Chapter 19
Markov Chains and Electrical Networks

We consider symmetric simple random walk on $\mathbb{Z}^2$. By Pólya's theorem (Theorem 17.39), this random walk is recurrent. However, is this still true if we remove a single edge from the lattice $\mathbb{L}^2$ of $\mathbb{Z}^2$? Intuitively, such a small local change should not make a difference for a global phenomenon such as recurrence. However, the computations used in Section 17.5 to prove recurrence are not very robust and would need a substantial improvement in order to cope with even a small change. The situation becomes even more puzzling if we restrict the random walk to, e.g., the upper half plane $\{(x, y) : x \in \mathbb{Z}, y \in \mathbb{N}_0\}$ of $\mathbb{Z}^2$. Is this random walk recurrent? Or consider *bond percolation* on $\mathbb{Z}^2$. Fix a parameter $p \in [0, 1]$ and independently declare any edge of $\mathbb{L}^2$ *open* with probability p and *closed* with probability $1 - p$. At a second stage, start a random walk on the random subgraph of open edges. At each step, the walker chooses one of the adjacent open edges at random (with equal probability) and traverses it. For $p > \frac{1}{2}$, there exists a unique infinite connected component of open edges (Theorem 2.47). The question that we answer at the end of this chapter is: Is a random walk on the infinite open cluster recurrent or transient?

The aim of this chapter is to establish a connection between certain Markov chains and electrical networks. This connection

- in some cases allows us to distinguish between recurrence and transience by means of easily computable quantities, and
- in other cases provides a comparison criterion that says that if a random walk on a graph is recurrent, then a random walk on any connected subgraph is recurrent. Any of the questions raised above can be answered using this comparison technique.

Some of the material of this chapter is taken from [36, 111].

19.1 Harmonic Functions

In this chapter, E is always a countable set and X is a discrete Markov chain on E with transition matrix p and Green function G. Recall that $F(x, y)$ is the probability

A. Klenke, *Probability Theory*, Universitext, DOI 10.1007/978-1-4471-5361-0_19,

of hitting y at least once when starting at x. Compare Section 17.4, in particular, Definitions 17.28 and 17.33.

Definition 19.1 Let $A \subset E$. A function $f : E \to \mathbb{R}$ is called *harmonic* on $E \setminus A$ if $pf(x) = \sum_{y \in E} p(x, y) f(y)$ exists and if $pf(x) = f(x)$ for all $x \in E \setminus A$.

Theorem 19.2 (Superposition principle) *Assume f and g are harmonic on $E \setminus A$ and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is also harmonic on $E \setminus A$.*

Proof This is trivial. □

Example 19.3 Let X be transient and let $a \in E$ be a transient state (that is, a is not absorbing). Then $f(x) := G(x, a)$ is harmonic on $E \setminus \{a\}$: For $x \neq a$, we have

$$pf(x) = p \sum_{n=0}^{\infty} p^n(x, a) = \sum_{n=1}^{\infty} p^n(x, a) = G(x, a) - \mathbb{1}_{\{a\}}(x) = G(x, a).$$

◊

Example 19.4 For $x \in E$, let $\tau_x := \inf\{n > 0 : X_n = x\}$. For $A \subset E$, let

$$\tau := \tau_A := \inf_{x \in A} \tau_x$$

be the stopping time of the first entrance to A. Assume that A is chosen so that $\mathbf{P}_x[\tau_A < \infty] = 1$ for every $x \in E$. Let $g : A \to \mathbb{R}$ be a bounded function. Define

$$f(x) := \begin{cases} g(x), & \text{if } x \in A, \\ \mathbf{E}_x[g(X_\tau)], & \text{if } x \in E \setminus A. \end{cases} \tag{19.1}$$

Then f is harmonic on $E \setminus A$. We give two proofs for this statement.

1st Proof By the Markov property, for $x \notin A$ and $y \in E$,

$$\mathbf{E}_x\big[g(X_\tau) \,\big|\, X_1 = y\big] = \left\{\begin{array}{ll} g(y), & \text{if } y \in A \\ \mathbf{E}_y[g(X_\tau)], & \text{if } y \in E \setminus A \end{array}\right\} = f(y).$$

Hence, for $x \in E \setminus A$,

$$\begin{aligned} f(x) = \mathbf{E}_x\big[g(X_\tau)\big] &= \sum_{y \in E} \mathbf{E}_x\big[g(X_\tau); X_1 = y\big] \\ &= \sum_{y \in E} p(x, y) \mathbf{E}_x\big[g(X_\tau) \,\big|\, X_1 = y\big] = \sum_{y \in E} p(x, y) f(y) = pf(x). \end{aligned}$$

2nd Proof We change the Markov chain by adjoining a cemetery state Δ. That is, the new state space is $\tilde{E} = E \cup \{\Delta\}$ and the transition matrix is

$$\tilde{p}(x, y) = \begin{cases} p(x, y), & \text{if } x \in E \setminus A,\ y \neq \Delta, \\ 0, & \text{if } x \in E \setminus A,\ y = \Delta, \\ 1, & \text{if } x \in A \cup \{\Delta\},\ y = \Delta. \end{cases} \tag{19.2}$$

The corresponding Markov chain $\tilde{X}$ is transient, and Δ is the only absorbing state. Furthermore, we have $pf = f$ on $E \setminus A$ if and only if $\tilde{p} f = f$ on $E \setminus A$. Since $\tilde{G}(y, y) = 1$ for all $y \in A$, we have (compare Theorem 17.34)

$$\mathbf{P}_x[X_\tau = y] = \mathbf{P}_x[\tilde{\tau}_y < \infty] = \tilde{F}(x, y) = \tilde{G}(x, y) \quad \text{for all } x \in E \setminus A,\ y \in A.$$

Now $x \mapsto \tilde{G}(x, y)$ is harmonic on $E \setminus A$. Hence, by the superposition principle,

$$f(x) = \sum_{y \in A} \tilde{G}(x, y) g(y) \tag{19.3}$$

is harmonic on $E \setminus A$. Due to the analogy of (19.3) to Green's formula in continuous space potential theory, the function $\tilde{G}$ is called the *Green function* for the equation $(p - I) f = 0$ on $E \setminus A$. ◊

Definition 19.5 The system of equations

$$\begin{aligned} (p - I) f(x) &= 0, \qquad && \text{for } x \in E \setminus A, \\ f(x) &= g(x), && \text{for } x \in A, \end{aligned} \tag{19.4}$$

is called the *Dirichlet problem* on $E \setminus A$ with respect to $p - I$ and with boundary value g on A.

We have shown the existence of solutions of the Dirichlet problem in Example 19.4. In order to show uniqueness (under certain conditions) we first derive the maximum principle for harmonic functions.

If $p = I$ then any function f that coincides with g on A is a solution of the Dirichlet problem. However, even in less extreme situations the solution of (19.4) may be ambiguous. This is the case if $E \setminus A$ decomposes into domains between which the chain that is stopped in A cannot change.

In order to describe formally the irreducibility condition that we have to impose, we introduce the transition matrix p_A of the chain stopped upon reaching A by

$$p_A(x, y) := \begin{cases} p(x, y), & \text{if } x \notin A, \\ \mathbb{1}_{\{x=y\}}, & \text{if } x \in A. \end{cases}$$

Further, define F_A for p_A similarly as F was defined for p. Finally, for $x \in E$ let

$$S_A^n(x) = \big\{ y \in E : (p_A)^n(x, y) > 0 \big\}, \quad \text{for } n \in \mathbb{N}_0$$

and

$$S_A(x) = \bigcup_{n=0}^{\infty} S_A^n(x) = \big\{ y \in E : F_A(x, y) > 0 \big\}.$$

Theorem 19.6 (Maximum principle) *Let f be a harmonic function on $E \setminus A$.*

(i) *If there exists an $x_0 \in E \setminus A$ such that*

$$f(x_0) = \sup f\big(S_A(x_0)\big), \tag{19.5}$$

then $f(y) = f(x_0)$ for any $y \in S_A(x_0)$.

(ii) *In particular, if $F_A(x, y) > 0$ for all $x, y \in E \setminus A$, and if there is an $x_0 \in E \setminus A$ such that $f(x_0) = \sup f(E)$, then $f(x_0) = f(y)$ for any $y \in E \setminus A$.*

Proof (i) Let $m := \sup f(S_A(x_0))$. As f is harmonic on $E \setminus A$, we have $p_A f = f$ on E. Hence, for any $n \in \mathbb{N}$,

$$f(x_0) = (p_A)^n f(x_0) = \sum_{y \in S_A^n(x_0)} p_A^n(x_0, y) f(y) \le m$$

with equality if and only if $f(y) = m$ for all $y \in S_A^n(x_0)$. Since (19.5) implies equality, we infer $f(x_0) = f(y)$ for all $y \in S_A(x_0)$.

(ii) This is a direct consequence of (i) since $S_A(x) \supset E \setminus A$ for any $x \in E \setminus A$. □

Theorem 19.7 (Uniqueness of harmonic functions) *Assume that $F(x, y) > 0$ for all $x, y \in E$. Let $A \subset E$ be such that $A \ne \emptyset$ and $E \setminus A$ is finite. Assume that f_1 and f_2 are harmonic on $E \setminus A$. If $f_1 = f_2$ on A, then $f_1 = f_2$.*
In other words, the Dirichlet problem (19.4) *has a unique solution given by* (19.3) (*or equivalently by* (19.1)).

Proof By the superposition principle, $f := f_1 - f_2$ is harmonic on $E \setminus A$ with $f|_A \equiv 0$.

We will show $f \le 0$. Then, by symmetry, also $f \ge 0$ and hence $f \equiv 0$. To this end, we assume that there exists an $x \in E$ such that $f(x) > 0$ and deduce a contradiction.

Since $f|_A \equiv 0$ and since $E \setminus A$ is finite, there is an $x_0 \in E \setminus A$ such that $f(x_0) = \max f(E) \ge f(x) > 0$.

Since $F(x, y) > 0$ for all $x, y \in E$, we have

$$n_0 := \min\big\{n \in \mathbb{N}_0 : p^n(x_0, y) > 0 \text{ for some } y \in A\big\} < \infty.$$

Clearly, we have $p^{n_0}(x_0, y) = (p_A)^{n_0}(x_0, y)$ for all $y \in A$. Hence, there exists a $y \in A$ such that $(p_A)^{n_0}(x_0, y) > 0$, i.e., $y \in S_A(x_0)$. By Theorem 19.6, this implies $f(x_0) = f(y) = 0$ contradicting the assumption. □

Exercise 19.1.1 Let $\overline{p}$ be the substochastic $E \times E$ matrix that is given by $\overline{p}(x, y) = \tilde{p}(x, y)$, $x, y \in E$ (with $\tilde{p}$ as in (19.2)). Hence $\overline{p}(x, y) = p(x, y)\mathbb{1}_{x \in E \setminus A}$. Let I be the unit matrix on E.

(i) Show that $I - \overline{p}$ is invertible.

(ii) Define $\overline{G} := (I - \overline{p})^{-1}$. Show that $\overline{G}(x, y) = \tilde{G}(x, y)$ for all $x, y \in E \setminus A$ and that $\overline{G}(x, y) = \mathbb{1}_{\{x=y\}}$ if $x \in A$. In particular,

$$\overline{G}(x, y) = \mathbf{P}_x[X_{\tau_A} = y] \quad \text{for } x \in E \setminus A \text{ and } y \in A.$$

19.2 Reversible Markov Chains

Definition 19.8 The Markov chain X is called *reversible* with respect to the measure π if

$$\pi\big(\{x\}\big)p(x, y) = \pi\big(\{y\}\big)p(y, x) \quad \text{for all } x, y \in E. \tag{19.6}$$

Equation (19.6) is sometimes called the equation of *detailed balance*. X is called reversible if there is a π with respect to which X is reversible.

Remark 19.9 If X is reversible with respect to π, then π is an invariant measure for X since

$$\pi p\big(\{x\}\big) = \sum_{y\in E} \pi\big(\{y\}\big)p(y, x) = \sum_{y\in E} \pi\big(\{x\}\big)p(x, y) = \pi\big(\{x\}\big).$$

If X is irreducible and recurrent, then, by Remark 17.50, π is thus unique up to constant multiples. ◊

Example 19.10 Let (E, K) be a graph with vertex set (or set of nodes) E and with edge set K (see p. 65). By $\langle x, y\rangle = \langle y, x\rangle \in K$, denote an (undirected) edge that connects x with y. Let $C := (C(x, y), x, y \in E)$ be a family of weights with $C(x, y) = C(y, x) \geq 0$ for all $x, y \in E$ and

$$C(x) := \sum_{y\in E} C(x, y) < \infty \quad \text{for all } x \in E.$$

If we define $p(x, y) := \frac{C(x,y)}{C(x)}$ for all $x, y \in E$, then X is reversible with respect to $\pi(\{x\}) = C(x)$. In fact,

$$\begin{aligned}\pi\big(\{x\}\big)p(x, y) &= C(x)\frac{C(x, y)}{C(x)} = C(x, y)\\ &= C(y, x) = C(y)\frac{C(y, x)}{C(y)} = \pi\big(\{y\}\big)p(y, x).\end{aligned}$$ ◊

Definition 19.11 Let (E, K), C and X be as in Example 19.10. Then X is called a random walk on E with weights C. In particular, if $C(x, y) = \mathbb{1}_{\{\langle x,y\rangle\in K\}}$, then X is called a *simple random walk* on (E, K).

Thus the random walk with weights C is reversible. However, the converse is also true.

Theorem 19.12 *If X is a reversible Markov chain and if π is an invariant measure, then X is a random walk on E with weights $C(x, y) = p(x, y)\pi(\{x\})$. If X is irreducible and recurrent, then π and hence C are unique up to a factor.*

Proof This is obvious. □

Exercise 19.2.1 Show that p is reversible with respect to π if and only if the linear map $L^2(\pi) \to L^2(\pi)$, $f \mapsto pf$ is self-adjoint.

Exercise 19.2.2 Let $\beta > 0$, $K \in \mathbb{N}$ and $W_1, \ldots, W_K \in \mathbb{R}$. Define

$$p(i, j) := \frac{1}{Z} \exp(-\beta W_j) \quad \text{for all } i, j = 1, \ldots, K,$$

where $Z := \sum_{j=1}^{K} \exp(-\beta W_j)$ is the normalising constant.

Assume that in K (enumerated) urns there are a total of N indistinguishable balls. At each step, choose one of the N balls uniformly at random. If i is the number of the urn from which the ball is drawn, then with probability $p(i, j)$ move the ball to the urn with number j.

(i) Give a formal description of this process as a Markov chain.
(ii) Determine the invariant distribution π and show that the chain is reversible with respect to π.

19.3 Finite Electrical Networks

An electrical network (E, C) consists of a set E of sites (the electrical contacts) and wires between pairs of sites. The *conductance* of the wire that connects the points $x \in E$ and $y \in E \setminus \{x\}$ is denoted by $C(x, y) \in [0, \infty)$. If $C(x, y) = 0$, then we could just as well assume that there is no wire connecting x and y. By symmetry, we have $C(x, y) = C(y, x)$ for all x and y. Denote by

$$R(x, y) = \frac{1}{C(x, y)} \in (0, \infty]$$

the *resistance* of the connection $\langle x, y\rangle$. A particular case is that of a graph (E, K) where all edges have the same conductance, say 1; that is, $C(x, y) = \mathbb{1}_{\{\langle x,y\rangle \in K\}}$. The corresponding network (E, C) will be called the *unit network* on (E, K).

In the remainder of this section, assume that (E, C) is a *finite* electrical network.

Now let $A \subset E$. At the points $x_0 \in A$, we apply the voltages $u(x_0)$ (e.g., using batteries). What is the voltage $u(x)$ at $x \in E \setminus A$?

Definition 19.13 A map $I : E \times E \to \mathbb{R}$ is called a *flow* on $E \setminus A$ if it is antisymmetric (that is, $I(x, y) = -I(y, x)$) and if it obeys *Kirchhoff's rule*:

$$\begin{aligned} I(x) &= 0, \quad \text{for } x \in E \setminus A, \\ I(A) &= 0. \end{aligned} \tag{19.7}$$

Here we denoted

$$I(x) := \sum_{y \in E} I(x, y) \quad \text{and} \quad I(A) := \sum_{x \in A} I(x).$$

Definition 19.14 A flow $I : E \times E \to \mathbb{R}$ on $E \setminus A$ is called a *current flow* if there exists a function $u : E \to \mathbb{R}$ with respect to which *Ohm's rule* is fulfilled:

$$I(x, y) = \frac{u(x) - u(y)}{R(x, y)} \quad \text{for all } x, y \in E, x \neq y.$$

In this case, $I(x, y)$ is called the flow from x to y and $u(x)$ is called the electrical *potential* (or voltage) at x.

Theorem 19.15 *An electrical potential u in (E, C) is a harmonic function on $E \setminus A$:*

$$u(x) = \sum_{y \in E} \frac{1}{C(x)} C(x, y) u(y) \quad \textit{for all } x \in E \setminus A.$$

In particular, if the network is irreducible, an electrical potential is uniquely determined by the values on A.

Proof By Ohm's rule and Kirchhoff's rule,

$$u(x) - \sum_{y \in E} \frac{C(x, y)}{C(x)} u(y) = \sum_{y \in E} \frac{C(x, y)}{C(x)} \big(u(x) - u(y)\big) = \frac{1}{C(x)} \sum_{y \in E} I(x, y) = 0.$$

Hence u is harmonic for the stochastic matrix $p(x, y) = C(x, y)/C(x)$. The claim follows by the uniqueness theorem for harmonic functions (Theorem 19.7). □

Corollary 19.16 *Let X be a Markov chain on E with edge weights C. Then $u(x) = \mathbf{E}_x[u(X_{\tau_A})]$.*

Assume $A = \{x_0, x_1\}$ where $x_0 \neq x_1$, and $u(x_0) = 0$, $u(x_1) = 1$. Then $I(x_1)$ is the total flow *into* the network and $-I(x_0)$ is the total flow *out of* the network. Kirchhoff's rule says that the flow is divergence-free and that the flows into and out of the network are equal. In other words, the net flow is $I(x_0) + I(x_1) = 0$.

Recall that, by Ohm's rule, the resistance of a wire is the quotient of the potential difference and the current flow. Hence we define the *effective resistance* between x_0

and x_1 as

$$R_{\text{eff}}(x_0 \leftrightarrow x_1) = \frac{u(x_1) - u(x_0)}{I(x_1)} = \frac{1}{I(x_1)} = -\frac{1}{I(x_0)}.$$

Correspondingly, the *effective conductance* is $C_{\text{eff}}(x_0 \leftrightarrow x_1) = R_{\text{eff}}(x_0 \leftrightarrow x_1)^{-1}$. As I and u are uniquely determined by x_0, x_1 and C, the quantities $C_{\text{eff}}(x_0 \leftrightarrow x_1)$ and $R_{\text{eff}}(x_0 \leftrightarrow x_1)$ are well-defined and can be computed from C.

Consider now two sets $A_0, A_1 \subset E$ with $A_0 \cap A_1 = \emptyset$, $A_0, A_1 \neq \emptyset$. Define $u(x) = 0$ for every $x \in A_0$ and $u(x) = 1$ for every $x \in A_1$. Let I be the corresponding current flow. In a manner similar to the above, we make the following definition.

Definition 19.17 We call $C_{\text{eff}}(A_0 \leftrightarrow A_1) := I(A_1)$ the *effective conductance* between A_0 and A_1 and $R_{\text{eff}}(A_0 \leftrightarrow A_1) := \frac{1}{I(A_1)}$ the *effective resistance* between A_0 and A_1.

Example 19.18

(i) Let $E = \{0, 1, 2\}$ with $C(0, 2) = 0$, and $A_0 = \{x_0\} = \{0\}$, $A_1 = \{x_1\} = \{2\}$. Define $u(0) = 0$ and $u(2) = 1$. Then (with $p(x, y) = C(x, y)/C(x)$),

$$\begin{aligned} u(1) &= 1 \cdot p(1, 2) + 0 \cdot p(1, 0) \\ &= \frac{C(1, 2)}{C(1, 2) + C(1, 0)} = \frac{R(1, 0)}{R(1, 0) + R(1, 2)} \\ &= \frac{R_{\text{eff}}(1 \leftrightarrow 0)}{R_{\text{eff}}(1 \leftrightarrow 0) + R_{\text{eff}}(1 \leftrightarrow 2)}. \end{aligned}$$

The total current flow is

$$I\big(\{2\}\big) = u(1)C(0, 1) = \frac{1}{R(0, 1) + R(1, 2)} = \frac{1}{\frac{1}{C(0,1)} + \frac{1}{C(1,2)}}.$$

Hence we have $R_{\text{eff}}(0 \leftrightarrow 2) = \frac{1}{I(\{2\})} = R(0, 1) + R(1, 2)$ and $C_{\text{eff}}(0 \leftrightarrow 2) = (C(0, 1)^{-1} + C(1, 2)^{-1})^{-1}$.

(ii) (*Series connection*) Let $n \in \mathbb{N}$, $n \geq 2$ and $E = \{0, \ldots, n\}$ with conductances $C(k - 1, k) > 0$ and $C(k, l) = 0$ if $|k - l| > 1$. By Kirchhoff's rule, we have $I(l, l + 1) = -I(x_1)$ for any $l = 0, \ldots, n - 1$. By Ohm's rule, we get $u(1) = u(0) + I(x_1)R(0, 1)$, $u(2) = u(1) + I(x_1)R(1, 2)$ and so on, yielding

$$u(k) - u(0) = I(x_1) \sum_{l=0}^{k-1} R(l, l + 1).$$

Hence

$$R_{\text{eff}}(0 \leftrightarrow k) = \frac{u(k) - u(0)}{I(x_1)} = \sum_{l=0}^{k-1} R(l, l + 1).$$

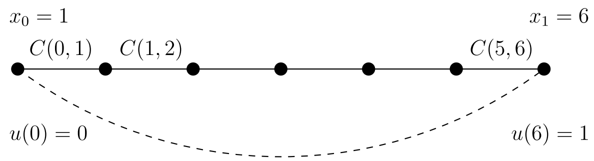


Fig. 19.1 Series connection of six resistors. The effective resistance is $R_{\text{eff}}(0 \leftrightarrow 6) = R(0, 1) + \ldots + R(5, 6)$

By symmetry, we also have

$$R_{\text{eff}}(k \leftrightarrow n) = \sum_{l=k}^{n-1} R(l, l+1)$$

and thus $R_{\text{eff}}(0 \leftrightarrow n) = R_{\text{eff}}(0 \leftrightarrow k) + R_{\text{eff}}(k \leftrightarrow n)$.

Finally, for $k \in \{1, \ldots, n-1\}$, we get

$$u(k) = \frac{R_{\text{eff}}(0 \leftrightarrow k)}{R_{\text{eff}}(0 \leftrightarrow k) + R_{\text{eff}}(k \leftrightarrow n)}.$$

Note that this yields the ruin probability of the corresponding Markov chain X on $\{0, \ldots, n\}$,

$$\mathbf{P}_k[\tau_n < \tau_0] = u(k) = \frac{R_{\text{eff}}(0 \leftrightarrow k)}{R_{\text{eff}}(0 \leftrightarrow n)} = \sum_{l=0}^{k-1} R(l, l+1) \Big/ \sum_{l=0}^{n-1} R(l, l+1). \quad (19.8)$$

(iii) (*Parallel connection*) Let $E = \{0, 1\}$. We extend the model a little by allowing for more than one wire to connect 0 and 1. Denote the conductances of these wires by $C_1, \ldots, C_n$. Then, by Ohm's rule, the current flow along the ith wire is $I_i = \frac{u(1)-u(0)}{R_i} = \frac{1}{R_i}$. Hence the total current is $I = \sum_{i=1}^n \frac{1}{R_i}$ and thus we have

$$C_{\text{eff}}(0 \leftrightarrow 1) = \sum_{i=1}^{n} C_i \quad \text{and} \quad R_{\text{eff}}(0 \leftrightarrow 1) = \left(\sum_{i=1}^{n} \frac{1}{R_i} \right)^{-1}. \qquad \Diamond$$

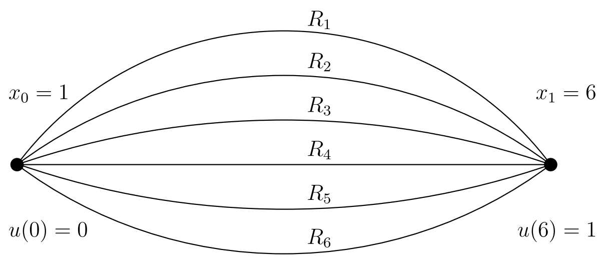


Fig. 19.2 Parallel connection of six resistors. The effective resistance is $R_{\text{eff}}(0 \leftrightarrow 1) = (R_1^{-1} + \ldots + R_6^{-1})^{-1}$

In each of the three preceding examples, the effective resistance is a monotone function of the individual resistances. This is more than just coincidence.

Theorem 19.19 (Rayleigh's monotonicity principle) *Let (E, C) and (E, C') be electrical networks with $C(x, y) \geq C'(x, y)$ for all $x, y \in E$.*
Then, for $A_0, A_1 \subset E$ with $A_0, A_1 \neq \emptyset$ and $A_0 \cap A_1 = \emptyset$,

$$C_{\text{eff}}(A_0 \leftrightarrow A_1) \geq C'_{\text{eff}}(A_0 \leftrightarrow A_1).$$

The remainder of this section is devoted to the proof of this theorem. We will need a theorem on conservation of energy and Thomson's principle (also called Dirichlet's principle) on the minimization of the energy dissipation.

Theorem 19.20 (Conservation of energy) *Let $A = A_0 \cup A_1$, and let I be a flow on $E \setminus A$ (but not necessarily a current flow; that is, Kirchhoff's rule holds but Ohm's rule need not). Further, let $w : E \to \mathbb{R}$ be a function that is constant both on A_0 and on A_1: $w|_{A_0} \equiv: w_0$ and $w|_{A_1} \equiv: w_1$. Then*

$$(w_1 - w_0) I(A_1) = \frac{1}{2} \sum_{x,y \in E} \big(w(x) - w(y)\big) I(x, y).$$

Note that this is a discrete version of Gauß's integral theorem for (wI). In fact, Kirchhoff's rule says that I is divergence-free on $E \setminus A$.

Proof We compute

$$\begin{aligned}
&\sum_{x,y \in E} \big(w(x) - w(y)\big) I(x, y) \\
&\quad = \sum_{x \in E} \Big(w(x) \sum_{y \in E} I(x, y)\Big) - \sum_{y \in E} \Big(w(y) \sum_{x \in E} I(x, y)\Big) \\
&\quad = \sum_{x \in A} \Big(w(x) \sum_{y \in E} I(x, y)\Big) - \sum_{y \in A} \Big(w(y) \sum_{x \in E} I(x, y)\Big) \\
&\quad = w_0 I(A_0) + w_1 I(A_1) - w_0\big(-I(A_0)\big) - w_1\big(-I(A_1)\big) \\
&\quad = 2(w_1 - w_0) I(A_1).
\end{aligned}$$

□

Definition 19.21 Let I be a flow on $E \setminus A$. Denote by

$$L_I := L_I^C := \frac{1}{2} \sum_{x,y \in E} I(x, y)^2 R(x, y)$$

the *energy dissipation* of I in the network (E, C).

Theorem 19.22 (Thomson's (or Dirichlet's) principle of minimization of energy dissipation) *Let I and J be unit flows from A_1 to A_0 (that is, $I(A_1) = J(A_1) = 1$). Assume in addition that I is a current flow (that is, it satisfies Ohm's rule with some potential u that is constant both on A_0 and on A_1). Then*

$$L_I \le L_J$$

with equality if and only if $I = J$. In particular, the unit current flow is uniquely determined.

Proof Let $D = J - I \not\equiv 0$ be the difference of the flows. Then clearly $D(A_0) = D(A_1) = 0$. We infer

$$\begin{aligned}
&\sum_{x,y\in E} J(x,y)^2 R(x,y)\\
&\quad= \sum_{x,y\in E} \big(I(x,y) + D(x,y)\big)^2 R(x,y)\\
&\quad= \sum_{x,y\in E} \big(I(x,y)^2 + D(x,y)^2\big) R(x,y) + 2\sum_{x,y\in E} I(x,y)D(x,y)R(x,y)\\
&\quad= \sum_{x,y\in E} \big(I(x,y)^2 + D(x,y)^2\big) R(x,y) + 2\sum_{x,y\in E} \big(u(x) - u(y)\big)D(x,y).
\end{aligned}$$

By the principle of conservation of energy, the last term equals

$$2\sum_{x,y\in E} \big(u(x) - u(y)\big)D(x,y) = 4D(A_1)(u_1 - u_0) = 0.$$

Therefore (since $D \not\equiv 0$),

$$L_J = L_I + \frac{1}{2}\sum_{x,y\in E} D(x,y)^2 R(x,y) > L_I.$$

□

Proof (Rayleigh's monotonicity principle, Theorem 19.19) Let I and I' be the unit current flows from A_1 to A_0 with respect to C and C', respectively. By Thomson's principle, the principle of conservation of energy and the assumption $R(x,y) \le R'(x,y)$ for all $x, y \in E$, we have

$$\begin{aligned}
R_{\mathrm{eff}}(A_0 \leftrightarrow A_1) &= \frac{u(1) - u(0)}{I(A_1)} = u(1) - u(0)\\
&= \frac{1}{2}\sum_{x,y\in E} I(x,y)^2 R(x,y)\\
&\le \frac{1}{2}\sum_{x,y\in E} I'(x,y)^2 R(x,y) \le \frac{1}{2}\sum_{x,y\in E} I'(x,y)^2 R'(x,y)\\
&= u'(1) - u'(0) = R'_{\mathrm{eff}}(A_0 \leftrightarrow A_1).
\end{aligned}$$

□

19.4 Recurrence and Transience

We consider the situation where E is countable and $A_1 = \{x_1\}$ for some $x_1 \in E$. Let X be a random walk on E with weights $C = (C(x, y), x, y \in E)$ and hence with transition probabilities $p(x, y) = C(x, y)/C(x)$ (compare Definition 19.11).

The main goal of this section is to express the probability $1 - F(x_1, x_1)$ that the random walk never returns to x_1 in terms of effective resistances in the network. In order to apply the results on *finite* electrical networks from the last section, we henceforth assume that $A_0 \subset E$ is such that $E \setminus A_0$ is finite. We will obtain $1 - F(x_1, x_1)$ as the limit of the probability that a random walk started at x_1 hits A_0 before returning to x_1 as $A_0 \downarrow \emptyset$.

Let $u = u_{x_1,A_0}$ be the unique potential function on E with $u(x_1) = 1$ and $u(x) = 0$ for any $x \in A_0$. By Theorem 19.7, u is harmonic and can be written as

$$\begin{aligned} u_{x_1,A_0}(x) &= \mathbf{E}_x[\mathbb{1}_{\{X_{\tau_{A_0 \cup \{x_1\}}} = x_1\}}] \\ &= \mathbf{P}_x[\tau_{x_1} < \tau_{A_0}] \quad \text{for every } x \in E \setminus \big(A_0 \cup \{x_1\}\big). \end{aligned}$$

Hence the current flow I with respect to u satisfies

$$\begin{aligned} -I(A_0) = I(x_1) &= \sum_{x \in E} I(x_1, x) = \sum_{x \in E} \big(u(x_1) - u(x)\big) C(x_1, x) \\ &= C(x_1) \sum_{x \in E} \big(1 - u(x)\big) p(x_1, x) \\ &= C(x_1) \Bigg(\sum_{x \notin A_0 \cup \{x_1\}} p(x_1, x) \mathbf{P}_x[\tau_{A_0} < \tau_{x_1}] + \sum_{x \in A_0} p(x_1, x) \Bigg) \\ &= C(x_1) \mathbf{P}_{x_1}[\tau_{A_0} < \tau_{x_1}]. \end{aligned}$$

Therefore,

$$\begin{aligned} p_F(x_1, A_0) &:= \mathbf{P}_{x_1}[\tau_{A_0} < \tau_{x_1}] \\ &= \frac{C_{\text{eff}}(x_1 \leftrightarrow A_0)}{C(x_1)} = \frac{1}{C(x_1)} \frac{1}{R_{\text{eff}}(x_1 \leftrightarrow A_0)}. \end{aligned} \tag{19.9}$$

Definition 19.23 We denote the *escape probability*

$$p_F(x_1) = \mathbf{P}_{x_1}[\tau_{x_1} = \infty] = 1 - F(x_1, x_1).$$

We denote the effective conductance from x_1 to ∞ by

$$C_{\text{eff}}(x_1 \leftrightarrow \infty) := C(x_1) \inf\big\{p_F(x_1, A_0) : A_0 \subset E \text{ with } |E \setminus A_0| < \infty,\, A_0 \not\ni x_1\big\}.$$

Lemma 19.24 *For any decreasing sequence $A_0^n \downarrow \emptyset$ such that $|E \setminus A_0^n| < \infty$ and $x_1 \notin A_0^n$ for all $n \in \mathbb{N}$, we have*

$$C_{\text{eff}}(x_1 \leftrightarrow \infty) = \lim_{n \to \infty} C_{\text{eff}}\big(x_1 \leftrightarrow A_0^n\big).$$

Proof This is obvious since

$$C_{\text{eff}}(x_1 \leftrightarrow \infty) = C(x_1) \inf\bigl\{p_F(x_1, A_0) : |E \setminus A_0| < \infty, A_0 \not\ni x_1\bigr\} \tag{19.10}$$

and since $p_F(x_1, A_0)$ is monotone decreasing in A_0. □

Theorem 19.25 *We have*

$$p_F(x_1) = \frac{1}{C(x_1)} C_{\text{eff}}(x_1 \leftrightarrow \infty). \tag{19.11}$$

In particular,

$$x_1 \textit{ is recurrent} \iff C_{\text{eff}}(x_1 \leftrightarrow \infty) = 0 \iff R_{\text{eff}}(x_1 \leftrightarrow \infty) = \infty.$$

Proof Let $A_0^n \downarrow \emptyset$ be a decreasing sequence such that $|E \setminus A_0^n| < \infty$ and $x_1 \notin A_0^n$ for all $n \in \mathbb{N}$. Define $F_n := \{\tau_{A_0^n} < \tau_{x_1}\}$. For every $M \in \mathbb{N}$, we have

$$\mathbf{P}_{x_1}[\tau_{A_0^n} \le M] \le \sum_{k=0}^{M} \mathbf{P}_{x_1}\bigl[X_k \in A_0^n\bigr] \overset{n\to\infty}{\longrightarrow} 0.$$

Hence $\tau_{A_0^n} \uparrow \infty$ almost surely, and thus $F_n \downarrow \{\tau_{x_1} = \infty\}$ (up to a null set). We conclude

$$\frac{1}{C(x_1)} C_{\text{eff}}(x_1 \leftrightarrow \infty) = \lim_{n\to\infty} \mathbf{P}_{x_1}[F_n] = \mathbf{P}_{x_1}[\tau_{x_1} = \infty] = p_F(x_1).$$ □

Example 19.26 Symmetric simple random walk on $E = \mathbb{Z}$ is recurrent. Here $C(x, y) = \mathbb{1}_{\{|x-y|=1\}}$. The effective resistance from 0 to ∞ can be computed by the formulas for parallel and sequence connections,

$$R_{\text{eff}}(0 \leftrightarrow \infty) = \frac{1}{2} \sum_{i=0}^{\infty} R(i, i+1) = \infty.$$ ◊

Example 19.27 Asymmetric simple random walk on $E = \mathbb{Z}$ with $p(x, x+1) = p \in (\frac{1}{2}, 1)$, $p(x, x-1) = 1 - p$ is transient. Here one choice (and thus up to multiples the unique choice) for the conductances is

$$C(x, x+1) = \left(\frac{p}{1-p}\right)^x \quad \text{for } x \in \mathbb{Z},$$

and $C(x, y) = 0$ if $|x - y| > 1$. By the monotonicity principle, the effective resistance from 0 to ∞ can be bounded by

$$R_{\text{eff}}(0 \leftrightarrow \infty) = \lim_{n\to\infty} R_{\text{eff}}\bigl(0 \leftrightarrow \{-n, n\}\bigr)$$

$$\leq \lim_{n\to\infty} R_{\text{eff}}(0 \leftrightarrow n)$$
$$= \sum_{n=0}^{\infty} \left(\frac{1-p}{p}\right)^n = \frac{p}{2p-1} < \infty. \qquad \Diamond$$

Example 19.28 Symmetric simple random walk on $E = \mathbb{Z}^2$ is recurrent. Here again $C(x, y) = \mathbb{1}_{\{|x-y|=1\}}$. Let $B_n = \{-n, \ldots, n\}^2$ and $\partial B_n = B_n \setminus B_{n-1}$. We construct a network C' with greater conductances by adding ring-shaped *superconductors* along ∂B. (See Figs. 19.3 and 19.4 for illustrations.) That is, we replace $C(x, y)$ by

$$C'(x, y) = \begin{cases} \infty, & \text{if } x, y \in \partial B_n \text{ for some } n \in \mathbb{N}, \\ C(x, y), & \text{else.} \end{cases}$$

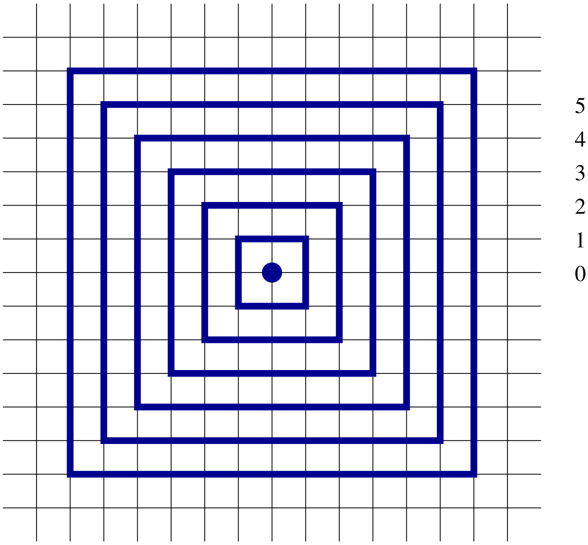


Fig. 19.3 Electrical network on $\mathbb{Z}^2$. The bold lines are *superconductors*. The nth and the $(n+1)$th superconductors are connected by $4(2n+1)$ edges

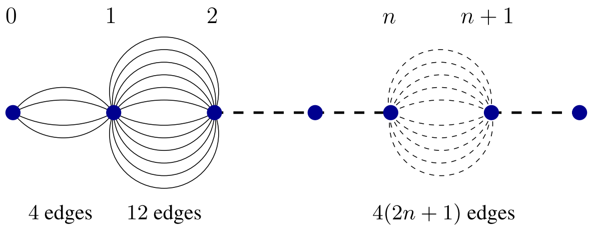


Fig. 19.4 Effective network after adding superconductors to $\mathbb{Z}^2$. The ring-shaped superconductors have melted down to single points

Then $R'_{\text{eff}}(B_n \leftrightarrow B_n^c) = \frac{1}{4(2n+1)}$ (note that there are $4(2n+1)$ edges that connect B_n with B_n^c), and thus

$$R'_{\text{eff}}(0 \leftrightarrow \infty) = \sum_{n=0}^{\infty} \frac{1}{4(2n+1)} = \infty.$$

By the monotonicity principle, we thus have $R_{\text{eff}}(0 \leftrightarrow \infty) \geq R'_{\text{eff}}(0 \leftrightarrow \infty) = \infty$. ◊

Example 19.29 Let (E, K) be an arbitrary connected subgraph of the square lattice $(\mathbb{Z}^2, \mathbb{L}^2)$. Then simple random walk on (E, K) (see Definition 19.11) is recurrent. Indeed, by the monotonicity principle, we have

$$R_{\text{eff}}^{(E,K)}(0 \leftrightarrow \infty) \geq R_{\text{eff}}^{(\mathbb{Z}^2,\mathbb{L}^2)}(0 \leftrightarrow \infty) = \infty.$$ ◊

We formulate the method used in the foregoing examples as a theorem.

Theorem 19.30 *Let C and C' be edge weights on E with $C'(x, y) \leq C(x, y)$ for all $x, y \in E$. If the Markov chain X with weights C is recurrent, then the Markov chain X' with weights C' is also recurrent.*
In particular, consider a graph (E, K) and a subgraph (E', K'). If simple random walk on (E, K) is recurrent, then so is simple random walk on (E', K').

Proof This follows from Theorem 19.25 and Rayleigh's monotonicity principle (Theorem 19.19). □

Example 19.31 Symmetric simple random walk on $\mathbb{Z}^3$ is transient. In order to prove this, we construct a subgraph for which we can compute $R'_{\text{eff}}(0 \leftrightarrow \infty) < \infty$.

Sketch. We consider the set of all infinite paths starting at 0 and that

- begin by taking one step in the x-direction, the y-direction or the z-direction,
- continue by choosing a possibly different direction x, y or z and make *two* steps in that direction, and
- at the nth stage choose a direction x, y or z and take 2^{n+1} steps in that direction.

For example, by $xyyxxxxzzzzzzzz\ldots$ we denote the path that starts with one step in direction x, then chooses y, then x, then z and so on. Note that after two paths follow different directions for the first time, they will not have any common *edge* again, though some of the *nodes* can be visited by both paths.

Consider the electrical network with unit resistors. Apply a voltage of 1 at the origin and 0 at the endpoints of the paths at the nth stage. By symmetry, the potential at a given node depends only on the distance (length of the shortest path) from the origin. We thus obtain an equivalent network if we replace multiply used nodes by multiple nodes (see Fig. 19.5). Thus we obtain a tree-shaped network: For any $n \in \mathbb{N}_0$, after 2^n steps each path splits into three (see Fig. 19.6). The 3^n paths leading from the nodes of the nth generation to those of the $(n+1)$th generation are disjoint

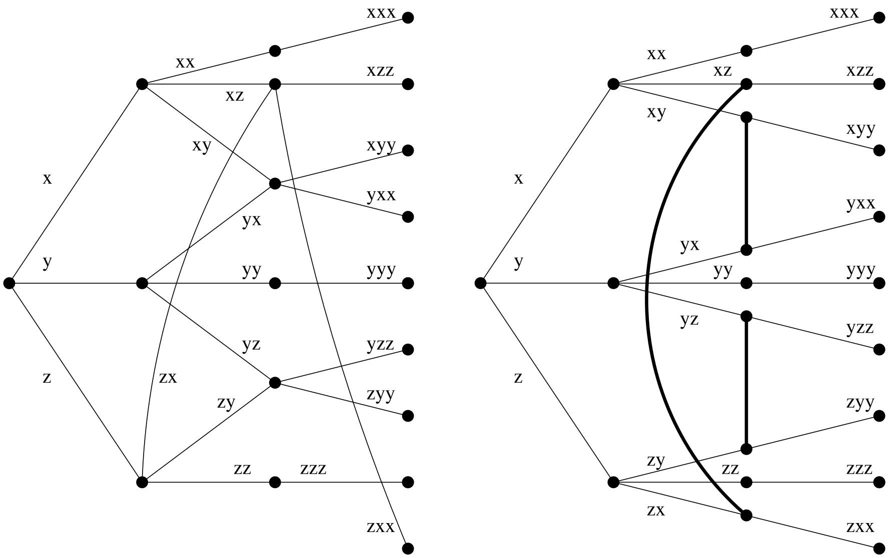


Fig. 19.5 Scheme of the first three steps (two stages) of the graph from Example 19.31. The left figure shows the actual edges where, e.g., xyy indicates that the first step is in direction x, the second step is in direction y and then the third step is necessarily also in direction y. In the right figure, the nodes at the ends of xz/zx, xy/yx and yz/zy are split into two nodes and then connected by a superconductor (bold line). If we remove the superconductors from the network, we end up with the network of Fig. 19.6 whose effective resistance $R'_{\text{eff}}(0 \leftrightarrow \infty)$ is not smaller than that of $\mathbb{Z}^3$. (If at the root we apply a voltage of 1 and at the points to the right the voltage 0, then by symmetry no current flows through the superconductors. Thus, in fact, the network is *equivalent* to that in Fig. 19.6)

paths, each of length 2^{n-1}. If $B(n)$ denotes the set of points up to the nth generation, then

$$R'_{\text{eff}}\big(0 \leftrightarrow B(n+1)^c\big) = \sum_{k=0}^{n-1} R'_{\text{eff}}\big(B(k) \leftrightarrow B(k)^c\big) = \sum_{k=0}^{n-1} 2^k 3^{-k}.$$

Therefore, $R'_{\text{eff}}(0 \leftrightarrow \infty) = \frac{1}{3}\sum_{k=0}^{\infty}(\frac{2}{3})^k = 1 < \infty$. On this tree, random walk is transient. Hence, by Theorem 19.30, random walk on $\mathbb{Z}^3$ is also transient. ◊

Exercise 19.4.1 Consider the electrical network on $\mathbb{Z}^d$ with unit resistors between neighboring points. Let X be a symmetric simple random walk on $\mathbb{Z}^d$. Finally, fix two arbitrary neighboring points $x_0, x_1 \in \mathbb{Z}^d$. Show the following:

(i) The effective conductance between x_0 and x_1 is $C_{\text{eff}}(x_0 \leftrightarrow x_1) = d$.
(ii) If $d \le 2$, then $\mathbf{P}_{x_0}[\tau_{x_1} < \tau_{x_0}] = \frac{1}{2}$.
(iii) If $d \ge 3$, then $\mathbf{P}_{x_0}[\tau_{x_1} < \tau_{x_0} \mid \tau_{x_0} \wedge \tau_{x_1} < \infty] = \frac{1}{2}$.

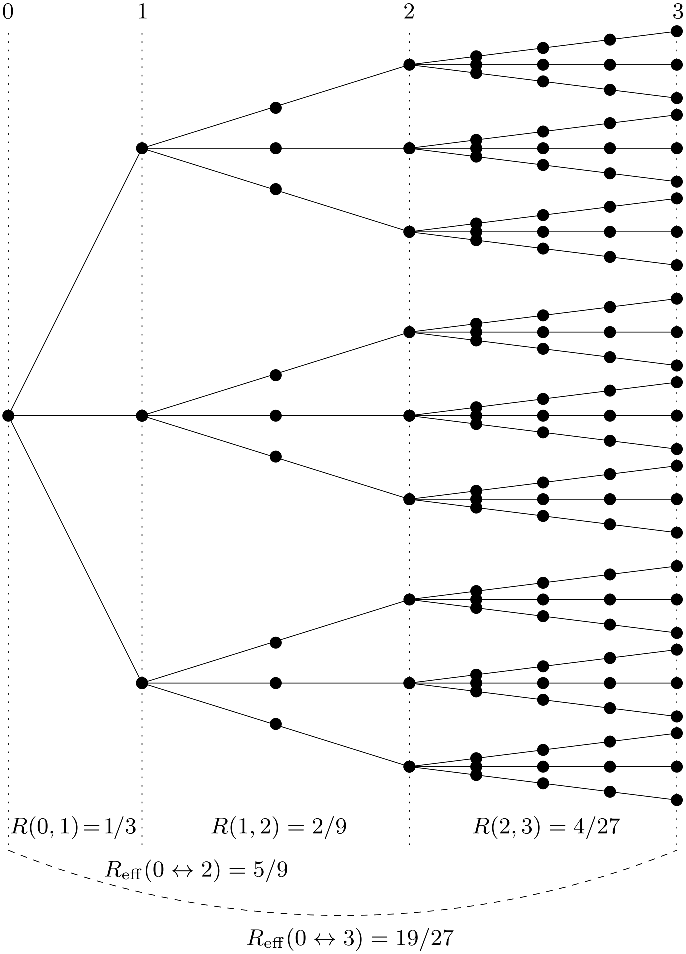


Fig. 19.6 A tree as a subgraph of $\mathbb{Z}^3$ on which random walk is still transient

19.5 Network Reduction

Example 19.32 Consider a random walk on the graph in Fig. 19.7 that starts at x and at each step jumps to one of its neighbors at random with equal probability. What is the probability P that this Markov chain visits 1 before it visits 0?

We can regard the graph as an electrical network with unit resistors at each edge, voltage 0 at 0 and voltage 1 at 1. Then P equals the voltage at point x:

$$P = u(x).$$

In order to compute $u(x)$, we replace the network step by step by simpler networks such that the effective resistances between 0, 1, and x remain unchanged. Hence in each step the voltage $u(x)$ at point x does not change. ◊

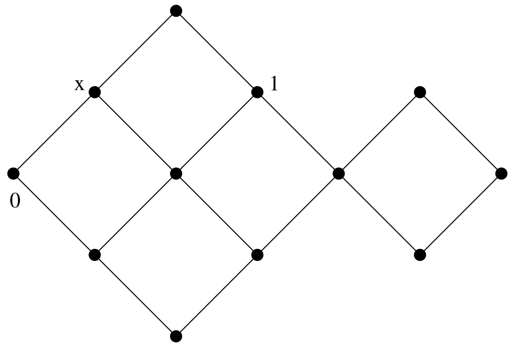


Fig. 19.7 Initial situation

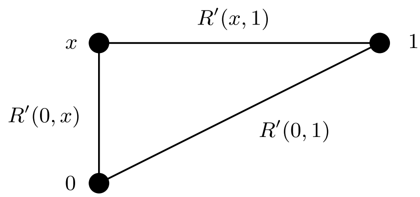


Fig. 19.8 Reduced network with three nodes

Reduced Network

Assume that we have already reduced the network to a network with the three points 0, 1 and x and with resistors between these points $R'(0,1)$, $R'(0,x)$ and $R'(1,x)$. See Fig. 19.8.

Clearly, we have

$$P = u(x) = \frac{R'(0,x)}{R'(0,x) + R'(1,x)}. \tag{19.12}$$

If we knew the effective resistances $R_{\text{eff}}(0 \leftrightarrow x)$, $R_{\text{eff}}(1 \leftrightarrow x)$ and $R_{\text{eff}}(0 \leftrightarrow 1)$, we could avoid the hassle of reducing the network and we could compute $u(x)$ directly. In order to derive the formula for $u(x)$, we make the following observations. In the reduced network, the effective resistances are easy to compute: If $\{a,b,c\} = \{0,1,x\}$, then

$$R_{\text{eff}}(a \leftrightarrow b) = \left(\frac{1}{R'(a,b)} + \frac{1}{R'(a,c) + R'(b,c)} \right)^{-1}. \tag{19.13}$$

Solving these three equations for $R'(0,1)$, $R'(0,x)$ and $R'(1,x)$ and plugging the values into (19.12) yields

$$P = u(x) = \frac{R_{\text{eff}}(0 \leftrightarrow 1) + R_{\text{eff}}(0 \leftrightarrow x) - R_{\text{eff}}(x \leftrightarrow 1)}{2R_{\text{eff}}(0 \leftrightarrow 1)}. \tag{19.14}$$

In particular, in the case $R'(0,1) = \infty$ (or equivalently $R_{\text{eff}}(0 \leftrightarrow 1) = R_{\text{eff}}(0 \leftrightarrow x) + R_{\text{eff}}(x \leftrightarrow 1)$), we have $R_{\text{eff}}(0 \leftrightarrow x) = R'(0, x)$ and $R_{\text{eff}}(1 \leftrightarrow x) = R'(1, x)$, hence

$$u(x) = \frac{R_{\text{eff}}(0 \leftrightarrow x)}{R_{\text{eff}}(0 \leftrightarrow x) + R_{\text{eff}}(x \leftrightarrow 1)}. \tag{19.15}$$

Since we always have $u(x) \in [0, 1]$, rearranging the terms yields (again in the general situation)

$$R_{\text{eff}}(1 \leftrightarrow x) \le R_{\text{eff}}(0 \leftrightarrow 1) + R_{\text{eff}}(0 \leftrightarrow x). \tag{19.16}$$

This is the triangle inequality for the effective resistances and it shows that the effective resistance is a metric in any electrical network.

Step-by-Step Reduction of the Network

Having seen how to compute $u(x)$ from the effective resistances, we now turn to the systematic computation of these effective resistances. Later we will come back to the introductory example and make the computations explicit.

There are four elementary transformations for the reduction of an electrical network:

1. **Deletion of loops.** The three points on the very right of the graph form a loop that can be deleted from the network without changing any of the remaining voltages. In particular, any edge that directly connects 0 to 1 can be deleted.
2. **Joining serial edges.** If two (or more) edges are in a row such that the nodes along them do not have any further adjacent edges, this sequence of edges can be substituted by a single edge whose resistance is the sum of the resistances of the single edges (see Fig. 19.1).
3. **Joining parallel edges.** Two (or more) edges with resistances $R_1, \ldots, R_n$ that connect the same two nodes can by replaced by a single edge with resistance $R = (R_1^{-1} + \ldots + R_n^{-1})^{-1}$ (see Fig. 19.2).
4. **Star–triangle transformation.** (See Exercise 19.5.1.) The star-shaped part of a network (left in Fig. 19.9) is equivalent to the triangle-shaped part (right in Fig. 19.9) if the resistances R_1, R_2, R_3, $\widetilde{R}_1$, $\widetilde{R}_2$, $\widetilde{R}_3$ satisfy the condition

$$R_i \tilde{R}_i = \delta \quad \text{for any } i = 1, 2, 3, \tag{19.17}$$

where

$$\delta = R_1 R_2 R_3 \big(R_1^{-1} + R_2^{-1} + R_3^{-1}\big) = \frac{\widetilde{R}_1 \widetilde{R}_2 \widetilde{R}_3}{\widetilde{R}_1 + \widetilde{R}_2 + \widetilde{R}_3}.$$

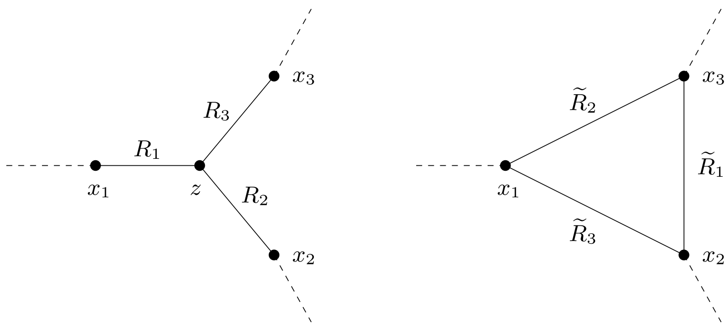


Fig. 19.9 Star–triangle transformation

Application to Example 19.32

With the four transformations at hand, we solve the problem of Example 19.32. Assume that initially all edges have resistance 1. In the figures we label each edge with its resistance if it differs (in the course of the reduction) from 1.

Step 1. Delete the loop at the right-hand side (left in Fig. 19.10).

Step 2. Replace the series on top, bottom and right by edges with resistance 2 (right in Fig. 19.10).

Step 3. Use the star-triangle transformation to remove the lower left node (left in Fig. 19.11). Here $R_1 = 1$, $R_2 = 2$, $R_3 = 1$, $\delta = 5$, $\widetilde{R}_1 = \delta/R_1 = 5$, $\widetilde{R}_2 = \delta/R_2 = 5/2$ and $\widetilde{R}_3 = \delta/R_3 = 5$.

Step 4. Replace the parallel edges with resistances $R_1 = 5$ and $R_2 = 1$ by one edge with $R = (\frac{1}{5} + 1)^{-1} = \frac{5}{6}$ (right in Fig. 19.11).

Step 5. Use the star-triangle transformation to remove the lower right node (left in Fig. 19.12). Here $R_1 = 5$, $R_2 = 2$, $R_3 = \frac{5}{6}$, $\delta = 95/6$, $\widetilde{R}_1 = \delta/R_1 = 19/6$, $\widetilde{R}_2 = \delta/R_2 = 95/12$ and $\widetilde{R}_3 = \delta/R_3 = 19$.

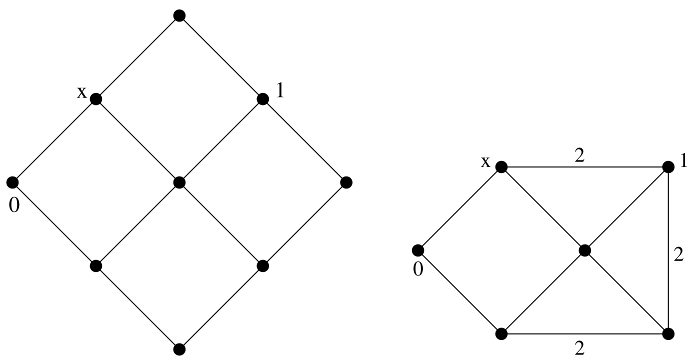


Fig. 19.10 Steps 1 and 2

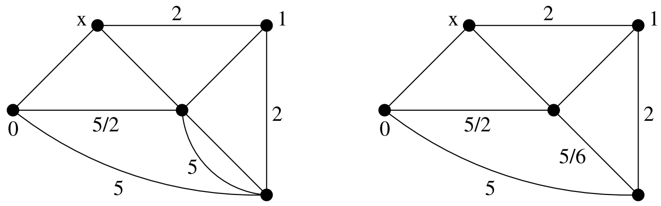


Fig. 19.11 Steps 3 and 4

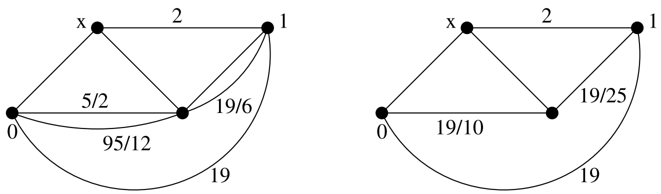


Fig. 19.12 Steps 5 and 6

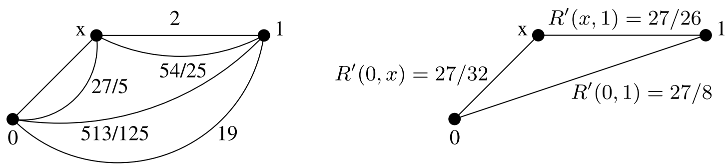


Fig. 19.13 Steps 7 and 8

Step 6. Replace the parallel edges by edges with resistances $(\frac{12}{95} + \frac{2}{5})^{-1} = \frac{19}{10}$ and $(\frac{6}{19} + 1)^{-1} = \frac{19}{25}$, respectively (right in Fig. 19.12).

Step 7. Use the star-triangle transformation to remove the lower right node (left in Fig. 19.13). Here $R_1 = \frac{19}{10}$, $R_2 = \frac{19}{25}$, $R_3 = 1$, $\delta = \frac{513}{125}$, $\widetilde{R}_1 = \delta/R_1 = \frac{54}{25}$, $\widetilde{R}_2 = \delta/R_2 = \frac{27}{5}$ and $\widetilde{R}_3 = \delta/R_3 = \frac{513}{125}$.

Step 8. Replace the three pairs of parallel edges by single edges with resistances $(\frac{5}{27} + 1)^{-1} = \frac{27}{32}$, $(\frac{25}{54} + \frac{1}{2})^{-1} = \frac{27}{26}$ and $(\frac{1}{19} + \frac{125}{513})^{-1} = \frac{27}{8}$, respectively.

In the reduced network, we have the resistances

$$R'(0, x) = \frac{27}{32} \quad \text{and} \quad R'(x, 1) = \frac{27}{26}.$$

Using (19.12), the probability that the random walk visits 1 before 0 is

$$P = \frac{\frac{27}{32}}{\frac{27}{32} + \frac{27}{26}} = \frac{13}{29}.$$

Using the values of $R'(0,x)$, $R'(1,x)$ and $R'(0,1)$ and Eq. (19.13), we compute the effective resistances in the reduced network (and hence in the original network):

$$R_{\text{eff}}(0 \leftrightarrow x) = \left(\frac{32}{27} + \frac{1}{\frac{27}{8} + \frac{27}{26}}\right)^{-1} = \frac{17}{24},$$

$$R_{\text{eff}}(1 \leftrightarrow x) = \left(\frac{26}{27} + \frac{1}{\frac{27}{32} + \frac{27}{8}}\right)^{-1} = \frac{5}{6},$$

$$R_{\text{eff}}(0 \leftrightarrow 1) = \left(\frac{8}{27} + \frac{1}{\frac{27}{26} + \frac{27}{32}}\right)^{-1} = \frac{29}{24}.$$

Using (19.14) we can use the values to compute $u(x)$:

$$P = u(x) = \frac{\frac{29}{24} + \frac{17}{24} - \frac{5}{6}}{2 \cdot \frac{29}{24}} = \frac{13}{29}.$$

Clearly, the latter computation is more complicated than using the resistances R' from the reduced network directly. However, it has the advantage that it can be performed without going through all the network reduction steps if, for some reason, we know the effective resistances already. For example, we could buy resistors in an electronic market, solder the network and measure the resistances with a multimeter.

Alternative Solution

A different approach to solving the problem of Example 19.32 is to use linear algebra instead of network reduction. It is a matter of taste as to which solution is preferable. First generate the transition matrix p of the Markov chain. To this end, enumerate the nodes of the graph from 1 to 12 as in Fig. 19.14.

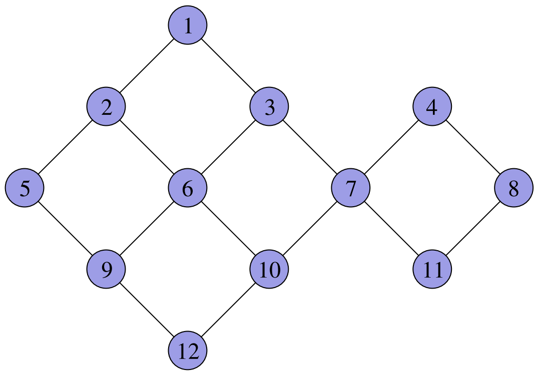


Fig. 19.14 Graph with enumerated nodes

The chain starts at 2, and we want to compute the probability that it visits 3 before 5.

Generate the matrix $\overline{p}$ of the chain that is killed at 3 and at 5 and compute $\overline{G} = (I - \overline{p})^{-1}$. By Exercise 19.1.1 (with $A = \{3, 5\}$, $x = 2$ and $y = 3$), the probability of visiting 3 before 5 is $P = \overline{G}(2, 3) = \frac{13}{29}$.

$$\overline{p} := \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix},$$

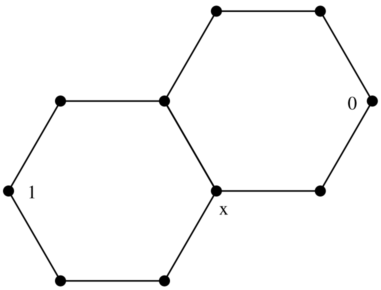
$$\overline{G} := (I - \overline{p})^{-1}$$

$$= \begin{pmatrix} \frac{143}{116} & \frac{81}{116} & \frac{21}{29} & \frac{3}{58} & \frac{8}{29} & \frac{19}{58} & \frac{3}{29} & \frac{3}{58} & \frac{15}{116} & \frac{9}{58} & \frac{3}{58} & \frac{11}{116} \\ \frac{27}{58} & \frac{81}{58} & \frac{13}{29} & \frac{3}{29} & \frac{16}{29} & \frac{19}{29} & \frac{6}{29} & \frac{3}{29} & \frac{15}{58} & \frac{9}{29} & \frac{3}{29} & \frac{11}{58} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{58} & \frac{9}{58} & \frac{24}{29} & \frac{165}{58} & \frac{5}{29} & \frac{15}{29} & \frac{78}{29} & \frac{68}{29} & \frac{21}{58} & \frac{30}{29} & \frac{107}{58} & \frac{27}{58} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{19}{116} & \frac{57}{116} & \frac{18}{29} & \frac{15}{58} & \frac{11}{29} & \frac{95}{58} & \frac{15}{29} & \frac{15}{58} & \frac{75}{116} & \frac{45}{58} & \frac{15}{58} & \frac{55}{116} \\ \frac{3}{58} & \frac{9}{58} & \frac{24}{29} & \frac{39}{29} & \frac{5}{29} & \frac{15}{29} & \frac{78}{29} & \frac{39}{29} & \frac{21}{58} & \frac{30}{29} & \frac{39}{29} & \frac{27}{58} \\ \frac{3}{58} & \frac{9}{58} & \frac{24}{29} & \frac{68}{29} & \frac{5}{29} & \frac{15}{29} & \frac{78}{29} & \frac{97}{29} & \frac{21}{58} & \frac{30}{29} & \frac{68}{29} & \frac{27}{58} \\ \frac{5}{58} & \frac{15}{58} & \frac{11}{29} & \frac{7}{29} & \frac{18}{29} & \frac{25}{29} & \frac{14}{29} & \frac{7}{29} & \frac{93}{58} & \frac{21}{29} & \frac{7}{29} & \frac{45}{58} \\ \frac{3}{29} & \frac{9}{29} & \frac{19}{29} & \frac{20}{29} & \frac{10}{29} & \frac{30}{29} & \frac{40}{29} & \frac{20}{29} & \frac{21}{29} & \frac{60}{29} & \frac{20}{29} & \frac{27}{29} \\ \frac{3}{58} & \frac{9}{58} & \frac{24}{29} & \frac{107}{58} & \frac{5}{29} & \frac{15}{29} & \frac{78}{29} & \frac{68}{29} & \frac{21}{58} & \frac{30}{29} & \frac{165}{58} & \frac{27}{58} \\ \frac{11}{116} & \frac{33}{116} & \frac{15}{29} & \frac{27}{58} & \frac{14}{29} & \frac{55}{58} & \frac{27}{29} & \frac{27}{58} & \frac{135}{116} & \frac{81}{58} & \frac{27}{58} & \frac{215}{116} \end{pmatrix}.$$

Exercise 19.5.1 Show the validity of the star-triangle transformation.

Exercise 19.5.2 Consider a random walk on the honeycomb graph shown below. Show that if the walk starts at x, then the probability of visiting 1 before 0 is $\frac{8}{17}$ using

(i) the method of network reduction, and
(ii) the method of matrix inversion.



Exercise 19.5.3 Consider the graph of Fig. 19.15.

(i) For the effective conductance between a and z, show that $C_{\text{eff}}(a \longleftrightarrow z) = \sqrt{3}$.
(ii) For a random walk started at a, show that the probability $\mathbf{P}_a[\tau_z < \tau_a]$ of visiting z before returning to a is $\mathbf{P}_a[\tau_z < \tau_a] = 1/\sqrt{3}$.

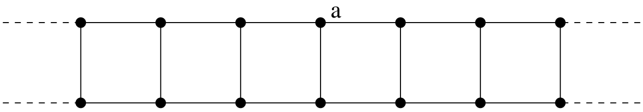


Fig. 19.15 Simple ladder graph

Exercise 19.5.4 For the graph of Fig. 19.16, determine $C_{\text{eff}}(a \longleftrightarrow z)$ and $\mathbf{P}_a[\tau_z < \tau_a]$. (This is simpler than in Exercise 19.5.3!)

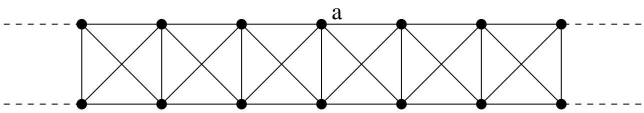


Fig. 19.16 Crossed ladder graph

Exercise 19.5.5 For a random walk on the graph of Fig. 19.17, determine the probability $\mathbf{P}_a[\tau_z < \tau_a]$.

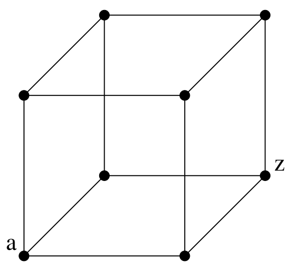


Fig. 19.17 Random walk on a hypercube

19.6 Random Walk in a Random Environment

(Compare [174], [142] and [76, 77], [93].) Consider a Markov chain X on $\mathbb{Z}$ that at each step makes a jump either to the left (with probability w_i^-) or to the right (with probability w_i^+) if X is at $i \in \mathbb{Z}$. Hence, let $w_i^- \in (0, 1)$ and $w_i^+ := 1 - w_i^-$ for $i \in \mathbb{Z}$. Then X is the Markov chain with transition matrix

$$p_w(i, j) = \begin{cases} w_i^-, & \text{if } j = i - 1, \\ w_i^+, & \text{if } j = i + 1, \\ 0, & \text{else.} \end{cases}$$

We consider $(w_i^-)_{i \in \mathbb{Z}}$ as an environment in which X walks and later choose the environment at random.

In order to describe X in terms of conductances of an electrical network, we define $\varrho_i := w_i^- / w_i^+$ for $i \in \mathbb{Z}$. Let $C_w(i, j) := 0$ if $|i - j| \neq 1$ and

$$C_w(i + 1, i) := C_w(i, i + 1) := \begin{cases} \prod_{k=0}^{i} \varrho_k^{-1}, & \text{if } i \geq 0, \\ \prod_{k=i}^{-1} \varrho_k, & \text{if } i < 0. \end{cases}$$

With this definition,

$$\frac{C_w(i, i + 1)}{C_w(i)} = \frac{1}{\varrho_i + 1} = w_i^+ \quad \text{and} \quad \frac{C_w(i, i - 1)}{C_w(i)} = \frac{\varrho_i}{\varrho_i + 1} = w_i^-.$$

Hence the transition probabilities p_w are indeed described by the C_w. Let

$$R_w^+ := \sum_{i=0}^{\infty} R_w(i, i + 1) = \sum_{i=0}^{\infty} \frac{1}{C_w(i, i + 1)} = \sum_{i=0}^{\infty} \prod_{k=0}^{i} \varrho_k$$

and

$$R_w^- := \sum_{i=0}^{\infty} R_w(-i, -i - 1) = \sum_{i=0}^{\infty} \frac{1}{C_w(-i, -i - 1)} = \sum_{i=1}^{\infty} \prod_{k=-i}^{1} \varrho_k^{-1}.$$

Note that R_w^+ and R_w^- are the effective resistances from 0 to $+\infty$ and from 0 to $-\infty$, respectively. Hence

$$R_{w,\text{eff}}(0 \leftrightarrow \infty) = \frac{1}{\frac{1}{R_\omega^-} + \frac{1}{R_\omega^+}}$$

is finite if and only if $R_w^- < \infty$ or $R_w^+ < \infty$. Therefore, by Theorem 19.25,

$$X \text{ is transient} \quad \Longleftrightarrow \quad R_w^- < \infty \text{ or } R_w^+ < \infty. \tag{19.18}$$

If X is transient, in which direction does it get lost?

Theorem 19.33

(i) *If* $R_w^- < \infty$ *or* $R_w^+ < \infty$, *then* (*agreeing on* $\frac{\infty}{\infty} = 1$)

$$\mathbf{P}_0[X_n \overset{n\to\infty}{\longrightarrow} -\infty] = \frac{R_w^+}{R_w^- + R_w^+} \quad \textit{and} \quad \mathbf{P}_0[X_n \overset{n\to\infty}{\longrightarrow} +\infty] = \frac{R_w^-}{R_w^- + R_w^+}.$$

(ii) *If* $R_w^- = \infty$ *and* $R_w^+ = \infty$, *then* $\liminf_{n\to\infty} X_n = -\infty$ *and* $\limsup_{n\to\infty} X_n = \infty$ *almost surely.*

Proof (i) Let $\tau_N := \inf\{n \in \mathbb{N}_0 : X_n \in \{-N, N\}\}$. As X is transient, we have $\mathbf{P}_0[\tau_N < \infty] = 1$ and (as in (19.8))

$$\mathbf{P}_0[X_{\tau_N} = -N] = \frac{R_{w,\text{eff}}(0 \leftrightarrow N)}{R_{w,\text{eff}}(-N \leftrightarrow N)} = \frac{R_{w,\text{eff}}(0 \leftrightarrow N)}{R_{w,\text{eff}}(0 \leftrightarrow -N) + R_{w,\text{eff}}(0 \leftrightarrow N)}.$$

Again, since X is transient, we infer

$$\begin{aligned}\mathbf{P}_0[X_n \overset{n\to\infty}{\longrightarrow} -\infty] &= \mathbf{P}\big[\sup\{X_n : n \in \mathbb{N}_0\} < \infty\big] \\ &= \lim_{N\to\infty} \mathbf{P}\big[\sup\{X_n : n \in \mathbb{N}_0\} < N\big] \\ &\le \limsup_{N\to\infty} \mathbf{P}[X_{\tau_N} = -N] \\ &= \frac{R_w^+}{R_w^- + R_w^+}.\end{aligned}$$

By symmetry (and since X is transient), we get

$$\mathbf{P}_0[X_n \overset{n\to\infty}{\longrightarrow} -\infty] = 1 - \mathbf{P}_0[X_n \overset{n\to\infty}{\longrightarrow} \infty] \ge 1 - \frac{R_w^-}{R_w^- + R_w^+} = \frac{R_w^+}{R_w^- + R_w^+}.$$

(ii) If $R_w^- = R_w^+ = \infty$, then X is recurrent and hence *every* point is visited infinitely often. That is, $\limsup_{n\to\infty} X_n = \infty$ and $\liminf_{n\to\infty} X_n = -\infty$ a.s. □

We now consider the situation where the sequence $w = (w_i^-)_{i\in\mathbb{Z}}$ is also random. That is, we consider a two-stage experiment: At the first stage we choose a realization of i.i.d. random variables $W = (W_i^-)_{i\in\mathbb{Z}}$ on $(0, 1)$ and let $W_i^+ := 1 - W_i^-$. At the second stage, given W, we construct a Markov chain X on $\mathbb{Z}$ with transition matrix

$$p_W(i, j) = \begin{cases} W_i^-, & \text{if } j = i - 1, \\ W_i^+, & \text{if } j = i + 1, \\ 0, & \text{else.} \end{cases}$$

Note that X is a Markov chain only given W; that is, under the probability measure $\mathbf{P}[X \in \cdot \mid W]$. However, it is not a Markov chain with respect to the so-called annealed measure $\mathbf{P}[X \in \cdot\,]$. In fact, if W is unknown, observing X gives an increasing amount of information on the true realization of W. This is precisely what memory is and is thus in contrast with the Markov property of X.

Definition 19.34 The process X is called a *random walk in the random environment* W.

We are now in the position to prove a theorem of Solomon [157]. Let $\varrho_i := W_i^- / W_i^+$ for $i \in \mathbb{Z}$ and R_W^- and R_W^+ be defined as above.

Theorem 19.35 (Solomon (1975)) *Assume that* $\mathbf{E}[|\log(\varrho_0)|] < \infty$.

(i) *If* $\mathbf{E}[\log(\varrho_0)] < 0$, *then* $X_n \overset{n\to\infty}{\longrightarrow} \infty$ *a.s.*
(ii) *If* $\mathbf{E}[\log(\varrho_0)] > 0$, *then* $X_n \overset{n\to\infty}{\longrightarrow} -\infty$ *a.s.*
(iii) *If* $\mathbf{E}[\log(\varrho_0)] = 0$, *then* $\liminf_{n\to\infty} X_n = -\infty$ *and* $\limsup_{n\to\infty} X_n = \infty$ *a.s.*

Proof (i) and (ii) By symmetry, it is enough to show (ii). Hence, let $c := \mathbf{E}[\log(\varrho_0)] > 0$. By the strong law of large numbers, there is an $n_0^- = n_0^-(\omega)$ with

$$\prod_{k=-n}^{1} \varrho_k^{-1} = \exp\left(-\sum_{k=-n}^{1} \log(\varrho_i)\right) < e^{-cn/2} \quad \text{for all } n \ge n_0^-.$$

Therefore,

$$R_W^- = \sum_{n=1}^{\infty} \prod_{k=-n}^{1} \varrho_k^{-1} \le \sum_{n=1}^{n_0^- -1} \prod_{k=-n}^{1} \varrho_k^{-1} + \sum_{n=n_0^-}^{\infty} e^{-cn/2} < \infty \quad \text{a.s.}$$

Similarly, there is an $n_0^+ = n_0^+(\omega)$ with

$$\prod_{k=0}^{n} \varrho_k > e^{cn/2} \quad \text{for all } n \ge n_0^+.$$

We conclude

$$R_W^+ = \sum_{n=0}^{\infty} \prod_{k=0}^{n} \varrho_k \ge \sum_{n=0}^{n_0^+ -1} \prod_{k=0}^{n} \varrho_k + \sum_{n=n_0^+}^{\infty} e^{cn/2} = \infty \quad \text{a.s.}$$

Now, by Theorem 19.33, we get $X_n \overset{n\to\infty}{\longrightarrow} -\infty$ almost surely.

(iii) In order to show $R_W^- = R_W^+ = \infty$ almost surely, it is enough to show $\limsup_{n\to\infty} \sum_{k=0}^{n} \log(\varrho_k) > -\infty$ and $\limsup_{n\to\infty} \sum_{k=-n}^{1} \log(\varrho_k^{-1}) > -\infty$ almost surely if $\mathbf{E}[\log(\varrho_0)] = 0$. If $\log(\varrho_0)$ has a finite variance, this follows by the central limit theorem. In the general case, it follows by Theorem 20.21. □

Exercise 19.6.1 Consider the situation of Theorem 19.35 but with the random walk restricted to $\mathbb{N}_0$. To this end, change the walk so that whenever it attempts to make a step from 0 to -1, it simply stays in 0. Show that this random walk in a random environment is

- a.s. transient if $\mathbf{E}[\log(\varrho_0)] < \infty$,
- a.s. null recurrent if $\mathbf{E}[\log(\varrho_0)] = \infty$, and
- a.s. positive recurrent if $\mathbf{E}[\log(\varrho_0)] > \infty$.