Chapter 13
The Perfect Gas with Radiation

13.1 Radiation Pressure

The pressure in a star is not only given by that of the gas because the photons in the stellar interior can contribute considerably to the pressure, and therefore our discussion of the perfect gas of Sect. 4.2 has to be extended. Since the radiation is practically that of a black body (see Sect. 5.1.1), its pressure P_{rad} is given by

$$P_{\text{rad}} = \frac{1}{3}U = \frac{a}{3}T^4 \ , \tag{13.1}$$

where U is the energy density and a is the radiation density constant $a = 7.56464 \times 10^{-15}\,\text{erg cm}^{-3}\,\text{K}^{-4}$. Then the total pressure P consists of the gas pressure P_{gas} and radiation pressure P_{rad}:

$$P = P_{\text{gas}} + P_{\text{rad}} = \frac{\Re}{\mu}\varrho T + \frac{a}{3}T^4 \ , \tag{13.2}$$

where on the right we have assumed that the gas is perfect. We now define a measure for the importance of the radiation pressure by

$$\beta := \frac{P_{\text{gas}}}{P} \ , \quad 1 - \beta = \frac{P_{\text{rad}}}{P} \ . \tag{13.3}$$

For $\beta = 1$ the radiation pressure is zero, while $\beta = 0$ means that the gas pressure is zero. The definition (13.3) can also be used if the gas is not perfect.

Two other relations which can be derived by differentiation of (13.3) are sometimes useful:

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$$\left(\frac{\partial \beta}{\partial T}\right)_P = -\left[\frac{\partial(1-\beta)}{\partial T}\right]_P = -\frac{4}{T}(1-\beta)\;, \tag{13.4}$$

$$\left(\frac{\partial \beta}{\partial P}\right)_T = -\left[\frac{\partial(1-\beta)}{\partial P}\right]_T = \frac{1}{P}(1-\beta)\;. \tag{13.5}$$

13.2 Thermodynamic Quantities

From (13.2) we obtain

$$\varrho = \frac{\mu}{\Re}\frac{1}{T}\left(P - \frac{a}{3}T^4\right)\;, \tag{13.6}$$

and with the definitions (6.6) with (13.4), and (13.5) we find that

$$\alpha = \frac{1}{\beta}\;, \qquad \delta = \frac{4-3\beta}{\beta}\;, \quad \varphi = 1\;. \tag{13.7}$$

Indeed, if the radiation pressure can be neglected ($\beta = 1$), we find $\alpha = \delta = 1$, as should be expected for a perfect monatomic gas.

If the gas components are monatomic, then the internal energy per unit mass is

$$u = \frac{3}{2}kT\frac{n}{\varrho} + \frac{aT^4}{\varrho} = \frac{3}{2}\frac{\Re}{\mu}T + \frac{aT^4}{\varrho} = \frac{\Re T}{\mu}\left[\frac{3}{2} + \frac{3(1-\beta)}{\beta}\right]\;, \tag{13.8}$$

so that according to the definition (4.4) of c_P we have

$$c_P = \left(\frac{\partial u}{\partial T}\right)_P + P\left(\frac{\partial v}{\partial T}\right)_P = \left(\frac{\partial u}{\partial T}\right)_P - \frac{P}{\varrho^2}\left(\frac{\partial \varrho}{\partial T}\right)_P\;. \tag{13.9}$$

Using (13.8), after some algebraic manipulations involving (13.4), we obtain

$$\left(\frac{\partial u}{\partial T}\right)_P = \frac{\Re}{\mu}\left[\frac{3}{2} + \frac{3(4+\beta)(1-\beta)}{\beta^2}\right]\;. \tag{13.10}$$

From the definition of δ with (13.7)–(13.9) we write

$$c_P = \frac{\Re}{\mu}\left[\frac{3}{2} + \frac{3(4+\beta)(1-\beta)}{\beta^2} + \frac{4-3\beta}{\beta^2}\right]\;, \tag{13.11}$$

and then the relation (4.21) may be applied in order to determine the adiabatic gradient ∇_{ad} for the perfect gas plus radiation:

$$\nabla_{ad} = \frac{\Re\delta}{\beta\mu c_P} = \left(1 + \frac{(1-\beta)(4+\beta)}{\beta^2}\right) \Big/ \left(\frac{5}{2} + \frac{4(1-\beta)(4+\beta)}{\beta^2}\right)\;. \tag{13.12}$$

For $\beta \rightarrow 1$, (13.11) and (13.12) give the well-known values for the perfect monatomic gas: $c_P = 5\Re/(2\mu)$ and $\nabla_{\rm ad} = 2/5$, while for $\beta \rightarrow 0$, one has $\nabla_{\rm ad} \rightarrow 1/4$ and c_P becomes infinite.

Sometimes the derivative

$$\frac{1}{\gamma_{\rm ad}} := \left(\frac{d \ln \varrho}{d \ln P}\right)_{\rm ad} \tag{13.13}$$

is required (4.37, 4.41). If in the definition

$$\frac{d\varrho}{\varrho} = \alpha \frac{dP}{P} - \delta \frac{dT}{T} \tag{13.14}$$

of α and δ the adiabatic condition $PdT/(TdP) = \nabla_{\rm ad}$ is introduced, one finds

$$\gamma_{\rm ad} = \frac{1}{\alpha - \delta \nabla_{\rm ad}} \, . \tag{13.15}$$

In the case of a perfect gas with radiation pressure we have to introduce the expressions (13.7), while for the limit $\beta = 1$, we find

$$\gamma_{\rm ad} = \frac{1}{1 - \nabla_{\rm ad}} \, . \tag{13.16}$$

For a monatomic gas without radiation pressure ($\beta = 1$) one has $\nabla_{\rm ad} = 0.4$ and therefore $\gamma_{\rm ad} = 5/3$, whereas in the limit $\beta \rightarrow 0$–after α, δ, and $\nabla_{\rm ad}$ are inserted from (13.7) and (13.12)–we find for a gas dominated by radiation pressure that

$$\gamma_{\rm ad} \rightarrow \frac{4}{3} \, , \qquad \nabla_{\rm ad} \rightarrow \frac{1}{4} \, . \tag{13.17}$$

Instead of $\gamma_{\rm ad}$, $\nabla_{\rm ad}$, one often uses the "adiabatic exponents" introduced by Chandrasekhar, which are defined by

$$\Gamma_1 := \left(\frac{d \ln P}{d \ln \varrho}\right)_{\rm ad} = \gamma_{\rm ad} \, , \tag{13.18}$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} := \left(\frac{d \ln P}{d \ln T}\right)_{\rm ad} = \frac{1}{\nabla_{\rm ad}} \, , \tag{13.19}$$

$$\Gamma_3 := \left(\frac{d \ln T}{d \ln \varrho}\right)_{\rm ad} + 1 \, , \tag{13.20}$$

and obey the relation

$$\frac{\Gamma_1}{\Gamma_3 - 1} = \frac{\Gamma_2}{\Gamma_2 - 1} \, . \tag{13.21}$$