Chapter 16
Infinitely Divisible Distributions

For every n, the normal distribution $\mathcal{N}_{\mu,\sigma^2}$ is the nth convolution power of a probability measure (namely, of $\mathcal{N}_{\mu/n,\sigma^2/n}$). This property is called infinite divisibility and is shared by other probability distributions such as the Poisson distribution and the Gamma distribution. In the first section, we study which probability measures on $\mathbb{R}$ are infinitely divisible and give an exhaustive description of this class of distributions by means of the Lévy–Khinchin formula.

Unlike the Poisson distribution, the normal distribution is the limit of *rescaled* sums of i.i.d. random variables (central limit theorem). In the second section, we investigate briefly which subclass of the infinitely divisible measures on $\mathbb{R}$ shares this property.

16.1 Lévy–Khinchin Formula

For the sake of brevity, in this section, we use the shorthand "CFP" for "characteristic function of a probability measure on $\mathbb{R}$".

Definition 16.1 A measure $\mu \in \mathcal{M}_1(\mathbb{R})$ is called *infinitely divisible* if, for every $n \in \mathbb{N}$, there is a $\mu_n \in \mathcal{M}_1(\mathbb{R})$ such that $\mu_n^{*n} = \mu$. Analogously, a CFP φ is called infinitely divisible if, for every $n \in \mathbb{N}$, there is a CFP φ_n such that $\varphi = \varphi_n^n$. A real random variable X is called infinitely divisible if, for every $n \in \mathbb{N}$, there exist i.i.d. random variables $X_{n,1}, \ldots, X_{n,n}$ such that $X \stackrel{\mathcal{D}}{=} X_{n,1} + \ldots + X_{n,n}$.

Manifestly, all three notions of infinite divisibility are equivalent, and we will use them synonymously. Note that the uniqueness of μ_n and φ_n, respectively, is by no means evident. Indeed, n-fold divisibility alone does not imply uniqueness of the nth convolution root $\mu^{*1/n} := \mu_n$ or of φ_n, respectively. As an example for even n, choose a real-valued CFP φ for which $|\varphi| \neq \varphi$ is also a CFP (see Examples 15.16 and 15.17). Then $\varphi^n = |\varphi|^n$ is n-fold divisible; however, the factors are not unique.

By virtue of Lévy's continuity theorem, one can show that (see Exercise 16.1.1) $\varphi(t) \neq 0$ for all $t \in \mathbb{R}$ if φ is infinitely divisible. The probabilistic meaning of this

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fact is that as a continuous function $\log(\varphi(t))$ is uniquely defined and thus there exists only one continuous function $\varphi^{1/n} = \exp(\log(\varphi)/n)$. The nth convolution roots are thus unique if the distribution is *infinitely* divisible.

Example 16.2

(i) δ_x is infinitely divisible with $\delta_{x/n}^{*n} = \delta_x$ for every $n \in \mathbb{N}$.
(ii) The normal distribution is infinitely divisible with $\mathcal{N}_{m,\sigma^2} = \mathcal{N}_{m/n,\sigma^2/n}^{*n}$.
(iii) The Cauchy distribution Cau_a with density $x \mapsto (a\pi)^{-1}\big(1+(x/a)^2\big)^{-1}$ is infinitely divisible with $\mathrm{Cau}_a = \mathrm{Cau}_{a/n}^{*n}$. Indeed, Cau_a has CFP $\varphi_a(t) = e^{-a|t|}$; hence $\varphi_{a/n}^n = \varphi_a$.
(iv) Every symmetric stable distribution with index $\alpha \in (0,2]$ and scale parameter $\gamma > 0$ (that is, the distribution with CFP $\varphi_{\alpha,\gamma}(t) = e^{-|\gamma t|^\alpha}$) is infinitely divisible. Indeed, $\varphi^n_{\alpha,\gamma/n^{1/\alpha}} = \varphi_{\alpha,\gamma}$. (To be precise, we have shown only for $\alpha \in (0,1]$ (in Corollary 15.25) and for $\alpha = 2$ (normal distribution) that $\varphi_{\alpha,\gamma}$ is in fact a CFP. In Section 16.2, we will show that this is true for all $\alpha \in (0,2]$. For $\alpha > 2$, $\varphi_{\alpha,\gamma}$ is not a CFP, see Exercise 15.4.3.)
(v) The Gamma distribution $\Gamma_{\theta,r}$ with CFP $\varphi_{\theta,r}(t) = \exp(r\psi_\theta(t))$, where $\psi_\theta(t) = \log(1 - it/\theta)$, is infinitely divisible with $\Gamma_{\theta,r} = \Gamma_{\theta,r/n}^{*n}$.
(vi) The Poisson distribution is infinitely divisible with $\mathrm{Poi}_\lambda = \mathrm{Poi}_{\lambda/n}^{*n}$.
(vii) The negative binomial distribution $b^-_{r,p}(\{k\}) = \binom{-r}{k}(-1)^k p^r (1-p)^k$, $k \in \mathbb{N}_0$, with parameters $r > 0$ and $p \in (0,1)$, is infinitely divisible with $b^-_{r,p} = (b^-_{r/n,p})^{*n}$. Indeed, $\varphi_{r,p}(t) = e^{r\psi_p(t)}$, where

$$\psi_p(t) = \log(p) - \log\big(1-(1-p)e^{it}\big).$$

(viii) Let X and Y be independent with $X \sim \mathcal{N}_{0,\sigma^2}$ and $Y \sim \Gamma_{\theta,r}$, where $\sigma^2, \theta, r > 0$. It can be shown that the random variable $Z := X/\sqrt{Y}$ is infinitely divisible (see [65] or [131]). In particular, Student's t-distribution with $k \in \mathbb{N}$ degrees of freedom is infinitely divisible (this is the case where $\sigma^2 = 1$ and $\theta = r = k/2$).
(ix) The binomial distribution $b_{n,p}$ with parameters $n \in \mathbb{N}$ and $p \in (0,1)$ is *not* infinitely divisible (why?).
(x) Somewhat more generally, there is no nontrivial infinitely divisible distribution that is concentrated on a bounded interval. ◊

A main goal of this section is to show that every infinitely divisible distribution can be composed of three generic ones:

- the Dirac measures δ_x with $x \in \mathbb{R}$,
- the normal distributions $\mathcal{N}_{\mu,\sigma^2}$ with $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, and
- (limits of) convolutions of Poisson distributions.

As convolutions of Poisson distributions play a special role, we will consider them separately.

If $\nu \in \mathcal{M}_1(\mathbb{R})$ with CFP φ_ν and if $\lambda > 0$, then one can easily check that $\varphi(t) = \exp(\lambda(\varphi_\nu(t) - 1))$ is the CFP of $\mu_\lambda = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \nu^{*k}$. Hence, formally we can write $\mu_\lambda = e^{*\lambda(\nu - \delta_0)}$. Indeed, μ_λ is infinitely divisible with $\mu_\lambda = \mu_{\lambda/n}^{*n}$. We want to combine the two parameters λ and ν into one parameter $\lambda\nu$. For $\nu \in \mathcal{M}_f(\mathbb{R})$, we can define $\nu^{*n} = \nu(\mathbb{R})^n (\nu/\nu(\mathbb{R}))^{*n}$ (and $\nu^{*n} = 0$ if $\nu = 0$). In both cases, let $\nu^{*0} := \delta_0$. Hence we make the following definition.

Definition 16.3 The *compound Poisson distribution* with intensity measure $\nu \in \mathcal{M}_f(\mathbb{R})$ is the following probability measure on $\mathbb{R}$:

$$\mathrm{CPoi}_\nu := e^{*(\nu - \nu(\mathbb{R})\delta_0)} := e^{-\nu(\mathbb{R})} \sum_{n=0}^{\infty} \frac{\nu^{*n}}{n!}.$$

The CFP of CPoi_ν is given by

$$\varphi_\nu(t) = \exp\left(\int \left(e^{itx} - 1\right) \nu(dx) \right). \tag{16.1}$$

In particular, $\mathrm{CPoi}_{\mu+\nu} = \mathrm{CPoi}_\mu * \mathrm{CPoi}_\nu$; hence CPoi_ν is infinitely divisible.

Example 16.4 For every measurable set $A \subset \mathbb{R} \setminus \{0\}$ and every $r > 0$,

$$r^{-1}\mathrm{CPoi}_{r\nu}(A) = e^{-r\nu(\mathbb{R})}\nu(A) + e^{-r\nu(\mathbb{R})} \sum_{k=2}^{\infty} \frac{r^{k-1}\nu^{*k}(A)}{k!} \xrightarrow{r \downarrow 0} \nu(A).$$

We use this in order to show that $b_{r,p}^- = \mathrm{CPoi}_{r\nu}$ for some $\nu \in \mathcal{M}_f(\mathbb{N})$. To this end, for $k \in \mathbb{N}$, we compute

$$r^{-1} b_{r,p}^-\big(\{k\}\big) = \frac{r(r+1)\cdots(r+k-1)}{rk!} p^r (1-p)^k \xrightarrow{r \downarrow 0} \frac{(1-p)^k}{k}.$$

If we had $b_{r,p}^- = \mathrm{CPoi}_{r\nu}$ for some $\nu \in \mathcal{M}_f(\mathbb{N})$, then we would have $\nu(\{k\}) = (1-p)^k/k$. We compute the CFP of $\mathrm{CPoi}_{r\nu}$ for this ν,

$$\varphi_{r\nu}(t) = \exp\left(r \sum_{k=1}^{\infty} \frac{\left((1-p)e^{it}\right)^k - (1-p)^k}{k} \right) = p^r \left(1 - (1-p)e^{it}\right)^{-r}.$$

However, this is the CFP of $b_{r,p}^-$; hence indeed $b_{r,p}^- = \mathrm{CPoi}_{r\nu}$. ◊

Not every infinitely divisible distribution is of the type CPoi_ν, however we have the following theorem.

Theorem 16.5 *A probability measure μ on $\mathbb{R}$ is infinitely divisible if and only if there is a sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_f(\mathbb{R} \setminus \{0\})$ such that* $\mathrm{CPoi}_{\nu_n} \xrightarrow{n\to\infty} \mu$.

Since every CPoi_{ν_n} is infinitely divisible, on the one hand we have to show that this property is preserved under weak limits. On the other hand, we show that, for infinitely divisible μ, the sequence $\nu_n = n\mu^{*1/n}$ does the trick. We prepare for the proof of Theorem 16.5 with a further theorem.

Theorem 16.6 *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of CFPs. Then the following are equivalent.*

(i) *For every $t \in \mathbb{R}$, the limit $\varphi(t) = \lim_{n\to\infty} \varphi_n^n(t)$ exists and φ is continuous at* 0.
(ii) *For every $t \in \mathbb{R}$, the limit $\psi(t) = \lim_{n\to\infty} n(\varphi_n(t) - 1)$ exists and ψ is continuous at* 0.

If (i) *and* (ii) *hold, then $\varphi = e^{\psi}$ is a CFP.*

Proof The proof is based on a Taylor expansion of the logarithm,

$$\big|\log(z) - (z-1)\big| \le \frac{1}{2}|z-1|^2 \quad \text{for } z \in \mathbb{C} \text{ with } |z-1| < \frac{1}{2}.$$

In particular, for $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$,

$$\limsup_{n\to\infty} n|z_n - 1| < \infty \quad \Longleftrightarrow \quad \limsup_{n\to\infty} \big|n\log(z_n)\big| < \infty, \tag{16.2}$$

and $\lim_{n\to\infty} n(z_n - 1) = \lim_{n\to\infty} n\log(z_n)$ if one of the limits exists.

Applying this to $z_n = \varphi_n(t)$, we see that (ii) implies (i). On the other hand, (i) implies (ii) if $\liminf_{n\to\infty} n\log(|\varphi_n(t)|) > -\infty$ and hence if $\varphi(t) \ne 0$ for all $t \in \mathbb{R}$.

Since φ is continuous at 0 and since $\varphi(0) = 1$, there is an $\varepsilon > 0$ with $|\varphi(t)| > \frac{1}{2}$ for all $t \in [-\varepsilon, \varepsilon]$. Since φ and φ_n are CFPs, $|\varphi|^2$ and $|\varphi_n|^2$ are also CFPs. Thus, since $|\varphi_n(t)|^{2n}$ converges to $|\varphi(t)|^2$ pointwise, Lévy's continuity theorem implies uniform convergence on compact sets. Now apply (16.2) with $z_n = |\varphi_n(t)|^2$. Thus $(n(1 - |\varphi_n(t)|^2))_{n\in\mathbb{N}}$ is bounded for $t \in [-\varepsilon, \varepsilon]$. Hence, by Lemma 15.11(v), $n(1 - |\varphi_n(2t)|^2) \le 4n(1 - |\varphi_n(t)|^2)$ also is bounded; thus

$$\big|\varphi(2t)\big|^2 \ge \liminf_{n\to\infty} \exp\big(4n\big(\big|\varphi_n(t)\big|^2 - 1\big)\big) = \big(\big|\varphi(t)\big|^2\big)^4.$$

Inductively, we get $|\varphi(t)| \ge 2^{-(4^k)}$ for $|t| \le 2^k\varepsilon$. Hence there is a $\gamma > 0$ such that

$$\big|\varphi(t)\big| > \frac{1}{2}e^{-\gamma t^2} \quad \text{for all } t \in \mathbb{R}. \tag{16.3}$$

If (i) and (ii) hold, then

$$\log\varphi(t) = \lim_{n\to\infty} n\log\big(\varphi_n(t)\big) = \lim_{n\to\infty} n\big(\varphi_n(t) - 1\big) = \psi(t).$$

By Lévy's continuity theorem, as a continuous limit of CFPs, φ is a CFP. □

Corollary 16.7 *If the conditions of Theorem 16.6 hold, then φ^r is a CFP for every $r > 0$. In particular, $\varphi = (\varphi^{1/n})^n$ is infinitely divisible.*

Proof If φ_n is the CFP of $\mu_n \in \mathcal{M}_1(\mathbb{R})$, then $e^{rn(\varphi_n-1)}$ is the CFP of $\mathrm{CPoi}_{rn\mu_n}$. Being a limit of CFPs that is continuous at 0, by Lévy's continuity theorem, $\varphi^r = e^{r\psi} = \lim_{n\to\infty} e^{rn(\varphi_n-1)}$ is a CFP. Letting $r = \frac{1}{n}$, we get that $\varphi = (\varphi^{1/n})^n$ is infinitely divisible. $\square$

Corollary 16.8 *Let $\varphi : \mathbb{R} \to \mathbb{C}$ be continuous at* 0. *φ is an infinitely divisible CFP if and only if there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of CFPs such that $\varphi_n^n(t) \to \varphi(t)$ for all $t \in \mathbb{R}$.*

Proof One implication has been shown already in Corollary 16.7. Hence, let φ be an infinitely divisible CFP. Then $\varphi_n = \varphi^{1/n}$ serves the purpose. $\square$

Corollary 16.9 *If $(\mu_n)_{n\in\mathbb{N}}$ is a (weakly) convergent sequence of infinitely divisible probability measures on $\mathbb{R}$, then $\mu = \lim_{n\to\infty} \mu_n$ is infinitely divisible.*

Proof Apply Theorem 16.6, where φ_n is the CFP of $\mu_n^{*1/n}$. $\square$

Corollary 16.10 *If $\mu \in \mathcal{M}_1(\mathbb{R})$ is infinitely divisible, then there exists a continuous convolution semigroup $(\mu_t)_{t\ge 0}$ with $\mu_1 = \mu$ and a stochastic process $(X_t)_{t\ge 0}$ with independent, stationary increments $X_t - X_s \sim \mu_{t-s}$ for $t > s$.*

Proof Let φ be the CFP of μ. The existence of the convolution semigroup follows by Corollaries 16.8 and 16.7 if we define μ_r by φ^r. Since $\varphi^r(t) \neq 0$ for all $t \in \mathbb{R}$, we have $\varphi^r \to 1$ for $r \to 0$ and thus the semigroup is continuous. Finally, Theorem 14.47 implies the existence of the process X. $\square$

Corollary 16.11 *If φ is an infinitely divisible CFP, then there exists a $\gamma > 0$ with $|\varphi(t)| \ge \frac{1}{2} e^{-\gamma t^2}$ for all $t \in \mathbb{R}$. In particular, for $\alpha > 2$, $t \mapsto e^{-|t|^\alpha}$ is not a CFP.*

Proof This is a direct consequence of (16.3). $\square$

Proof of Theorem 16.5 As every CPoi_{ν_n} is infinitely divisible, by Corollary 16.9, the weak limit is also infinitely divisible.

Now let μ be infinitely divisible with CFP φ. Fix probability measures μ_n with CFP φ_n as in Corollary 16.8. By Theorem 16.6, $e^{n(\varphi_n-1)} \overset{n\to\infty}{\longrightarrow} \varphi$; hence we have $\mathrm{CPoi}_{n\mu_n} \overset{n\to\infty}{\longrightarrow} \nu$. $\square$

Without proof, we quote the following strengthening of Corollary 16.8 that relies on a finer analysis using the arguments from the proof of Theorem 16.6.

Theorem 16.12 *Let $(\varphi_{n,l};\, l = 1, \ldots, k_n, n \in \mathbb{N})$ be an array of CFPs with the property*

$$\sup_{L>0} \limsup_{n\to\infty} \sup_{t\in[-L,L]} \sup_{l=1,\ldots,k_n} \left|\varphi_{n,l}(t) - 1\right| = 0. \tag{16.4}$$

Assume that, for every $t \in \mathbb{R}$, the limit $\varphi(t) := \lim_{n\to\infty} \prod_{l=1}^{k_n} \varphi_{n,l}(t)$ exists and that φ is continuous at 0. Then φ is an infinitely divisible CFP.

Proof See, e.g., [54, Chapter XV.7]. □

In the special case where for every n, the individual $\varphi_{n,l}$ are equal and where $k_n \overset{n\to\infty}{\longrightarrow} \infty$, Eq. (16.4) holds automatically if the product converges to a continuous function. Thus, the theorem is in fact an improvement of Corollary 16.8.

The benefit of this theorem will become clear through the following observation. Let $(X_{n,l};\, l = 1, \ldots, k_n, n \in \mathbb{N})$ be an array of real random variables with CFPs $\varphi_{n,l}$. This array is a null array if and only if (16.4) holds. In fact, if $\mathbf{P}[|X_{n,l}| > \varepsilon] < \delta$, then we have $|\varphi_{n,l}(t) - 1| \le 2\varepsilon + \delta$ for all $t \in [-1/\varepsilon, 1/\varepsilon]$. Hence (16.4) holds if the array $(X_{n,l})$ is a null array. On the other hand, (16.4) implies $\varphi_{n,l_n} \overset{n\to\infty}{\longrightarrow} 1$ for every sequence (l_n) with $l_n \le k_n$. Hence $X_{n,l_n} \overset{n\to\infty}{\longrightarrow} 0$ in probability.

From these considerations and from Theorem 16.12, we conclude the following theorem.

Theorem 16.13 *Let $(X_{n,l};\, l = 1, \ldots, k_n, n \in \mathbb{N})$ be an independent null array of real random variables. If there exists a random variable S with*

$$X_{n,1} + \ldots + X_{n,k_n} \quad \overset{n\to\infty}{\Longrightarrow} \quad S,$$

then S is infinitely divisible.

As a direct application of Theorem 16.5, we give a complete description of the class of infinitely divisible probability measures on $[0, \infty)$ in terms of their Laplace transforms. The following theorem is of independent interest. Here, however, it is primarily used to provide familiarity with the techniques that will be needed for the more challenging classification of the infinitely divisible probability measures on $\mathbb{R}$.

Theorem 16.14 (Lévy–Khinchin formula on $[0, \infty)$) *Let $\mu \in \mathcal{M}_1([0, \infty))$ and let $u : [0, \infty) \to [0, \infty)$, $t \mapsto -\log \int e^{-tx} \mu(dx)$ be the* log-*Laplace transform μ. μ is infinitely divisible if and only if there exists an $\alpha \ge 0$ and a σ-finite measure $\nu \in \mathcal{M}((0, \infty))$ with*

$$\int (1 \wedge x)\nu(dx) < \infty \tag{16.5}$$

and such that

$$u(t) = \alpha t + \int \left(1 - e^{-tx}\right)\nu(dx) \quad \textit{for } t \ge 0. \tag{16.6}$$

In this case, the pair (α, ν) is unique. ν is called the canonical measure or Lévy measure of μ, and α is called the deterministic part.

Proof "$\Longrightarrow$" First assume μ is infinitely divisible. The case $\mu = \delta_0$ is trivial. Now let $\mu \neq \delta_0$; hence $u(1) > 0$.

By Theorem 16.5, there exist $\nu_1, \nu_2, \ldots \in \mathcal{M}_f(\mathbb{R} \setminus \{0\})$ with $\mathrm{CPoi}_{\nu_n} \overset{n\to\infty}{\longrightarrow} \mu$. Evidently, we can assume $\nu_n((-\infty, 0)) = 0$. If we define $u_n(t) := \int (1 - e^{-tx})\nu_n(dx)$, then (as in (16.1)) $u_n(t) \overset{n\to\infty}{\longrightarrow} u(t)$ for all $t \geq 0$. In particular, $u_n(1) > 0$ for sufficiently large n. Define $\tilde{\nu}_n \in \mathcal{M}_1([0,\infty))$ by $\tilde{\nu}_n(dx) := \frac{1-e^{-x}}{u_n(1)}\nu_n(dx)$. Hence, for all $t \geq 0$,

$$\int e^{-tx}\tilde{\nu}_n(dx) = \frac{u_n(t+1) - u_n(t)}{u_n(1)} \overset{n\to\infty}{\longrightarrow} \frac{u(t+1) - u(t)}{u(1)}.$$

Therefore, the weak limit $\tilde{\nu} := \text{w-lim}\,\tilde{\nu}_n$ in $\mathcal{M}_1([0,\infty))$ exists and is uniquely determined by u. Let $\alpha := \tilde{\nu}(\{0\})u(1)$ and define $\nu \in \mathcal{M}((0,\infty))$ by

$$\nu(dx) = u(1)\big(1 - e^{-x}\big)^{-1}\mathbb{1}_{(0,\infty)}(x)\tilde{\nu}(dx).$$

Since $1 \wedge x \leq 2(1 - e^{-x})$ for all $x \geq 0$, clearly

$$\int (1 \wedge x)\nu(dx) \leq 2\int \big(1 - e^{-x}\big)\nu(dx) \leq 2u(1) < \infty.$$

For all $t \geq 0$, the function (compare (15.8))

$$f_t : [0,\infty) \to [0,\infty), \qquad x \mapsto \begin{cases} \frac{1-e^{-tx}}{1-e^{-x}}, & \text{if } x > 0, \\ t, & \text{if } x = 0, \end{cases}$$

is continuous and bounded (by $t \wedge 1$). Hence we have

$$\begin{aligned} u(t) &= \lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} u_n(1)\int f_t\, d\tilde{\nu}_n \\ &= u(1)\int f_t\, d\tilde{\nu} = \alpha t + \int \big(1 - e^{-tx}\big)\nu(dx). \end{aligned}$$

"$\Longleftarrow$" Now assume that α and ν are given. Define the intervals $I_0 = [1,\infty)$ and $I_k = [1/(k+1), 1/k)$ for $k \in \mathbb{N}$. Let $X_0, X_1, \ldots$ be independent random variables with $\mathbf{P}_{X_k} = \mathrm{CPoi}_{(\nu|_{I_k})}$ for $k = 0, 1, \ldots$, and let $X := \alpha + \sum_{k=0}^{\infty} X_k$. For every $k \in \mathbb{N}$, we have $\mathbf{E}[X_k] = \int_{I_k} x\,\nu(dx)$; hence $\sum_{k=1}^{\infty}\mathbf{E}[X_k] = \int_{(0,1)} x\,\nu(dx) < \infty$. Thus $X < \infty$ almost surely and $\alpha + \sum_{k=0}^{n} X_k \overset{n\to\infty}{\Longrightarrow} X$. Therefore,

$$-\log \mathbf{E}\big[e^{-tX}\big] = \alpha t - \sum_{k=0}^{\infty} \log \mathbf{E}\big[e^{-tX_k}\big] = \alpha t + \int \big(1 - e^{-tx}\big)\nu(dx). \qquad \square$$

Example 16.15 For an infinitely divisible distribution μ on $[0,\infty)$, we can compute the Lévy measure ν by the vague limit

$$\nu = \underset{n\to\infty}{\text{v-lim}}\, n\mu^{*1/n}\big|_{(0,\infty)}. \tag{16.7}$$

Often α is also easy to obtain (e.g., via the representation from Exercise 16.1.3). For example, for the Gamma distribution, we get $\alpha = 0$ and

$$n\Gamma_{\theta,1/n}(A) = \frac{\theta^{1/n}}{\Gamma(1/n)/n} \int_A x^{(1/n)-1} e^{-\theta x}\, dx \overset{n\to\infty}{\longrightarrow} \int_A x^{-1} e^{-\theta x}\, dx,$$

hence $\nu(dx) = x^{-1} e^{-\theta x}\, dx$. $\Diamond$

For infinitely divisible distributions on $\mathbb{R}$, we would like to obtain a description similar to that in the preceding theorem. However, an infinitely divisible real random variable X is not simply the difference of two infinitely divisible nonnegative random variables, as the normal distribution shows. In addition, we have more freedom if, as in the last proof, we want to express X as a sum of independent random variables X_k.

Hence we define the real random variable X as the sum of independent random variables,

$$X = b + X^N + X_0 + \sum_{k=1}^{\infty} (X_k - \alpha_k), \tag{16.8}$$

where $b \in \mathbb{R}$, $X^N = \mathcal{N}_{0,\sigma^2}$ for some $\sigma^2 \ge 0$ and $\mathbf{P}_{X_k} = \mathrm{CPoi}_{\nu_k}$ with intensity measure ν_k that is concentrated on $I_k := (-1/k, -1/(k+1)] \cup [1/(k+1), 1/k)$ (with the convention $1/0 = \infty$), $k \in \mathbb{N}_0$. Furthermore, $\alpha_k = \mathbf{E}[X_k] = \int x\,\nu_k(dx)$ for $k \ge 1$. In order for the series to converge almost surely, it is sufficient (and also necessary, as a simple application of Kolmogorov's three-series theorem shows) that

$$\sum_{k=1}^{\infty} \mathbf{Var}[X_k] < \infty. \tag{16.9}$$

(In contrast to the situation in Theorem 16.14, here it is not necessary to have $\sum_{k=1}^{\infty} \mathbf{E}[|X_k - \alpha_k|] < \infty$. This allows for greater freedom in the choice of ν than in the case of nonnegative random variables.) Now $\mathbf{Var}[X_k] = \int x^2 \nu_k(dx)$. Hence, if we let $\nu = \sum_{k=0}^{\infty} \nu_k$, then (16.9) is equivalent to $\int_{(-1,1)} x^2 \nu(dx) < \infty$. As ν_0 is always finite, this in turn is equivalent to $\int (x^2 \wedge 1)\nu(dx) < \infty$.

Definition 16.16 A σ-finite measure ν on $\mathbb{R}$ is called a *canonical measure* if $\nu(\{0\}) = 0$ and

$$\int \big(x^2 \wedge 1\big)\nu(dx) < \infty. \tag{16.10}$$

If $\sigma^2 \ge 0$ and $b \in \mathbb{R}$, then (σ^2, b, ν) is called a *canonical triple*.

To every canonical triple, by (16.8) there corresponds an infinitely divisible random variable. Define

$$\psi_0(t) = \log \mathbf{E}\big[e^{itX_0}\big] = \int_{I_0} \big(e^{itx} - 1\big)\nu(dx).$$

For $k \in \mathbb{N}$, let

$$\psi_k(t) = \log \mathbf{E}\big[e^{it(X_k-\alpha_k)}\big] = \int_{I_k} \big(e^{itx} - 1 - itx\big)\nu(dx).$$

Hence

$$\psi(t) := \log \mathbf{E}\big[e^{itX}\big] = -\frac{\sigma^2}{2}t^2 + ibt + \sum_{k=0}^{\infty} \psi_k(t)$$

satisfies the Lévy–Khinchin formula

$$\psi(t) = -\frac{\sigma^2}{2}t^2 + ibt + \int \big(e^{itx} - 1 - itx\mathbb{1}_{\{|x|<1\}}\big)\nu(dx). \tag{16.11}$$

Theorem 16.17 (Lévy–Khinchin formula) *Let $\mu \in \mathcal{M}_1(\mathbb{R})$ and*

$$\psi(t) := \log \int e^{itx}\mu(dx).$$

μ is infinitely divisible if and only if there exists a canonical triple (σ^2, b, ν) such that (16.11) holds. By (16.11), this triple is uniquely determined.

Again, ν is called the Lévy measure of μ, σ^2 is called the Gaussian coefficient and b is called the centering constant.

Proof We have shown already that via (16.11) every canonical triple (σ^2, b, ν) corresponds to an infinitely divisible distribution μ. It remains to show:

(i) A canonical triple is uniquely determined by (16.11).
(ii) For every infinitely divisible distribution, there exists a canonical triple such that (16.11) holds.

(i) *Uniqueness*. Define $g_t(x) = e^{itx} - 1 - itx\mathbb{1}_{\{|x|<1\}}$. For every $x \neq 0$, we have

$$2 \geq \left|\frac{g_t(x)}{t^2(1 \wedge x^2)}\right| \overset{t\to\infty}{\longrightarrow} 0.$$

Since (16.10) holds, by the dominated convergence theorem,

$$\begin{aligned}\lim_{t\to\infty} \frac{\psi(t)}{t^2} &= -\frac{\sigma^2}{2} + \lim_{t\to\infty} \frac{ib}{t} + \lim_{t\to\infty} \int_{-\infty}^{\infty} \left(\frac{g_t(x)}{t^2(1 \wedge x^2)}\right)\big(1 \wedge x^2\big)\nu(dx) \\ &= -\frac{\sigma^2}{2}.\end{aligned} \tag{16.12}$$

This implies the uniqueness of σ^2. Thus we can and will assume $\sigma^2 = 0$ in the following. Define

$$\overline{\psi}(t) = \psi(t) - \frac{1}{2}\int_{t-1}^{t+1} \psi(s)\,ds. \tag{16.13}$$

Then

$$\overline{\psi}(t) = \int_{\mathbb{R}} e^{itx}\left(1 - \frac{1}{2}\int_{-1}^{1} e^{isx}\,ds\right)\nu(dx) = \int e^{itx} h(x)\nu(dx), \tag{16.14}$$

where $h(x) = 1 - \frac{\sin(x)}{x}$ for $x \neq 0$ and $h(0) = 0$. Define $\hat{h}(x) = h(x)/(1 \wedge x^2)$ for $x \neq 0$ and $\hat{h}(0) = 1/6$. Clearly, h and $\hat{h}$ are bounded and continuous and

$$0 < 1 - \sin(1) \leq \hat{h}(x) \leq \frac{3}{2} \quad \text{for all } x \in \mathbb{R}.$$

$\overline{\psi}$ is the characteristic function of $\tilde{\nu} \in \mathcal{M}_f(\mathbb{R})$, where $\tilde{\nu}(dx) = h(x)\nu(dx)$. Hence $\tilde{\nu}$ is uniquely determined by ψ. Since $\nu(dx) = (\mathbb{1}_{\{x\neq 0\}}/h(x))\tilde{\nu}(dx)$, ν is also uniquely determined by ψ. Now the number b is the difference of the remaining terms.

(ii) *Existence of a canonical triple*. Let μ be infinitely divisible and let

$$\psi(t) = \log \int e^{itx}\mu(dx).$$

Clearly, $\operatorname{Im}(\psi)$ is odd and $\operatorname{Re}(\psi(t)) \leq 0$ for all $t \in \mathbb{R}$. Hence $\overline{\psi}(0) \geq 0$ (with $\overline{\psi}$ from (16.13)) and $\overline{\psi}(0) = 0$ if $\operatorname{Re}\psi(t) =$ for all $t \in [-1, t]$. By Exercise 15.2.4, this is the case if and only if $\mu = \delta_b$ for some $b \in \mathbb{R}$. In this case, $(0, b, 0)$ is the corresponding canonical triple.

Now assume $\overline{\psi}(0) > 0$. By Theorem 16.5, there exists a sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_f(\mathbb{R})$ with $\mathrm{CPoi}_{\nu_n} \xrightarrow{n\to\infty} \mu$ and $\nu_n(\{0\}) = 0$ for any $n \in \mathbb{N}$. Define

$$b_n = \int x\mathbb{1}_{\{|x|<1\}}\nu_n(dx).$$

Then, by (16.1) and with g_t from (i),

$$\psi_n(t) := \log \int e^{itx}\mathrm{CPoi}_{\nu_n}(dx) = \int \left(e^{itx} - 1\right)\nu_n(dx) = \int g_t\,d\nu_n + ib_n t.$$

As in (16.14), we have

$$\overline{\psi}_n(t) := \psi_n(t) - \frac{1}{2}\int_{t-1}^{t+1} \psi_n(s)\,ds = \int e^{itx}h(x)\nu_n(dx).$$

As $\psi_n \xrightarrow{n\to\infty} \psi$ converges uniformly on compact sets (Theorem 15.23(i)), and since ψ is continuous and thus locally bounded, we have $\overline{\psi}_n \xrightarrow{n\to\infty} \overline{\psi}$ pointwise. Therefore,

$$\int e^{itx} h(x)\nu_n(dx) \overset{n\to\infty}{\longrightarrow} \overline{\psi}(t). \tag{16.15}$$

In particular, $\overline{\psi}_n(0) > 0$ for large n. If we let $\tilde{\nu}_n(dx) = (h(x)/\overline{\psi}_n(0))\nu_n(dx) \in \mathcal{M}_1(\mathbb{R})$, then $\int e^{itx}\tilde{\nu}_n(dx) \overset{n\to\infty}{\longrightarrow} \overline{\psi}(t)/\overline{\psi}(0)$ and the right-hand side is continuous. Hence, by Lévy's continuity theorem, there is a $\tilde{\nu} \in \mathcal{M}_1(\mathbb{R})$ with $\tilde{\nu}_n \overset{n\to\infty}{\longrightarrow} \tilde{\nu}$ and

$$\overline{\psi}(t) = \overline{\psi}(0) \int e^{itx}\tilde{\nu}(dx).$$

Let $\sigma^2 := -6\overline{\psi}(0)\tilde{\nu}(\{0\})$ and define a canonical measure ν by

$$\nu(dx) = \frac{\overline{\psi}(0)}{h(x)} \mathbb{1}_{\{x\neq 0\}}\tilde{\nu}(dx).$$

The map (compare (15.8))

$$f_t : \mathbb{R} \to \mathbb{C}, \quad x \mapsto \begin{cases} \frac{g_t(x)}{h(x)}, & \text{if } x \neq 0, \\ -3t^2, & \text{if } x = 0, \end{cases}$$

is bounded and continuous. By construction, we have

$$\int g_t\, d\nu_n = \overline{\psi}_n(0) \int f_t\, d\tilde{\nu}_n \overset{n\to\infty}{\longrightarrow} \overline{\psi}(0) \int f_t\, d\tilde{\nu} = -\frac{\sigma^2}{2}t^2 + \int g_t\, d\nu.$$

Hence also the limit

$$itb := \lim_{n\to\infty} itb_n = \lim_{n\to\infty}\left(\psi_n(t) - \int g_t\, d\nu_n\right) = \psi(t) + \frac{\sigma^2}{2}t^2 - \int g_t\, d\nu$$

exists, and we have

$$\psi(t) = -\frac{\sigma^2}{2}t^2 + ibt + \int g_t\, d\nu. \qquad \square$$

Remark 16.18 There are many versions of the Lévy–Khinchin formula

$$\psi(t) = -\frac{\sigma^2}{2}t^2 + ibt + \int \big(e^{itx} - 1 - itf(x)\big)\nu(dx)$$

that differ in the function $itf(x)$ that is subtracted for the centering in the integral. We chose $f(x) = x\mathbb{1}_{\{|x|<1\}}$ since this fits best to the construction with the random variables X_k. However, for a given canonical measure ν, any function $\tilde{f}$ for which $\int |f - \tilde{f}|\, d\nu < \infty$ holds is possible; that is, every $\tilde{f}$ for which

$|f(x) - \tilde{f}(x)|/(1 \wedge x^2)$ is bounded. One common function is, e.g., $\tilde{f}(x) = \sin(x)$. The Lévy measure and the Gaussian coefficient σ^2 do not change but the b differs:

$$\tilde{b} - b = \int (f - \tilde{f})\, d\nu.$$

If ν is a measure that is concentrated on $(0, \infty)$ and such that $\int (1 \wedge x)\nu(dx) < \infty$ holds, then this f is integrable with respect to ν and can thus be replaced by $\tilde{f} = 0$. Hence we recover Theorem 16.14 as a special case. However, condition (16.10) is weaker than $\int (1 \wedge x)\, \nu(dx) < \infty$ and thus describes a larger class of measures than is considered in Theorem 16.14. This implies that to a canonical triple $(b, 0, \nu)$ with $\nu((-\infty, 0)) = 0$ and $\int (1 \wedge x)\, \nu(dx) = \infty$, there corresponds an infinitely divisible probability distribution μ that is not concentrated on $[0, \infty)$, no matter how b is chosen. ♢

For a given infinitely divisible distribution μ, we can compute the canonical measure ν as the vague limit

$$\nu = \operatorname*{v\text{-}lim}_{n\to\infty} n\mu^{*1/n}\big|_{(0,\infty)}. \tag{16.16}$$

Example 16.19 For the Cauchy distribution Cau_a with $\psi(t) = -a|t|$, by symmetry, we get $b = 0$ and, by (16.12), $\sigma^2 = -2\lim_{t\to\infty} \psi(t)/t^2 = 0$. Finally, if $A \subset \mathbb{R}$ with $(-\varepsilon, \varepsilon) \cap A = \emptyset$ for some $\varepsilon > 0$, then

$$n\,\mathrm{Cau}_{1/n}(A) = \frac{1}{\pi}\int_A \frac{n^2}{1 + (nx)^2}\, dx \overset{n\to\infty}{\longrightarrow} \frac{1}{\pi}\int_A \frac{1}{x^2}\, dx.$$

Hence Cau_1 has the canonical triple $(0, 0, (\pi x^2)^{-1} dx)$. ♢

Exercise 16.1.1 Use a variance argument to show that an infinitely divisible distribution that is concentrated on a bounded interval is a Dirac measure.

Exercise 16.1.2 Let φ be infinitely divisible, and for every $n \in \mathbb{N}$, let φ_n be a CFP with $\varphi_n^n = \varphi$. Use Lévy's continuity theorem to show that $\varphi_n \overset{n\to\infty}{\longrightarrow} 1$ uniformly on compact sets $\varphi_n \overset{n\to\infty}{\longrightarrow} 1$. Conclude that $\varphi(t) \neq 0$ for all $t \in \mathbb{R}$.

Exercise 16.1.3 Under the conditions of Theorem 16.14, show that

$$\alpha = \sup\bigl\{x \geq 0 : \mu\bigl([0, x)\bigr) = 0\bigr\}.$$

16.2 Stable Distributions

A distribution μ on the real numbers is called stable if for any $n \in \mathbb{N}$, the n-fold convolution μ^{*n} equals μ up to an affine linear transformation. Hence stability can be interpreted as self-similarity. We first show that the class of stable distributions is rather simple and can easily be parameterized. Then we quote results which say that stable distributions are exactly those distributions that occur as limits of sums of i.i.d. random variables.

Symmetric Stable Distributions

For $\alpha \in (0, 2)$, let

$$\theta_\alpha := \int_{\mathbb{R}} \big(1-\cos(x)\big)|x|^{-\alpha-1}\,dx = \begin{cases} -2\Gamma(-\alpha)\cos(\alpha\pi/2), & \text{if } \alpha \neq 1, \\ \pi, & \text{if } \alpha = 1. \end{cases}$$

(Note that the integral diverges for $\alpha \in \mathbb{R} \setminus (0, 2)$.) Then $\nu_\alpha(dx) = \theta_\alpha^{-1}|x|^{-\alpha-1}\,dx$ is a canonical measure since

$$\int \big(1 \wedge x^2\big)\nu_\alpha(dx) = 2\theta_\alpha^{-1}\big(\alpha^{-1} + (2-\alpha)^{-1}\big) < \infty.$$

Let ψ_α be the logarithm of the characteristic function that corresponds to the infinitely divisible measure μ_α with canonical triple $(0, 0, \nu_\alpha)$. By the Lévy–Khinchin formula, we have

$$\begin{aligned} \psi_\alpha(t) &= \int_{-\infty}^{\infty} \big(e^{itx} - 1 - itx\mathbb{1}_{\{|x|<1\}}\big)\theta_\alpha^{-1}|x|^{-\alpha-1}\,dx \\ &= -\theta_\alpha^{-1}\int_{-\infty}^{\infty}\big(1-\cos(tx)\big)|x|^{-\alpha-1}\,dx \\ &= -|t|^\alpha. \end{aligned}$$

Hence $\varphi_\alpha(t) := e^{-|t|^\alpha}$ is the characteristic function of the infinitely divisible measure μ_α, which is called the symmetric *stable distribution* with index α. The name is due to the fact that, for i.i.d. random variables $X_1, X_2, \ldots$ that are μ_α-distributed, we have

$$X_1 + \ldots + X_n \stackrel{\mathcal{D}}{=} n^{1/\alpha} X_n \quad \text{for all } n \in \mathbb{N}. \tag{16.17}$$

General Stable Distributions

Motivated by equation (16.17), we present a somewhat more general notion of stability of a distribution.

Definition 16.20 (Stable distribution) Let $\mu \in \mathcal{M}_1(\mathbb{R})$ be a probability distribution on the real numbers that is not concentrated in one point. Assume that $X_1, X_2, \ldots$ are i.i.d. random variables with distribution μ. The distribution μ is said to be *stable in the broad sense* if there exist nonnegative numbers $a_1, a_2, \ldots$ and real numbers $d_1, d_2, \ldots$ such that

$$X_1 + \ldots + X_n \stackrel{\mathcal{D}}{=} a_n X_1 + d_n \quad \text{for all } n \in \mathbb{N}. \tag{16.18}$$

μ is called *stable* (in the strict sense), if (16.18) holds with $d_1 = d_2 = \ldots = 0$.

μ is called stable in the broad sense with index $\alpha \in (0, 2]$, if (16.18) holds with $a_n = n^{1/\alpha}$, $n \in \mathbb{N}$. It is called stable (in the strict sense) with index $\alpha \in (0, 2]$, if in addition, we can choose $d_1 = d_2 = \ldots = 0$.

Remark 16.21 If μ is stable in the broad sense, then it is infinitely divisible. $\diamondsuit$

Theorem 16.22 *Let μ be stable in the broad sense.*

(i) *There is an $\alpha \in (0, 2]$ such that μ is stable in the broad sense with index α.*
(ii) *If $\alpha = 2$, then μ is a normal distribution.*
(iii) *If $\alpha \in (0, 2)$, then the Lévy measure ν of μ has the density*

$$\frac{\nu(dx)}{dx} = \begin{cases} c^-(-x)^{-\alpha-1}, & \text{if } x < 0, \\ c^+ x^{-\alpha-1}, & \text{if } x > 0, \end{cases} \tag{16.19}$$

for some $c^-, c^+ \geq 0$, $c^- + c^+ > 0$.
(iv) *If $\alpha \neq 1$, then there exists a $b \in \mathbb{R}$ such that $\mu * \delta_{-b}$ is stable with index α.*
(v) *If $\alpha = 1$, then $d_n = (c^+ - c^-) n \log(n)$, $n \in \mathbb{N}$. If $c^- = c^+$, then μ is a Cauchy distribution.*

Remark 16.23 If μ is infinitely divisible with Lévy measure ν given by (16.19), then $\psi(t) := \log \int e^{itx} \mu(dx)$ is given by

$$\psi(t) = \begin{cases} |t|^\alpha \Gamma(-\alpha)\big[(c^+ + c^-)\cos\big(\frac{\pi\alpha}{2}\big) + i(c^+ - c^-)\sin\big(\frac{\pi\alpha}{2}\big)\big], & \alpha \neq 1, \\ -|t|(c^+ + c^-)\big[\frac{\pi}{2} + i\,\mathrm{sign}(t)(c^+ - c^-)\log(|t|)\big], & \alpha = 1. \end{cases} \tag{16.20}$$

$\diamondsuit$

Lemma 16.24 *Let μ be infinitely divisible with canonical triple (σ^2, b, ν); that is, with log-characteristic function $\psi(t) := \log(\int e^{itx} \mu(dx))$ given by*

$$\psi(t) = -\frac{\sigma^2}{2} t^2 + ibt + \int \big(e^{itx} - 1 - itx \mathbb{1}_{\{|x|<1\}}\big) \nu(dx).$$

Further, let $a > 0$, $d \in \mathbb{R}$, $n \in \mathbb{N}$ and let $X, X_1, \ldots, X_n$ be i.i.d. random variables with distribution μ.

(i) *The canonical triple of $X_1 + \ldots + X_n$ is $(n\sigma^2, nb, n\nu)$.*

(ii) *The canonical triple of* $aX+d$ *is* $(a^2\sigma^2, \tilde{b}, \nu \circ m_a^{-1})$, *where* $m_a : \mathbb{R} \to \mathbb{R}$, $x \mapsto ax$ *is the multiplication by* a *and*

$$\tilde{b} := ab + d + a \int \big(\mathbb{1}_{\{|x|<1/a\}} - \mathbb{1}_{\{|x|<1\}}\big) x \,\nu(dx). \tag{16.21}$$

Proof (i) The log-characteristic function of $X_1 + \ldots + X_n$ is $n\psi$.

(ii) The log-characteristic function of $aX+d$ is

$$\begin{aligned}
\psi_{aX+d}(t) &= \psi(at) + idt \\
&= -\frac{a^2\sigma^2}{2} t^2 + i(ab+d)t + \int \big(e^{iatx} - 1 - iatx\mathbb{1}_{\{|x|<1\}}\big)\nu(dx) \\
&= -\frac{a^2\sigma^2}{2} t^2 + i\tilde{b}t + \int \big(e^{iatx} - 1 - iatx\mathbb{1}_{\{|x|<1/a\}}\big)\nu(dx) \\
&= -\frac{a^2\sigma^2}{2} t^2 + i\tilde{b}t + \int \big(e^{itx} - 1 - itx\mathbb{1}_{\{|x|<1\}}\big)\nu \circ m_a^{-1}(dx). \qquad \square
\end{aligned}$$

Lemma 16.25 (Scaling of the canonical triple) *Under the assumptions of Theorem 16.22, let* (σ^2, b, ν) *be the canonical triple of* μ.

(i) *We have*

$$\big(a_n^2 - n\big)\sigma^2 = 0 \quad \textit{for all } n \in \mathbb{N} \tag{16.22}$$

and (*with* m_{a_n} *as in Lemma 16.24*)

$$n\nu = \nu \circ m_{a_n}^{-1} \quad \textit{for all } n \in \mathbb{N}. \tag{16.23}$$

(ii) *If* $\nu = 0$, *then* $a_n = n^{1/2}$ *for all* $n \in \mathbb{N}$ *and*

$$d_n = b\big(n - n^{1/2}\big). \tag{16.24}$$

(iii) *Assume that* $\alpha \in (0,2)$, $a_n = n^{1/\alpha}$, *and that* ν *is given by* (16.19). *Then we have*

$$d_n = \left(b + \frac{c^+ - c^-}{\alpha - 1}\right)\big(n - n^{1/\alpha}\big) \quad \textit{if } \alpha \neq 1, \tag{16.25}$$

and

$$d_n = \big(c^+ - c^-\big) n \log(n) \quad \textit{if } \alpha = 1. \tag{16.26}$$

Proof (i) Let $(a_n^2\sigma^2, \tilde{b}_n, \nu \circ m_{a_n}^{-1})$ be the canonical triple of $a_nX + d_n$ as determined in the preceding lemma and let $(n\sigma^2, nb, n\nu)$ be the canonical triple of $X_1 + \ldots + X_n$. By (16.18) and due to the uniqueness of the canonical triple (Theorem 16.17), we infer $a_n^2\sigma^2 = n\sigma^2$, $\tilde{b}_n = nb$ and $\nu \circ m_{a_n}^{-1} = n\nu$.

(ii) If $\nu = 0$, then $\sigma^2 > 0$, since by assumption, μ is not concentrated in one point. Hence, by (16.22), we get $a_n = n^{1/2}$. By virtue of Lemma 16.24(ii), we have $nb = \tilde{b}_n = bn^{1/2} + d_n$ and thus (16.24) holds.

(iii) Using (16.21), we compute $\tilde{b}_n$ more explicitly:

$$\begin{aligned} nb = \tilde{b}_n &= bn^{1/\alpha} + d_n - n^{1/\alpha} \int \mathbb{1}_{\{n^{-1/\alpha} \le |x| < 1\}} x \, \nu(dx) \\ &= bn^{1/\alpha} + d_n - n^{1/\alpha}\left(c^+ - c^-\right) \int_{n^{-1/\alpha}}^1 x^{-\alpha}\, dx \\ &= bn^{1/\alpha} + d_n - \left(c^+ - c^-\right) \begin{cases} (1-\alpha)^{-1}(n^{1/\alpha} - n), & \text{if } \alpha \ne 1, \\ n \log(n), & \text{if } \alpha = 1. \end{cases} \end{aligned}$$

Rearranging terms yields (16.25) and (16.26). □

Proof of Theorem 16.22 We distinguish the cases $\liminf_{n\to\infty} a_n n^{-1/2} < \infty$ and "$= \infty$".

Case 1. Assume that $\liminf_{n\to\infty} a_n n^{-1/2} < \infty$. Let $C \in [1, \infty)$ and let $(n_k)_{k\in\mathbb{N}}$ be a subsequence such that $a_{n_k} n_k^{-1/2} \le C$ for any $k \in \mathbb{N}$. Then for any $x \in \mathbb{R} \setminus \{0\}$, we have

$$C^2 \ge n_k^{-1}\left(1 \vee a_{n_k}^2\right) \ge \frac{n_k^{-1}(1 \wedge a_{n_k}^2 x^2)}{1 \wedge x^2} \overset{k\to\infty}{\longrightarrow} 0.$$

Using (16.23) and (16.10), the dominated convergence theorem yields

$$\int_{-\infty}^{\infty} \left(1 \wedge x^2\right)\nu(dx) = \int_{-\infty}^{\infty} \frac{n_k^{-1}(1 \wedge a_{n_k}^2 x^2)}{1 \wedge x^2}\left(1 \wedge x^2\right)\nu(dx) \overset{k\to\infty}{\longrightarrow} 0.$$

That is, we have $\nu = 0$. By Lemma 16.25(ii), we see that $\mu * \delta_{-b}$ is stable with index 2. This shows (ii).

Case 2. Assume that

$$a_n n^{-1/2} \overset{n\to\infty}{\longrightarrow} \infty. \tag{16.27}$$

By (16.22), we have $\sigma^2 = 0$ and hence $\nu \ne 0$. We define the function

$$F(x) = \begin{cases} \nu([x, \infty)), & \text{if } x > 0, \\ \nu((-\infty, x]), & \text{if } x < 0. \end{cases}$$

Since we have $\nu \ne 0$, there is an $x_0 \in \mathbb{R} \setminus \{0\}$ such that $F(x_0) > 0$. By symmetry, we may assume that $x_0 > 0$. Using (16.23), we infer

$$nF(x) = F(x/a_n) \quad \text{for any } x \in \mathbb{R} \setminus \{0\}, n \in \mathbb{N},$$

and thus

$$F\left(\left(\frac{a_{n+1}}{a_n}\right)^k x_0\right) = \left(\frac{n}{n+1}\right)^k F(x_0) \quad \text{for any } k \in \mathbb{Z}.$$

We can rephrase this as

$$F(x) = (x/x_0)^{-\alpha_n} F(x_0) \quad \text{for any } x \in \left\{(a_{n+1}/a_n)^k x_0 : k \in \mathbb{Z}\right\},$$

where $\alpha_n := \log((n+1)/n)/\log(a_{n+1}/a_n)$. Since F is monotone decreasing and since $F(x) \xrightarrow{x\to\infty} 0$, we have $\alpha_n > 0$ for all $n \in \mathbb{N}$, and

$$\left(\frac{m}{m+1}\right)\left(\frac{x}{x_0}\right)^{-\alpha_m} \le \frac{F(x)}{F(x_0)} \le \left(\frac{n+1}{n}\right)\left(\frac{x}{x_0}\right)^{-\alpha_n} \quad \text{for } x > 0,\ m, n \in \mathbb{N}.$$

Letting $x \to \infty$, we obtain $\alpha_m \ge \alpha_n$. By symmetry, we also get $\alpha_m \le \alpha_n$. Hence, we define $\alpha := \alpha_1 > 0$ and get $a_n = n^{1/\alpha}$ for all $n \in \mathbb{N}$ (note that (16.18) implies $a_1 = 1$). By the assumption (16.27), we have $\alpha < 2$. This shows (i).

We have $F(1) = x_0^\alpha F(x_0) > 0$ and $F(x) = x^{-\alpha} F(1)$ for all $x > 0$. Similarly, we get $F(x) = (-x)^{-\alpha} F(-1)$ for $x < 0$ (with the same $\alpha \in (0, 2)$ since it is determined by the sequence $(a_n)_{n\in\mathbb{N}}$). Defining $c^+ = \alpha\nu([1,\infty))$ and $c^- := \alpha\nu((-\infty,-1])$, we get (16.19) and thus (iii) and (i).

The statements (iv) and (v) are immediate consequences of Lemma 16.25. □

Convergence to Stable Distributions

To complete the picture, we cite theorems from [54, Chapter XVII.5] (see also [62] and [128]) that state that only stable distributions occur as limiting distributions of rescaled sums of i.i.d. random variables $X_1, X_2, \ldots$.

In the following, let $X, X_1, X_2, \ldots$ be i.i.d. random variables and for $n \in \mathbb{N}$, let $S_n = X_1 + \ldots + X_n$.

Definition 16.26 (Domain of attraction) Let $\mu \in \mathcal{M}_1(\mathbb{R})$ be nontrivial. The *domain of attraction* $\mathrm{Dom}(\mu) \subset \mathcal{M}_1(\mathbb{R})$ is the set of all distributions $\mathbf{P}_X$ with the property that there exist sequences of numbers $(a_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ with

$$\frac{S_n - d_n}{a_n} \quad \overset{n\to\infty}{\Longrightarrow} \quad \mu.$$

If μ is stable (in the broader sense) with index $\alpha \in (0, 2]$, then $\mathbf{P}_X$ is said to be in the *domain of normal attraction* if we can choose $a_n = n^{1/\alpha}$.

Theorem 16.27 *Let $\mu \in \mathcal{M}_1(\mathbb{R})$ be nontrivial. Then* $\mathrm{Dom}(\mu) \neq \emptyset$ *if and only if μ is stable* (*in the broader sense*). *In this case*, $\mu \in \mathrm{Dom}(\mu)$.

In the following, an important role is played by the function

$$U(x) := \mathbf{E}\left[X^2 \mathbb{1}_{\{|X|\le x\}}\right]. \tag{16.28}$$

A function $H : (0, \infty) \to (0, \infty)$ is called *slowly varying* at ∞ if

$$\lim_{x\to\infty} \frac{H(\gamma x)}{H(x)} = 1 \quad \text{for all } \gamma > 0.$$

In the following, we assume that there exists an $\alpha \in (0, 2]$ such that

$$U(x)x^{\alpha-2} \quad \text{is slowly varying at } \infty. \tag{16.29}$$

Theorem 16.28

(i) *If* $\mathbf{P}_X$ *is in the domain of attraction of some distribution, then there exists an* $\alpha \in (0, 2]$ *such that* (16.29) *holds.*
(ii) *In the case* $\alpha = 2$, *we have: If* $\mathbf{P}_X$ *is not concentrated at one point, then* (16.29) *implies that* $\mathbf{P}_X$ *is in the domain of attraction of some distribution.*
(iii) *In the case* $\alpha \in (0, 2)$, *we have:* $\mathbf{P}_X$ *is in the domain of attraction of some distribution if and only if* (16.29) *holds and the limit*

$$p := \lim_{x\to\infty} \frac{\mathbf{P}[X \geq x]}{\mathbf{P}[|X| \geq x]} \quad \textit{exists}. \tag{16.30}$$

Theorem 16.29 *Let* $\mathbf{P}_X$ *be in the domain of attraction of an* α*-stable distribution (that is, assume that condition* (ii) *or* (iii) *of Theorem* 16.28 *holds), and assume that* $(a_n)_{n\in\mathbb{N}}$ *is such that*

$$C := \lim_{n\to\infty} \frac{nU(a_n)}{a_n^2} \in (0, \infty)$$

exists. Further, let μ *be the stable distribution with index* α *whose characteristic function is given by* (16.20) *with* $c^+ = Cp$ *and* $c^- = C(1 - p)$.

(i) *In the case* $\alpha \in (0, 1)$, *let* $b_n \equiv 0$.
(ii) *In the case* $\alpha = 2$ *and* $\mathbf{Var}[X] < \infty$, *let* $\mathbf{E}[X] = 0$.
(iii) *In the case* $\alpha \in (1, 2]$, *let* $d_n = n\mathbf{E}[X]$ *for all* $n \in \mathbb{N}$.
(iv) *In the case* $\alpha = 1$, *let* $d_n = na_n\mathbf{E}[\sin(X/a_n)]$ *for all* $n \in \mathbb{N}$.

Then

$$\frac{S_n - d_n}{a_n} \quad \overset{n\to\infty}{\Longrightarrow} \quad \mu.$$

Corollary 16.30 *If* $\mathbf{P}_X$ *is in the domain of attraction of a stable distribution with index* α, *then* $\mathbf{E}[|X|^\beta] < \infty$ *for all* $\beta \in (0, \alpha)$ *and* $\mathbf{E}[|X|^\beta] = \infty$ *if* $\beta > \alpha$ *and* $\alpha < 2$.

Exercise 16.2.1 Let μ be an α-stable distribution and let φ be its characteristic function.

(i) Show by a direct computation using only the definition of stability that $|\varphi(t) - 1| \leq C|t|^\alpha$ for t close to 0 (for some $C < \infty$).
(ii) Use Exercise 15.3.2 to infer that $\mu = \delta_0$ if $\alpha > 2$.

(iii) Modify the argument in order to show that for $\alpha > 2$, the α-stable distributions in the broad sense are also necessarily trivial.

Exercise 16.2.2 Show that the distribution on $\mathbb{R}$ with density

$$f(x) = \frac{1 - \cos(x)}{\pi x^2}$$

is not infinitely divisible.

Exercise 16.2.3 Let Φ be the distribution function of the standard normal distribution $\mathcal{N}_{0,1}$ and let $F : \mathbb{R} \to [0, 1]$ be defined by

$$F(x) = \begin{cases} 2(1 - \Phi(x^{-1/2})), & \text{if } x > 0, \\ 0, & \text{else.} \end{cases}$$

Show the following.

(i) F is the distribution function of a $\frac{1}{2}$-stable distribution.
(ii) If $X_1, X_2, \ldots$ are i.i.d. with distribution function F, then $\frac{1}{n}\sum_{k=0}^{n} X_k$ diverges almost surely for $n \to \infty$.

Hint: Compute the density of F, and show that the Laplace transform is given by $\lambda \mapsto e^{-\sqrt{2\lambda}}$.

Exercise 16.2.4 Which of the following distributions is in the domain of attraction of a stable distribution and for which parameter?

(i) The distribution on $\mathbb{R}$ with density

$$f(x) = \begin{cases} \varrho \frac{1}{1+\alpha} |x|^{\alpha}, & \text{if } x < -1, \\ (1 - \varrho) \frac{1}{1+\beta} x^{\beta}, & \text{if } x > 1, \\ 0, & \text{else.} \end{cases}$$

Here $\alpha, \beta < -1$ and $\varrho \in [0, 1]$.
(ii) The exponential distribution $\exp_\theta$ for $\theta > 0$.
(iii) The distribution on $\mathbb{N}$ with weights cn^{α} if n is even and cn^{β} if n is odd. Here $\alpha, \beta < -1$ and $c = (2^{\alpha}\zeta(-\alpha) + (1 - 2^{\beta})\zeta(-\beta))^{-1}$ (ζ is the Riemann zeta function) is the normalization constant.