostatic interaction between ions at high densities, in the limit of $\Gamma_C \gg 1$. At the other extreme, when $\Gamma_C \ll 1$, the gas is close to being a perfect one, but not quite so. Consider an ionized gas, which consists of positively charged ions and unbound electrons. As long as these are not degenerate, they can move freely and will feel the Coulomb forces in the plasma. In particular, ions will attract electrons and it is plausible that clouds of electrons gather around ions such that from a sufficiently large distance the ion electron cloud will appear as being electrically neutral. This picture of *electron shielding (in the weak limit)* requires low particle densities, because the inter-ion distances must be larger than the typical electron cloud size. The physical effect is usually treated within the *Debye–Hückel theory* (Landau and Lifshitz 1980, Chap. 78; Weiss et al. 2004, Chap. 17.15), which we will encounter in detail in Sect. 18.4. Here it suffices to state that based on a shielded Coulomb potential around the ions,

$$\Phi(r) = \frac{Ze}{r} \cdot e^{-r/r_D}, \tag{16.29}$$

where r_D is the Debye-radius (18.50), the resulting attractive electrostatic forces lead to a reduction of the gas pressure according to

$$P = nkT \left[1 - 3.2 \times 10^7 \frac{\varrho^{1/2}}{T^{3/2}} \mu \zeta^{3/2} \right], \tag{16.30}$$

where

$$\zeta = \sum_i \frac{Z_i (Z_i + 1)}{A_i} X_i \tag{16.31}$$

is, as in (18.47), the mass weighted average of free electrons times ionic charge Z_i of all ion species i.

For the centre of the Sun, $\mu \approx 0.8$, $\zeta \approx 1.7$, $T \approx 15 \times 10^6$ K, and $\varrho \approx 140\,\mathrm{g/cm^{-3}}$, and the correction to P is 1.6 %. Although this effect appears to be small, it has turned out that equations of state with an accuracy of this order are needed for modern solar and stellar models.

These are the most important non-ideal effects that modify the equation of state. In addition there are even more interaction forces of quantum nature (such as spin–spin interaction), which may in some situations become important. Some of these effects are considered in equation of states published by specialized groups. The most important ones are the OPAL and MHD equations of state (Rogers et al. 1996 and Mihalas et al. 1988, and later improvements), both available in tabular form and for a variety of chemical mixtures. They are widely used in current stellar evolution calculations, and have helped to improve the solar model considerably. More on this issue can be found in Weiss et al. (2004), Chap. 15-A.