Chapter 7
Transport of Energy by Convection

Convective transport of energy means an exchange of energy between hotter and cooler layers in a dynamically unstable region through the exchange of macroscopic mass elements ("blobs", "bubbles", "convective elements"), the hotter of which move upwards while the cooler ones descend. The moving mass elements will finally dissolve in their new surroundings and thereby deliver their excess (or deficiency) of heat. Owing to the high density in stellar interiors, convective transport can be very efficient. However, this energy transfer can operate only if it finds a sufficient driving mechanism in the form of the buoyancy forces.

A thorough theoretical treatment of convective motions and transport of energy is extremely difficult. It is the prototype of the many astrophysical problems in which the bottleneck preventing decisive progress is the difficulty involved in solving the well-known hydrodynamic equations. For simplifying assumptions, solutions are available that may even give reasonable approximations for certain convective flows in the laboratory (or in the kitchen). Unfortunately, convection in stars proceeds under rather malicious conditions: turbulent motion transports enormous fluxes of energy in a very compressible gas, which is stratified in density, pressure, temperature, and gravity over many powers of ten. Nevertheless, large efforts have been made over many years to solve this notorious problem, and they have partly arrived at promising results. Canuto (2008) summarizes the state of the art of models for the underlying Navier-Stokes equations, which in the field of oceanography and atmospheric sciences have had great success, and which aim at modelling the fluctuations around an average state. None of these so-called *Reynolds stress models*, however, has reached a stage where it could provide a procedure easy enough to be handled in everyday stellar-structure calculations, and at the same time would describe the full properties of convection accurately enough. On the other hand, full two- and three-dimensional hydrodynamical simulations have also made large progress, thanks to the impressive advances in supercomputer technology and efficient numerical algorithms (see the review by Kupka 2008). They give valuable hints to the true nature of convection and often serve as numerical experiments to test the dynamical methods. Nevertheless, these numerical simulations are still limited in their size and thus can follow convection in most cases only for a limit

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time and only for thin convection zones. But even if these restrictions can be foreseen to get relaxed with time, such full hydrodynamical simulations will never be used in full stellar evolution models, as they would unnecessarily follow the star's evolution on a dynamical timescale, which is so much shorter than the dominant nuclear one. Therefore, we limit ourselves exclusively to the description of the old so-called "mixing-length" theory. The reason for this is not that we believe it to be sufficient, but it does provide at least a simple method for treating convection locally, at any given point of a star. Moreover, empirical tests of the resulting stellar models show a surprisingly good agreement with observations. And, finally, even this poor approximation shows without any doubt that in the very deep interior of a star, a detailed theory is normally not necessary.

Note that in the following we are dealing only with convection in stars that are in hydrostatic equilibrium. We furthermore assume that the convection is time independent, which means that it is fully adjusted to the present state of the star. Otherwise, a convection theory for rapidly changing regions (time-dependent convection) has to be developed.

Equation (5.28) gives the gradient $\nabla_{\rm rad}$ that would be maintained in a star if the whole luminosity l had to be transported outwards by radiation only. If convection contributes to the energy transport, the actual gradient ∇ will be different (namely smaller). It is the purpose of this section to estimate ∇ in the case of convection.

7.1 The Basic Picture

The mixing-length theory goes back to Ludwig Prandtl, who in 1925 modelled a simple picture of convection in complete analogy to molecular heat transfer: the transporting "particles" are macroscopic mass elements ("blobs") instead of molecules; their mean free path is the "mixing length" after which the blobs dissolve in their new surroundings. Prandtl's theory was adapted for stars by L. Biermann. There exist different variations and formulations of the mixing-length theory in the literature. Two widely used versions are those by Böhm-Vitense (1958) and Cox (see Weiss et al. 2004). We follow here the former one.

The total energy flux $l/4\pi r^2$ at a given point in the star consists of the radiative flux $F_{\rm rad}$ (in which the conductive flux may already be incorporated) plus the convective flux $F_{\rm con}$. Their sum defines according to (5.28) the gradient $\nabla_{\rm rad}$ that would be necessary to transport the whole flux by radiation:

$$F_{\rm rad} + F_{\rm con} = \frac{4acG}{3}\frac{T^4 m}{\kappa P r^2}\nabla_{\rm rad}. \tag{7.1}$$

However, part of the flux is transported by convection. If the actual gradient of the stratification is ∇, then the radiative flux is obviously only

$$F_{\rm rad} = \frac{4acG}{3}\frac{T^4 m}{\kappa P r^2}\nabla. \tag{7.2}$$

Note that ∇ is not yet known; in fact, we hope to obtain it as the result of this consideration. The first step is to derive an expression for $F_{\rm con}$.

Consider a convective element (a blob) with an excess temperature DT over its surroundings. It moves radially with velocity v and remains in complete balance of pressure, that is, $DP = 0$ [see (6.2) and Fig. 6.1]. This gives a local flux of convective energy

$$F_{\rm con} = \varrho v c_P DT, \tag{7.3}$$

which we can take immediately as the correct equation for the average convective flux, if we consider vDT replaced by the proper mean over the whole concentric sphere. One should be aware that this "proper mean" comprises most of the difficulties for a strict treatment. We adopt the following simple model.

All elements may have started their motion as very small perturbations only, that is, with initial values that can be approximated by $DT_0 = 0$ and $v_0 = 0$. Because of differences in temperature gradients and buoyancy forces, DT and v increase as the element rises (or sinks) until, after moving over a distance $\ell_{\rm m}$, the element mixes with the surroundings and loses its identity. $\ell_{\rm m}$ is called the *mixing length.* The elements passing at a given moment through a sphere of constant r will have different values of v and DT since they have started their motion at quite different distances, from zero to $\ell_{\rm m}$. We assume, therefore, that the "average" element has moved $\ell_{\rm m}/2$ when passing through the sphere. Then,

$$\begin{aligned} \frac{DT}{T} &= \frac{1}{T}\frac{\partial(DT)}{\partial r}\frac{\ell_{\rm m}}{2} \\ &= (\nabla - \nabla_{\rm e})\frac{\ell_{\rm m}}{2}\frac{1}{H_P}. \end{aligned} \tag{7.4}$$

The density difference [for $DP = D\mu = 0$, see (6.3) and (6.5)] is simply $D\varrho/\varrho = -\delta DT/T$ and the (radial) buoyancy force (per unit mass), $k_r = -g \cdot D\varrho/\varrho$. On average, half of this value may have acted on the element over the whole of its preceding motion ($\ell_{\rm m}/2$), such that the work done is

$$\frac{1}{2}k_r\frac{\ell_{\rm m}}{2} = g\delta(\nabla - \nabla_{\rm e})\frac{\ell_{\rm m}^2}{8H_P}. \tag{7.5}$$

Let us suppose that half of this work goes into the kinetic energy of the element ($v^2/2$ per unit mass), while the other half is transferred to the surroundings, which have to be "pushed aside". Then, we have for the average velocity v of the elements passing our sphere

$$v^2 = g\delta(\nabla - \nabla_{\rm e})\frac{\ell_{\rm m}^2}{8H_P}. \tag{7.6}$$

Inserting this and (7.4) into (7.3), we obtain for the average convective flux

$$F_{\mathrm{con}} = \varrho c_P T \sqrt{g\delta} \frac{\ell_{\mathrm{m}}^2}{4\sqrt{2}} H_P^{-3/2} (\nabla - \nabla_{\mathrm{e}})^{3/2} \ . \tag{7.7}$$

Finally, we shall consider the change of temperature T_{e} inside the element (diameter d, surface S, volume V) when it moves with velocity v. This change has two causes, one being the adiabatic expansion (or compression), and the other being the radiative exchange of energy with the surroundings. The total energy loss λ per unit time is given by (6.22); the corresponding temperature decrease per unit length over which the element rises is $\lambda/\varrho V c_P v$, and the total change per unit length is then

$$\left(\frac{dT}{dr}\right)_{\mathrm{e}} = \left(\frac{dT}{dr}\right)_{\mathrm{ad}} - \frac{\lambda}{\varrho V c_P v}. \tag{7.8}$$

Multiplying this by H_P/T, we have

$$\nabla_{\mathrm{e}} - \nabla_{\mathrm{ad}} = \frac{\lambda H_P}{\varrho V c_P v T} \ . \tag{7.9}$$

Here, λ may be replaced by (6.22), with the average DT given by (7.4). The resulting equation then contains a "form factor" $\ell_{\mathrm{m}} S/Vd$, which would be $6/\ell_{\mathrm{m}}$ for a sphere of diameter ℓ_{m}. In the literature, one often finds

$$\frac{\ell_{\mathrm{m}} S}{Vd} \approx \frac{9/2}{\ell_{\mathrm{m}}} \ , \tag{7.10}$$

which we will use in the following.

Equation (7.9), with the help of (6.22) and (7.10), then becomes

$$\frac{\nabla_{\mathrm{e}} - \nabla_{\mathrm{ad}}}{\nabla - \nabla_{\mathrm{e}}} = \frac{6acT^3}{\kappa \varrho^2 c_P \ell_{\mathrm{m}} v} \ . \tag{7.11}$$

Let us now summarize what we have achieved and describe what is still lacking. To start with the latter, we have obviously not yet used any physics that could *determine* the mixing length ℓ_{m}. Since we do not know a reasonable approach for this, we shall simply treat ℓ_{m} as a free parameter and make (more or less) plausible assumptions for its value (This is typical for all versions of the mixing-length approach and in fact also for many others that seem to be less arbitrary at a first glance.). In any case, the heat transfer mainly operates via the largest possible elements, and they can scarcely move over much more than their own diameter before differential forces destroy their identity.

Now, however, the prospect looks quite favourable: we have obtained the five equations (7.1), (7.2), (7.6), (7.7) and (7.11), which we can solve for the five quantities F_{rad}, F_{con}, v, ∇_{e}, and ∇, if the usual local quantities ($P, T, \varrho, l, m, c_P, \nabla_{\mathrm{ad}}, \nabla_{\mathrm{rad}}$, and g) are given.

7.2 Dimensionless Equations

For a simpler treatment of the five equations obtained from the mixing-length theory, we define two dimensionless quantities:

$$U := \frac{3acT^3}{c_P \varrho^2 \kappa \ell_m^2} \sqrt{\frac{8H_P}{g\delta}} , \tag{7.12}$$

$$W := \nabla_{rad} - \nabla_{ad} . \tag{7.13}$$

The meaning of U will become clear later; that of W is obvious. Note that both can be calculated immediately for any point in the star when the usual variables and the mixing length ℓ_m are given.

If v is eliminated with the help of (7.6), then (7.11) becomes

$$\nabla_e - \nabla_{ad} = 2U \sqrt{\nabla - \nabla_e}. \tag{7.14}$$

Eliminating F_{rad}, F_{con} from (7.1), (7.2) and (7.7) and using (2.4) and (6.8), we arrive at

$$(\nabla - \nabla_e)^{3/2} = \frac{8}{9} U(\nabla_{rad} - \nabla) . \tag{7.15}$$

We have thus replaced the set of five equations by the two equations (7.14) and (7.15) for ∇ and ∇_e, and we will now even reduce them to one final equation.

Rewriting the left-hand side of (7.14) as $(\nabla - \nabla_{ad}) - (\nabla - \nabla_e)$, one sees immediately that this is a quadratic equation for $(\nabla - \nabla_e)^{1/2}$ with the solution

$$\sqrt{\nabla - \nabla_e} = -U + \xi , \tag{7.16}$$

where ξ is a new variable given by the positive root of

$$\xi^2 = \nabla - \nabla_{ad} + U^2 . \tag{7.17}$$

In (7.15), we insert (7.16) on the left-hand side, eliminate ∇ on the right-hand side with (7.17), and obtain

$$(\xi - U)^3 + \frac{8U}{9}(\xi^2 - U^2 - W) = 0 . \tag{7.18}$$

So we have arrived at a cubic equation for ξ that can be solved for any given set of parameters U and W. It turns out that (7.18) has only one real solution. The resulting ξ, together with (7.17), then gives the decisive quantity ∇, that is, the average temperature gradient to which the layer settles in the presence of convection.

Other characteristic quantities of the convection are then also easily calculable, for example, the velocity v from (7.6) and (7.14).

We note for completeness that the cubic equation (7.18) should be solved numerically and not by the analytical formulae for the solution of third order equations, because the individual terms appearing therein can be many magnitudes larger than the root of the formula.

7.3 Limiting Cases, Solutions, Discussion

For a given difference $W = \nabla_{\mathrm{rad}} - \nabla_{\mathrm{ad}}$, the convection depends decisively on the value of U. Let us write (7.2) as $F_{\mathrm{rad}} = \sigma_{\mathrm{rad}}\nabla$, and (7.7) as $F_{\mathrm{con}} = \sigma_{\mathrm{con}}(\nabla - \nabla_{\mathrm{e}})^{3/2}$. Then, U, defined in (7.12), is essentially the ratio of the "conductivities": $\sigma_{\mathrm{rad}}/\sigma_{\mathrm{con}}$.

The dimensionless quantity U can also be written in terms of the time τ_{ff} it takes a mass element to fall freely over the distance H_P. With $\tau_{\mathrm{ff}} = (2H_P/g)^{1/2}$ and (6.25), we have

$$U \approx \frac{\tau_{\mathrm{ff}}}{\tau_{\mathrm{adj}}} \frac{d^2}{\ell_{\mathrm{m}}^2}\ , \tag{7.19}$$

where we have ignored a factor $3/(8\delta^{1/2})$, which is of order 1. One normally assumes that $\ell_{\mathrm{m}} \approx d$, and therefore, $U \approx \tau_{\mathrm{ff}}/\tau_{\mathrm{adj}}$.

The quantity U is also related to another dimensionless quantity Γ defined by

$$\Gamma := \frac{(\nabla - \nabla_{\mathrm{e}})^{1/2}}{2U} = \frac{\nabla - \nabla_{\mathrm{e}}}{\nabla_{\mathrm{e}} - \nabla_{\mathrm{ad}}}\ , \tag{7.20}$$

where we have made use of (7.14). Numerator and denominator have simple meanings as can easily be shown. For a roughly spherical convective element of radius $\ell_{\mathrm{m}}/2$, cross-section A, volume V, lifetime $\tau_l = \ell_{\mathrm{m}}/v$, and thermal energy $e_{\mathrm{th}} = \varrho V c_P T$, one finds from (7.3) and (7.4) that

$$\nabla - \nabla_{\mathrm{e}} = \frac{(F_{\mathrm{con}}A)\tau_l}{e_{\mathrm{th}}} \frac{4H_P}{3\ell_{\mathrm{m}}} \tag{7.21}$$

and from (7.9) that

$$\nabla_{\mathrm{e}} - \nabla_{\mathrm{ad}} = \frac{\lambda\tau_l}{e_{\mathrm{th}}} \frac{H_P}{\ell_{\mathrm{m}}}, \tag{7.22}$$

and therefore,

$$\Gamma = \frac{4}{3} \frac{F_{\mathrm{con}}A}{\lambda} \approx \frac{\text{energy transported}}{\text{energy lost}}\ . \tag{7.23}$$

For an average element, Γ gives the convective energy flowing through A relative to the radiative energy loss per second. It is a measure for the *efficiency of convection.* Large values of Γ (small U) are typical for very dense matter, where radiation losses are relatively unimportant compared to the convective flux. In regions of small density, however, the radiative losses can be so large that even very violent movements are ineffective for energy transport; the elements then lose nearly all of their excess heat through radiation to the surroundings, and cool down to $DT \approx 0$. In this case, Γ is very small (i.e. U is very large). The meaning of Γ can also be represented in terms of two typical timescales for the elements, namely, lifetime and adjustment time: in the second equation (6.25), replace DT by (7.4) and solve for $\nabla - \nabla_e$. This expression is then divided by (7.22) giving

$$\Gamma = \frac{\nabla - \nabla_e}{\nabla_e - \nabla_{ad}} = 2\frac{\tau_{adj}}{\tau_l} \; . \tag{7.24}$$

Let us consider the limiting cases for very large and very small U (or Γ). One should keep in mind that all gradients are finite; except for ∇_{rad}, they are all smaller than unity. And for the discussion in terms of Γ, one can easily rewrite (7.14) and (7.15) with the help of (7.20).

$\underline{U \to 0}$ (or $\underline{\Gamma \to \infty}$): Equation (7.14) gives $\nabla_e \to \nabla_{ad}$, and thus, (7.15) yields $\nabla \to \nabla_{ad}$. A negligible excess of ∇ over the adiabatic value is sufficient to transport the whole luminosity. This is the case in the very dense central part of a star. Here, we do not need to solve the mixing-length equations ($\nabla = \nabla_{ad}$ is known), and the uncertainties of this theory do not arise.

$\underline{U \to \infty}$ (or $\underline{\Gamma \to 0}$): In (7.15), the gradients on the left-hand side must be finite, and therefore on the right-hand side, $\nabla \to \nabla_{rad}$. Convection is ineffective and cannot transport a substantial fraction of the luminosity. Therefore, $F \to F_{rad}$, and the gradient ∇ is again known without further calculations. This is the case near the photosphere of a star.

The situation is difficult where the two limiting cases do not apply, for example, in the upper part of an outer convective envelope. There the equations of the mixing-length theory have to be solved, and they will yield a value for ∇ somewhere between ∇_{ad} and ∇_{rad}, the convection being said to be *superadiabatic*.

The following gives a more detailed discussion of the solutions of (7.18), which depend strongly on the (given) parameters U and W. We illustrate them in a diagram, where $\lg W$ is plotted over $\lg U$ (Fig. 7.1).

Instead of using the variable ξ, the solutions may be discussed in terms of the over-adiabaticity

$$x := \nabla - \nabla_{ad} = \xi^2 - U^2 \; , \tag{7.25}$$

which describes the gradient ∇ of the stratification relative to the (known) adiabatic gradient. With this definition, the cubic equation (7.18) is transformed to

$$\left[\sqrt{x + U^2} - U\right]^3 + \frac{8}{9}U(x - W) = 0 \; . \tag{7.26}$$

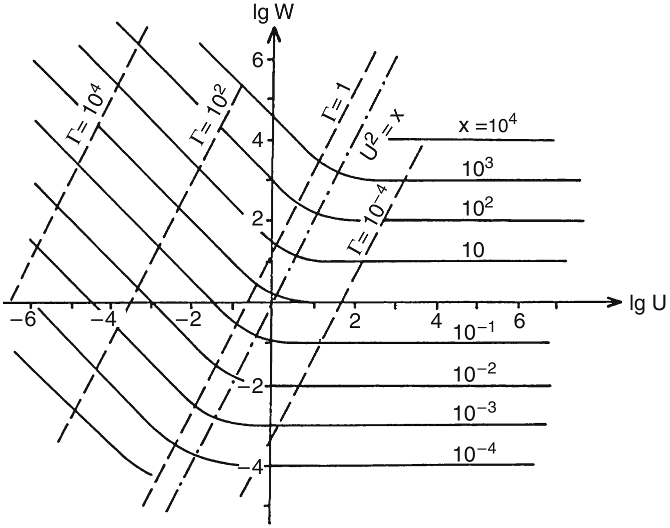

Fig. 7.1 The plane of the parameters U, W (on logarithmic scales) that determine the convection. The lines $x = \nabla - \nabla_{\mathrm{ad}}$ = constant are *solid*; the line where $U^2 = x$ is *dot-dashed*. Some lines Γ = constant are *dashed*

1. $\underline{\Gamma = 1}$: Let us first derive the line which separates the regimes of effective convection (at small U) and ineffective convection (at large U). Equation (7.20) for $\Gamma = 1$ is introduced into (7.16), which gives $\xi = 3U$ such that from (7.25), we have $x = 8U^2$. Inserting this into (7.26), we find the condition for $\Gamma = 1$ to be
$$W = 17\, U^2. \tag{7.27}$$
The corresponding straight line lg $W = 2$ lg $U + 1.23$ is shown by dashes in Fig. 7.1 (Lines for other values of Γ are obtained by a parallel shift.). We will now derive the lines on which x is constant. This is easily done by considering the following two limiting cases.
2. $\underline{U^2 \gg x}$: In (7.26), the term in square brackets on the left, divided by U, goes to zero, and one has
$$x = W\ . \tag{7.28}$$
Therefore, x = constant on straight lines parallel to the abscissa (right part of Fig. 7.1).
3. $\underline{U^2 \ll x}$: In (7.26), the term in square brackets goes to $x^{3/2} \gg Ux$, such that
$$x^{3/2} = \frac{8}{9} U W \tag{7.29}$$
and x = constant on the lines lg $W = -$ lg U + lg(9/8) + (3/2) lg x (left part of Fig. 7.1).

Finally, we derive the equation for the border between the regimes $U^2 \gg x$ and $U^2 \ll x$.

4. $\underline{U^2 = x}$: With this condition, (7.26) gives

$$W = U^2 \left[\frac{9}{8}(\sqrt{2} - 1)^3 + 1 \right] . \tag{7.30}$$

The corresponding straight line $\lg W = 2 \lg U + 0.033$ (dot-dashed line in Fig. 7.1) is below and parallel to that for $\Gamma = 1$.

The meaning of the different regions in Fig. 7.1 is now quite clear. Below and left of a line of sufficiently small x (say, $x = 10^{-2}$), we have nearly $\nabla = \nabla_{\text{ad}}$; above that line, the convection is superadiabatic. Not too far to the right of the line $\Gamma = 1$, the efficiency is so small that $\nabla \approx \nabla_{\text{rad}}$.

For an estimate for the interior of a star, let us assume a perfect monatomic gas with $\delta = \mu = 1, c_P/\Re = 5/2$ and a mixing-length $\ell_{\text{m}} = H_P$. For an average point in a star like the Sun, we may take $r = R_\odot/2, m = M_\odot/2, T = 10^7$ K, $\kappa = 1\,\text{cm}^2\,\text{g}^{-1}$ and $\varrho = 1\,\text{g}\,\text{cm}^{-3}$. Then, we obtain $U \approx 10^{-8}$, which is so far to the left in Fig. 7.1 that, for reasonable values of $W = \nabla_{\text{rad}} - \nabla_{\text{ad}}$ (say between 1 and 10^2), $\nabla - \nabla_{\text{ad}} \approx 10^{-5} \dots 10^{-4}$. For the central region of the Sun, ϱ and κ are larger by factors of 10^2 and 10, respectively. Then, $U \approx 10^{-13}$, and (for the same values of W) the difference $\nabla - \nabla_{\text{ad}}$ is even smaller by a factor 10^3 or more, that is, $< 10^{-7}$. The stratification of such convective zones is indeed very close to an adiabatic one, and we can simply set $\nabla = \nabla_{\text{ad}}$, independent of the uncertainties of the theory (The situation is difficult only near the interface between convective and radiative zones, where one should have a smooth transition between the two modes of transport.).

Convective elements in such dense layers are so effective ($\Gamma \approx 10^6 \dots 10^9$) that they can transport the whole luminosity with surprisingly little effort. Compared with the surroundings, they only need very small excesses of the T gradient, $D(dT/dr) \approx 10^{-12} \dots 10^{-10}\,\text{K cm}^{-1}$, and an average temperature excess $DT \approx 10^{-2} \dots 1$ K; their velocities are typically $v \approx 1 \dots 100\,\text{m}\,\text{s}^{-1}$ (which is $10^{-6} \dots 10^{-4}$ times the velocity of sound), and their lifetime is between 1 and 10^2 days.

The Reynolds number decides whether the flow of an incompressible viscous fluid is turbulent or laminar (Landau, Lifshitz, vol. 6, 1987). It is defined as

$$Re = \frac{v \varrho \ell_{\text{m}}}{\eta}. \tag{7.31}$$

Here, η is the viscosity of the fluid and ℓ_{m} and v are the typical distance elements travel and their velocity. For high Reynolds numbers, the flow is turbulent. In spite of the small velocities of convective elements, the Reynolds number is $\gg 1$, since the flow extends over such a large distance ℓ_{m}. The situation is quite different for convection near the surface of the star, where the density is low. This gives small effectivity and positive lg U. Here, the cubic equation for ξ (or x) has to be solved for each point to find the proper ∇ for that place, and the results are affected by the uncertainties of the theory.

In any case, we use the resulting value of ∇ in the transport equation written in the form

$$\frac{dT}{dm} = -\frac{T}{P}\frac{Gm}{4\pi r^4}\nabla \ . \tag{7.32}$$

(Here, we have replaced *dP/dm* by the right-hand side of the hydrostatic equation since the theory is suitable only for hydrostatic equilibrium.) For convection in the very deep interior, $\nabla = \nabla_{\rm ad}$, where $\nabla_{\rm ad}$ is given by (4.21), while for envelope convection, we take ∇ as given by the solution of the mixing-length theory. And we can even take the same equation (7.32) for transport by radiation, if we set $\nabla = \nabla_{\rm rad}$ (compare Sect. 5.2).

Aside from the more or less effective (and more or less well-determined) transport of energy, turbulent convection, if it occurs, has a side effect that is important for the life of the star: it mixes the stellar matter very thoroughly and rapidly compared to other relevant timescales, and thus, it contributes directly to the long-lasting chemical record of the star's history.

7.4 Extensions of the Mixing-Length Theory

The mixing-length theory, as described above, has many open and hidden assumptions. Most prominent is the mixing length itself, usually expressed as the "mixing length parameter" $\alpha_{\rm MLT}$, which is the mixing-length in units of the pressure scale height H_P. It is generally assumed that $\alpha_{\rm MLT}$ is both constant within a star and does vary neither with stellar mass, composition, nor with evolutionary stage. Its value is not known better than that from general physical arguments, it should be of order 1. To determine a reasonable numerical value, a comparison of the effective temperature or radius of stellar models with observed stars is done, preferentially in the case of the Sun. This yields values for $\alpha_{\rm MLT}$ between 1.5 and 2.0. Ludwig et al. (1999) have done a comparison with numerical simulations of convection and found only a weak dependence of the order of 20 % on stellar parameters for stars of solar metallicity.

But there are even more hidden parameters and assumptions. The theory contains, for example, several mean values entering the equations from (7.4) on. For (7.10), we assumed a certain geometrical form of the blobs to obtain the ratio of surface to volume. Different formulations of the mixing-length theory may make different assumptions for all this. Therefore, the mixing-length parameter may not be directly comparable between such different formulations.

The most basic limitation in all these variants of the theory is the assumption of one single size (and form) for the convective elements. The theory of turbulence, numerical simulations, laboratory experiments and astrophysical observations all show that this is certainly not the case. Instead, numerical simulations and helioseismology showed that convection often operates by extended funnel-like downdrafts and turbulent updrafts. Convective energy is thus realistically not transported in laminar flows of blobs of identical size and energy content, but by turbulent elements ("eddies") of all sizes. F_{con} can therefore not be calculated as in (7.3) but rather results from an integration over the full spectrum of convective eddies.

Canuto and Mazzitelli (1991) developed and presented an extension of the mixing-length theory, in which the full turbulent kinetic energy spectrum is taken into account. This "full spectrum turbulence" theory (FST) can be formulated in much the same way as the mixing-length theory, which is the limiting case for a δ-function like energy spectrum. It also results in a cubic equation to be solved. Canuto and Mazzitelli use the formulation of Cox and Giuli (Weiss et al. 2004, Chap. 14, and eq. 14.82); in this formulation, the cubic equation reads

$$\frac{9}{4}\Gamma'^3 + \Gamma'^2 + \Gamma' - \frac{1}{U^2}(\nabla_{\mathrm{rad}} - \nabla_{\mathrm{ad}}) = 0, \tag{7.33}$$

where we have replaced already some terms by quantities of our own formulation. Γ' is defined as

$$2\Gamma' + 1 = \left(1 + \frac{\nabla - \nabla_{\mathrm{ad}}}{U^2}\right)^{1/2} \tag{7.34}$$

and corresponds to the convective efficiency.

After modelling the convective flux in the FST model, (7.33) is modified by multiplying the Γ'^3-term with a function $\Omega(\Gamma')$, which is the new turbulent convective flux relative to that of the mixing-length theory. Ω rises monotonically from 0 to 1 for Γ' going from $\Gamma' \approx 0$ to $\Gamma' \to \infty$ and can be approximated by an analytical fitting function.

As a rule of thumb convective fluxes in this theory are larger than predicted by the mixing-length theory in case of efficient convection, and superadiabatic regions are narrower but more superadiabatic. The temperature gradient in the solar convection zone predicted by the FST model agrees much better with that obtained in numerical simulations than it does in the mixing-length case. The numerical value for the mixing length parameter in this case is around 0.7.