Chapter 3
The Virial Theorem

3.1 Stars in Hydrostatic Equilibrium

While the virial theorem generally plays a relatively minor role in physics, it is of vital importance for the understanding of stars. It connects two important energy reservoirs of a star and allows predictions and interpretations of certain evolutionary phases.

If we multiply (2.5) by $4\pi r^3$ and integrate over dm in the interval $[0, M]$, i.e. from centre to surface, we obtain on the left-hand side an integral which can be simplified by partial integration:

$$\int_0^M 4\pi r^3 \frac{\partial P}{\partial m} dm = \left[4\pi r^3 P\right]_0^M - \int_0^M 12\pi r^2 \frac{\partial r}{\partial m} P\, dm \,, \tag{3.1}$$

where the term in brackets vanishes, since $r = 0$ at the centre and $P = 0$ at the surface. With (1.6) the integrand of the last term in (3.1) is reduced to $3P/\varrho$. Therefore, after multiplication by $4\pi r^3$ and integration, (2.5) gives

$$\int_0^M \frac{Gm}{r} dm = 3 \int_0^M \frac{P}{\varrho} dm \,. \tag{3.2}$$

Both sides of (3.2) have the dimensions of energy and can be easily interpreted. We define the *gravitational energy* E_g by

$$E_g := -\int_0^M \frac{Gm}{r} dm \,. \tag{3.3}$$

Consider a unit mass at the position r. Its potential energy due to the gravitational field of the mass m inside r is $-Gm/r$. Therefore E_g is the potential energy of all mass elements dm of the star (normalized to zero at infinity). The energy $-E_g (> 0)$

R. Kippenhahn et al., *Stellar Structure and Evolution*, Astronomy and Astrophysics Library, DOI 10.1007/978-3-642-30304-3_3, © Springer-Verlag Berlin Heidelberg 2012

is necessary to expand all mass shells into infinity, and it is released when the stellar configuration forms out of an infinitely distributed medium.

We see that E_g varies if the configuration undergoes expansion or contraction: if all mass shells inside the configuration expand or contract simultaneously, then E_g increases or decreases, respectively. And the same must be true for the integral on the right of (3.2). Note that these radial motions must be slow compared to $\tau_{\rm hydr}$ in order that hydrostatic equilibrium is always maintained, otherwise (3.2) would not hold.

In order to understand the meaning of the term on the right of (3.2) we first assume a perfect gas. Then

$$\frac{P}{\varrho} = \frac{\Re}{\mu} T = (c_P - c_v)T = (\gamma - 1)c_v T, \tag{3.4}$$

where c_P, c_v are the specific heats per unit mass (and we make use of $\Re/\mu = c_P - c_v$ and replace c_P/c_v by γ). For a monatomic gas $\gamma = 5/3$, and we have

$$\frac{P}{\varrho} = \frac{2}{3} u \ , \tag{3.5}$$

where $u = c_v T$ is the internal energy per unit mass of the perfect gas. Therefore (3.2) can be written as

$$E_g = -2E_i \tag{3.6}$$

with the total internal energy of the star

$$E_i := \int_0^M u \, dm \ . \tag{3.7}$$

Equation (3.6) is the *virial theorem* for a perfect monatomic gas. For a general equation of state we define a quantity ζ by

$$\zeta u := 3\frac{P}{\varrho} \ . \tag{3.8}$$

For a perfect gas $\zeta = 3(\gamma - 1)$, in the monatomic case $\gamma = 5/3$, and therefore $\zeta = 2$. For a pure photon gas, $P = aT^4/3$, and $u\varrho = aT^4$ (a = radiation density constant), giving $\zeta = 1$. If ζ is constant throughout the star, (3.2) leads to the more general virial theorem:

$$\zeta E_i + E_g = 0 \ . \tag{3.9}$$

We now define the *total energy* W of our configuration,

$$W = E_i + E_g \ , \tag{3.10}$$

where for a gravitationally bound system $W < 0$, and with (3.9) we find that

$$W = (1 - \zeta)E_\mathrm{i} = \frac{\zeta - 1}{\zeta} E_\mathrm{g} \, . \tag{3.11}$$

In the case of $\zeta = 1$ ($\gamma = 4/3$) the total energy vanishes.

But in general W, E_g, and E_i are coupled. A change of the total energy of the configuration is then connected with a change of its internal energy and with expansion or shrinking. A gas of finite temperature must radiate and W must decrease. Let L be the *luminosity* of the star, i.e. the total energy loss per unit time by radiation; then conservation of energy demands that $(dW/dt) + L = 0$, so that with (3.11) we obtain

$$L = (\zeta - 1)\frac{dE_\mathrm{i}}{dt} = -\frac{\zeta - 1}{\zeta}\frac{dE_\mathrm{g}}{dt} \, . \tag{3.12}$$

We have seen that $\dot{E}_\mathrm{g} < 0$ for contraction of all mass shells (where the dot denotes a derivative with respect to time t). For a perfect gas (3.12) gives $L = -\dot{E}_\mathrm{g}/2 = \dot{E}_\mathrm{i}$, which means that half of the energy liberated by the contraction is radiated away and the other half is used to heat the star ($L > 0$, $\dot{E}_\mathrm{i} > 0$). The surprising fact that a star heats up while losing energy can be described by saying that the star has a negative specific heat (cf. the gravothermal specific heat defined in Sect. 25.3.4).

We have to keep in mind that it is the luminosity that causes the shrinking: a configuration in hydrostatic equilibrium has a finite temperature and therefore radiates into the (cold) universe.

3.2 The Virial Theorem of the Piston Model

Let us consider the situation for the piston model of Sect. 2.7 for the case of a perfect gas. Assuming $M^* \gg m^*$, we define $E_\mathrm{g} := +G^* h$, where the free additional constant is chosen such that $E_\mathrm{g} = 0$ for $h = 0$. Hydrostatic equilibrium (2.33) with $m^* = Ah\varrho$ and (3.4) demands that

$$hG^* = \frac{P}{\varrho} m^* = (\gamma - 1)c_v T \, m^* \, . \tag{3.13}$$

The internal energy E_i of the gas is $E_\mathrm{i} = c_v T m^*$, and we find that

$$E_\mathrm{g} = (\gamma - 1)E_\mathrm{i} \, , \tag{3.14}$$

which is the virial theorem for the piston model. Differentiating with respect to time, with $\gamma = 5/3$, results in

$$\frac{dE_g}{dt} = \frac{2}{3}\frac{dE_i}{dt} . \tag{3.15}$$

Hence we see that in contrast to the situation in stars, a reduction of E_g is connected with *cooling* of the gas. Indeed the piston can only sink if the gas cools.

This different behaviour comes from the fact that the gravitational field is assumed to be constant here. In order to demonstrate this we now assume the weight G^* to be a function of h and differentiate (3.13) with respect to h :

$$G^*(1 + G^*_h) = (\gamma - 1)\frac{dE_i}{dh} \tag{3.16}$$

with $G^*_h := (d \ln G^*/d \ln h)$. Indeed, if $G^*_h = 0$ (constant gravity), we see that E_i increases with h. If, however, G^* decreases sufficiently with increasing h (such that $G^*_h < -1$), then E_i increases with decreasing h, corresponding to the behaviour of stars. In fact in an expanding star each mass shell also loses weight with increasing r.

3.3 The Kelvin–Helmholtz Timescale

Returning now to consider stars, since according to (3.12) L is of the order of $|dE_g/dt|$, we can define a characteristic time-scale

$$\tau_{KH} := \frac{|E_g|}{L} \approx \frac{E_i}{L} \tag{3.17}$$

called the *Kelvin–Helmholtz timescale* (after the two physicists who estimated this as the evolutionary timescale for a contracting or cooling star).

A rough estimate for $|E_g|$ is

$$|E_g| \approx \frac{G\overline{m}^2}{\bar{r}} \approx \frac{GM^2}{2R} , \tag{3.18}$$

where quantities with a bar indicate mean values for m and r (which we have replaced by $M/2$ and $R/2$). Then we have

$$\tau_{KH} \approx \frac{GM^2}{2RL} . \tag{3.19}$$

For the Sun, with $L = 3.827 \times 10^{33}$ erg/s, we find $\tau_{KH} \approx 1.6 \times 10^7$ years. In the early days of astrophysics the source of stellar energy was still uncertain, and it was suggested, among other proposals, that the Sun "lived" from its gravitational energy E_g. Our estimate shows that this can work only for some 10^7 years, after which time it would have contracted to a very condensed body. As it became obvious that the Sun has been radiating in roughly the same way for some 10^9 years, the contraction hypothesis had to be abandoned. But there are phases in a stellar life when E_g is the main or even the only stellar energy source (Chap. 28); then the star evolves on the timescale τ_{KH}. A more detailed discussion of the evolution of a star in time appears in Sect. 4.5.

3.4 The Virial Theorem for Non-vanishing Surface Pressure

One often needs the virial theorem for gaseous spheres imbedded in a medium of finite pressure. In this case, at the surface ($m = M$), $P = P_0 > 0$ instead of $P = 0$. Consequently the first term on the right of (3.1) does not vanish at the surface, and (3.2) is modified to

$$\int_0^M \frac{Gm}{r} dm = 3 \int_0^M \frac{P}{\varrho} dm - 4\pi R^3 P_0 \ . \qquad (3.20)$$

Correspondingly we find, rather than (3.9), that

$$\zeta E_i + E_g = 4\pi R^3 P_0 \ . \qquad (3.21)$$