Chapter 10
Optional Sampling Theorems

In Chapter 9 we saw that martingales are transformed into martingales if we apply certain admissible gambling strategies. In this chapter, we establish a similar stability property for martingales that are stopped at a random time. In order also to obtain these results for submartingales and supermartingales, in the first section, we start with a decomposition theorem for adapted processes. We show the optional sampling and optional stopping theorems in the second section. The chapter finishes with the investigation of random stopping times with an infinite time horizon.

10.1 Doob Decomposition and Square Variation

Let $X = (X_n)_{n\in\mathbb{N}_0}$ be an adapted process with $\mathbf{E}[|X_n|] < \infty$ for all $n \in \mathbb{N}_0$. We will decompose X into a sum consisting of a martingale and a predictable process. To this end, for $n \in \mathbb{N}_0$, define

$$M_n := X_0 + \sum_{k=1}^{n} \bigl(X_k - \mathbf{E}[X_k \mid \mathcal{F}_{k-1}]\bigr) \tag{10.1}$$

and

$$A_n := \sum_{k=1}^{n} \bigl(\mathbf{E}[X_k \mid \mathcal{F}_{k-1}] - X_{k-1}\bigr).$$

Evidently, $X_n = M_n + A_n$. By construction, A is predictable with $A_0 = 0$, and M is a martingale since

$$\mathbf{E}[M_n - M_{n-1} \mid \mathcal{F}_{n-1}] = \mathbf{E}\bigl[X_n - \mathbf{E}[X_n \mid \mathcal{F}_{n-1}] \bigm| \mathcal{F}_{n-1}\bigr] = 0.$$

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Theorem 10.1 (Doob decomposition) *Let $X = (X_n)_{n\in\mathbb{N}_0}$ be an adapted integrable process. Then there exists a unique decomposition $X = M + A$, where A is predictable with $A_0 = 0$ and M is a martingale. This representation of X is called the* Doob decomposition. *X is a submartingale if and only if A is monotone increasing.*

Proof We only have to show uniqueness of the decomposition. Hence, let $X = M + A = M' + A'$ be two such decompositions. Then $M - M' = A' - A$ is a predictable martingale; hence (see Exercise 9.2.2) $M_n - M'_n = M_0 - M'_0 = 0$ for all $n \in \mathbb{N}_0$. □

Example 10.2 Let $I = \mathbb{N}_0$ or $I = \{0, \dots, N\}$. Let $(X_n)_{n\in I}$ be a square integrable $\mathbb{F}$-martingale (that is, $\mathbf{E}[X_n^2] < \infty$ for all $n \in I$). By Theorem 9.35, $Y := (X_n^2)_{n\in I}$ is a submartingale. Let $Y = M + A$ be the Doob decomposition of Y. Then $(X_n^2 - A_n)_{n\in I}$ is a martingale. Furthermore, $\mathbf{E}[X_{i-1}X_i \mid \mathcal{F}_{i-1}] = X_{i-1}\mathbf{E}[X_i \mid \mathcal{F}_{i-1}] = X_{i-1}^2$; hence (as in (10.1))

$$\begin{aligned} A_n &= \sum_{i=1}^{n} \Big(\mathbf{E}\big[X_i^2 \,\big|\, \mathcal{F}_{i-1}\big] - X_{i-1}^2\Big) \\ &= \sum_{i=1}^{n} \Big(\mathbf{E}\big[(X_i - X_{i-1})^2 \,\big|\, \mathcal{F}_{i-1}\big] - 2X_{i-1}^2 + 2\mathbf{E}[X_{i-1}X_i \mid \mathcal{F}_{i-1}]\Big) \\ &= \sum_{i=1}^{n} \mathbf{E}\big[(X_i - X_{i-1})^2 \,\big|\, \mathcal{F}_{i-1}\big]. \end{aligned}$$

◊

Definition 10.3 Let $(X_n)_{n\in I}$ be a square integrable $\mathbb{F}$-martingale. The unique predictable process A for which $(X_n^2 - A_n)_{n\in I}$ becomes a martingale is called the *square variation process* of X and is denoted by $\big(\langle X\rangle_n\big)_{n\in I} := A$.

By the preceding example, we conclude the following theorem.

Theorem 10.4 *Let X be as in Definition 10.3. Then, for $n \in \mathbb{N}_0$,*

$$\langle X\rangle_n = \sum_{i=1}^{n} \mathbf{E}\big[(X_i - X_{i-1})^2 \,\big|\, \mathcal{F}_{i-1}\big] \tag{10.2}$$

and

$$\mathbf{E}\big[\langle X\rangle_n\big] = \mathbf{Var}[X_n - X_0]. \tag{10.3}$$

Remark 10.5 If Y and A are as in Example 10.2, then A is monotone increasing since $(X_n^2)_{n\in I}$ is a submartingale (see Theorem 10.1). Therefore, A is sometimes called the *increasing process* of Y. ◊

Example 10.6 Let $Y_1, Y_2, \ldots$ be independent, square integrable, centered random variables. Then $X_n := Y_1 + \ldots + Y_n$ defines a square integrable martingale with $\langle X\rangle_n = \sum_{i=1}^n \mathbf{E}[Y_i^2]$. In fact, $A_n = \sum_{i=1}^n \mathbf{E}[Y_i^2 \mid Y_1, \ldots, Y_{i-1}] = \sum_{i=1}^n \mathbf{E}[Y_i^2]$ (as in Example 10.2).

Note that in order for $\langle X\rangle$ to have the simple form as in Example 10.6, it is not enough for the random variables $Y_1, Y_2, \ldots$ to be uncorrelated. ◊

Example 10.7 Let $Y_1, Y_2, \ldots$ be independent, square integrable random variables with $\mathbf{E}[Y_n] = 1$ for all $n \in \mathbb{N}$. Let $X_n := \prod_{i=1}^n Y_i$ for $n \in \mathbb{N}_0$. Then $X = (X_n)_{n\in\mathbb{N}_0}$ is a square integrable martingale with respect to $\mathbb{F} = \sigma(X)$ (why?) and

$$\mathbf{E}\big[(X_n - X_{n-1})^2 \,\big|\, \mathcal{F}_{n-1}\big] = \mathbf{E}\big[(Y_n - 1)^2 X_{n-1}^2 \,\big|\, \mathcal{F}_{n-1}\big] = \mathbf{Var}[Y_n] X_{n-1}^2.$$

Hence $\langle X\rangle_n = \sum_{i=1}^n \mathbf{Var}[Y_i] X_{i-1}^2$. We see that the square variation process can indeed be a truly random process. ◊

Example 10.8 Let $(X_n)_{n\in\mathbb{N}_0}$ be the one-dimensional symmetric simple random walk

$$X_n = \sum_{i=1}^n R_i \quad \text{for all } n \in \mathbb{N}_0,$$

where $R_1, R_2, R_3, \ldots$ are i.i.d. and $\sim \mathrm{Rad}_{1/2}$; that is,

$$\mathbf{P}[R_i = 1] = 1 - \mathbf{P}[R_i = -1] = \frac{1}{2}.$$

Clearly, X is a martingale and hence $|X|$ is a submartingale. Let $|X| = M + A$ be Doob's decomposition of $|X|$. Then

$$A_n = \sum_{i=1}^n \Big(\mathbf{E}\big[|X_i| \,\big|\, \mathcal{F}_{i-1}\big] - |X_{i-1}|\Big).$$

Now

$$|X_i| = \begin{cases} |X_{i-1}| + R_i, & \text{if } X_{i-1} > 0, \\ |X_{i-1}| - R_i, & \text{if } X_{i-1} < 0, \\ 1, & \text{if } X_{i-1} = 0. \end{cases}$$

Therefore,

$$\mathbf{E}\big[|X_i| \,\big|\, \mathcal{F}_{i-1}\big] = \begin{cases} |X_{i-1}|, & \text{if } |X_{i-1}| \neq 0, \\ 1, & \text{if } |X_{i-1}| = 0. \end{cases}$$

The process

$$A_n = \#\{i \le n-1 : |X_i| = 0\}$$

is the so-called *local time* of X at 0. We conclude that (since $\mathbf{P}[X_{2j}=0]=\binom{2j}{j}4^{-j}$ and $\mathbf{P}[X_{2j+1}=0]=0$)

$$\begin{aligned}\mathbf{E}\big[|X_n|\big] &= \mathbf{E}\big[\#\{i\le n-1: X_i=0\}\big]\\ &= \sum_{i=0}^{n-1}\mathbf{P}[X_i=0] = \sum_{j=0}^{\lfloor (n-1)/2\rfloor}\binom{2j}{j}4^{-j}.\end{aligned}$$

◊

Example 10.9 We want to generalize the preceding example further. Evidently, we did not use (except in the last formula) the fact that X is a random walk. Rather, we just used the fact that the differences $(\Delta X)_n := X_n - X_{n-1}$ take only the values -1 and $+1$. Hence, now let X be a martingale with $|X_n - X_{n-1}| = 1$ almost surely for all $n\in\mathbb{N}$ and with $X_0 = x_0\in\mathbb{Z}$ almost surely. Let $f:\mathbb{Z}\to\mathbb{R}$ be an arbitrary map. Then $Y := (f(X_n))_{n\in\mathbb{N}_0}$ is an integrable adapted process (since $|f(X_n)|\le \max_{x\in\{x_0-n,\ldots,x_0+n\}}|f(x)|$). In order to compute Doob's decomposition of Y, define the first and second discrete derivatives of f:

$$f'(x) := \frac{f(x+1)-f(x-1)}{2}$$

and

$$f''(x) := f(x-1)+f(x+1)-2f(x).$$

Further, let $F'_n := f'(X_{n-1})$ and $F''_n := f''(X_{n-1})$. By computing the cases $X_n = X_{n-1}-1$ and $X_n = X_{n-1}+1$ separately, we see that for all $n\in\mathbb{N}$

$$\begin{aligned}f(X_n)-f(X_{n-1}) &= \frac{f(X_{n-1}+1)-f(X_{n-1}-1)}{2}(X_n - X_{n-1})\\ &\quad + \frac{1}{2}f(X_{n-1}-1)+\frac{1}{2}f(X_{n-1}+1)-f(X_{n-1})\\ &= f'(X_{n-1})(X_n-X_{n-1})+\frac{1}{2}f''(X_{n-1})\\ &= F'_n\cdot(X_n-X_{n-1})+\frac{1}{2}F''_n.\end{aligned}$$

Summing up, we get the *discrete Itô formula*:

$$\begin{aligned}f(X_n) &= f(x_0)+\sum_{i=1}^{n} f'(X_{i-1})(X_i-X_{i-1})+\sum_{i=1}^{n}\frac{1}{2}f''(X_{i-1})\\ &= f(x_0)+\big(F'{\cdot}X\big)_n+\sum_{i=1}^{n}\frac{1}{2}F''_i.\end{aligned}\tag{10.4}$$

Here $F'{\cdot}X$ is the discrete stochastic integral (see Definition 9.37). Now $M := f(x_0)+F'{\cdot}X$ is a martingale by Theorem 9.39 since F' is predictable (and since $|F'_n| \le \max_{x\in\{x_0-n,\dots,x_0+n\}} |F'(x)|$), and $A := (\sum_{i=1}^n \frac{1}{2} F''_i)_{n\in\mathbb{N}_0}$ is predictable. Hence $f(X) := (f(X_n))_{n\in\mathbb{N}_0} = M + A$ is the Doob decomposition of $f(X)$. In particular, $f(X)$ is a submartingale if $f''(x) \ge 0$ for all $x \in \mathbb{Z}$; that is, if f is convex. We knew this already from Theorem 9.35; however, here we could also quantify how much $f(X)$ differs from a martingale.

In the special cases $f(x) = x^2$ and $f(x) = |x|$, the second derivative is $f''(x) = 2$ and $f''(x) = 2 \cdot \mathbb{1}_{\{0\}}(x)$, respectively. Thus, from (10.4), we recover the statements of Theorem 10.4 and Example 10.8.

Later we will derive a formula similar to (10.4) for stochastic processes in continuous time (see Section 25.3). ◊

10.2 Optional Sampling and Optional Stopping

Lemma 10.10 *Let $I \subset \mathbb{R}$ be countable, let $(X_t)_{t\in I}$ be a martingale, let $T \in I$ and let τ be a stopping time with $\tau \le T$. Then $X_\tau = \mathbf{E}[X_T \mid \mathcal{F}_\tau]$ and, in particular, $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$.*

Proof It is enough to show that $\mathbf{E}[X_T \mathbb{1}_A] = \mathbf{E}[X_\tau \mathbb{1}_A]$ for all $A \in \mathcal{F}_\tau$. By the definition of $\mathcal{F}_\tau$, we have $\{\tau = t\} \cap A \in \mathcal{F}_t$ for all $t \in I$. Hence

$$\begin{aligned}\mathbf{E}[X_\tau \mathbb{1}_A] &= \sum_{t\le T} \mathbf{E}[X_t \mathbb{1}_{\{\tau=t\}\cap A}] = \sum_{t\le T} \mathbf{E}\big[\mathbf{E}[X_T \mid \mathcal{F}_t]\mathbb{1}_{\{\tau=t\}\cap A}\big] \\ &= \sum_{t\le T} \mathbf{E}[X_T \mathbb{1}_A \mathbb{1}_{\{\tau=t\}}] = \mathbf{E}[X_T \mathbb{1}_A].\end{aligned}$$

□

Theorem 10.11 (Optional sampling theorem) *Let $X = (X_n)_{n\in\mathbb{N}_0}$ be a supermartingale and let $\sigma \le \tau$ be stopping times.*

(i) *Assume there exists a $T \in \mathbb{N}$ with $\tau \le T$. Then*

$$X_\sigma \ge \mathbf{E}[X_\tau \mid \mathcal{F}_\sigma],$$

and, in particular, $\mathbf{E}[X_\sigma] \ge \mathbf{E}[X_\tau]$. If X is a martingale, then equality holds in each case.

(ii) *If X is nonnegative and if $\tau < \infty$ a.s., then we have $\mathbf{E}[X_\tau] \le \mathbf{E}[X_0] < \infty$, $\mathbf{E}[X_\sigma] \le \mathbf{E}[X_0] < \infty$ and $X_\sigma \ge \mathbf{E}[X_\tau \mid \mathcal{F}_\sigma]$.*

(iii) *Assume that, more generally, X is only adapted and integrable. Then X is a martingale if and only if $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ for any bounded stopping time τ.*

Proof (i) Let $X = M + A$ be Doob's decomposition of X. Hence A is predictable and monotone decreasing, $A_0 = 0$, and M is a martingale. Applying Lemma 10.10

to M yields

$$\begin{aligned} X_\sigma = A_\sigma + M_\sigma &= \mathbf{E}[A_\sigma + M_T \mid \mathcal{F}_\sigma] \\ &\geq \mathbf{E}[A_\tau + M_T \mid \mathcal{F}_\sigma] = \mathbf{E}\big[A_\tau + \mathbf{E}[M_T \mid \mathcal{F}_\tau] \,\big|\, \mathcal{F}_\sigma\big] \\ &= \mathbf{E}[A_\tau + M_\tau \mid \mathcal{F}_\sigma] = \mathbf{E}[X_\tau \mid \mathcal{F}_\sigma]. \end{aligned}$$

Here we used $\mathcal{F}_\tau \supset \mathcal{F}_\sigma$, the tower property and the monotonicity of the conditional expectation (see Theorem 8.14).

(ii) We have $X_{\tau\wedge n} \overset{n\to\infty}{\longrightarrow} X_\tau$ almost surely. By (i), we get $\mathbf{E}[X_{\tau\wedge n}] \leq \mathbf{E}[X_0]$ for any $n \in \mathbb{N}$. Using Fatou's lemma, we infer

$$\mathbf{E}[X_\tau] \leq \liminf_{n\to\infty} \mathbf{E}[X_{\tau\wedge n}] \leq \mathbf{E}[X_0] < \infty.$$

Similarly, we can show that $\mathbf{E}[X_\sigma] \leq \mathbf{E}[X_0]$.

Now, let $m, n \in \mathbb{N}$ with $m \geq n$. Part (i) applied to the bounded stopping times $\tau \wedge m \geq \sigma \wedge n$ yields

$$X_{\sigma\wedge n} \geq \mathbf{E}[X_{\tau\wedge m} \mid \mathcal{F}_{\sigma\wedge n}].$$

Now $\{\sigma < n\} \cap A \in \mathcal{F}_{\sigma\wedge n}$ for $A \in \mathcal{F}_\sigma$. Hence

$$\mathbf{E}[X_\sigma \mathbb{1}_{\{\sigma<n\}\cap A}] = \mathbf{E}[X_{\sigma\wedge n} \mathbb{1}_{\{\sigma<n\}\cap A}] \geq \mathbf{E}[X_{\tau\wedge m} \mathbb{1}_{\{\sigma<n\}\cap A}].$$

Using Fatou's lemma, we get

$$\mathbf{E}[X_\tau \mathbb{1}_{\{\sigma<n\}\cap A}] \leq \liminf_{m\to\infty} \mathbf{E}[X_{\tau\wedge m} \mathbb{1}_{\{\sigma<n\}\cap A}] \leq \mathbf{E}[X_\sigma \mathbb{1}_{\{\sigma<n\}\cap A}].$$

Monotone convergence (for $n \to \infty$) thus yields $\mathbf{E}[X_\tau \mathbb{1}_A] \leq \mathbf{E}[X_\sigma \mathbb{1}_A]$.

(iii) If X is a martingale, then the claim follows from Lemma 10.10. Now assume that $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ for any bounded stopping time τ. Let $t > s$ and $A \in \mathcal{F}_s$. It is enough to show that $\mathbf{E}[X_t \mathbb{1}_A] = \mathbf{E}[X_s \mathbb{1}_A]$. Define $\tau = s\mathbb{1}_A + t\mathbb{1}_{A^c}$. Then τ is a bounded stopping time. However, by assumption,

$$\mathbf{E}[X_t \mathbb{1}_A] = \mathbf{E}[X_t] - \mathbf{E}[X_t \mathbb{1}_{A^c}] = \mathbf{E}[X_0] - \mathbf{E}[X_\tau] + \mathbf{E}[X_s \mathbb{1}_A] = \mathbf{E}[X_s \mathbb{1}_A]. \quad \square$$

Corollary 10.12 *Let X be a martingale* (*respectively a submartingale*), *and assume $(\tau_N)_{N\in\mathbb{N}}$ is a monotone increasing sequence of bounded stopping times* (*hence $\tau_N \leq T_N$, $N \in \mathbb{N}$ for some $T_N \in \mathbb{N}$*). *Then $(X_{\tau_N})_{N\in\mathbb{N}}$ is a martingale* (*respectively a submartingale*) *with respect to the filtration $(\mathcal{F}_{\tau_N})_{N\in\mathbb{N}}$.*

Definition 10.13 (Stopped process) Let $I \subset \mathbb{R}$ be countable, let $(X_t)_{t\in I}$ be adapted and let τ be a stopping time. We define the *stopped process* X^τ by

$$X_t^\tau = X_{\tau\wedge t} \quad \text{for any } t \in I.$$

Further, let $\mathbb{F}^\tau$ be the filtration $\mathbb{F}^\tau = (\mathcal{F}_t^\tau)_{t\in I} = (\mathcal{F}_{\tau\wedge t})_{t\in I}$.

Remark 10.14 X^τ is adapted both to $\mathbb{F}$ and to $\mathbb{F}^\tau$. ◊

Theorem 10.15 (Optional stopping) *Let $(X_n)_{n\in\mathbb{N}_0}$ be a (sub-, super-) martingale with respect to $\mathbb{F}$ and let τ be a stopping time. Then X^τ is a (sub-, super-) martingale both with respect to $\mathbb{F}$ and with respect to $\mathbb{F}^\tau$.*

Proof We give the proof only for the case where X is a submartingale. The other cases are similar since there $(-X)$ is a submartingale.

For each $n \in \mathbb{N}_0$, we have

$$\begin{aligned}\mathbf{E}\big[|X_n^\tau|\big] &\le \mathbf{E}\big[\max\big\{|X_m| : m \le n\big\}\big] \\ &\le \mathbf{E}\big[|X_0|\big] + \ldots + \mathbf{E}\big[|X_n|\big] < \infty.\end{aligned}$$

Hence X^τ is integrable.

Let X be a submartingale. Since $\{\tau > n-1\} \in \mathcal{F}_{n-1}$, we have

$$\begin{aligned}\mathbf{E}\big[X_n^\tau - X_{n-1}^\tau \,\big|\, \mathcal{F}_{n-1}\big] &= \mathbf{E}[X_{\tau\wedge n} - X_{\tau\wedge(n-1)} \,|\, \mathcal{F}_{n-1}] \\ &= \mathbf{E}\big[(X_n - X_{n-1})\mathbb{1}_{\{\tau>n-1\}} \,\big|\, \mathcal{F}_{n-1}\big] \\ &= \mathbb{1}_{\{\tau>n-1\}}\mathbf{E}[X_n - X_{n-1} \,|\, \mathcal{F}_{n-1}] \\ &\ge 0, \quad \text{since } X \text{ is an } \mathbb{F}\text{-submartingale.}\end{aligned}$$

Therefore, X^τ is an $\mathbb{F}$-submartingale. As X^τ is adapted to $\mathbb{F}^\tau$ and since $\mathbb{F}^\tau$ is the smaller filtration, X^τ is also an $\mathbb{F}^\tau$-submartingale (see Remark 9.29). □

Example 10.16 Let X be a symmetric simple random walk on $\mathbb{Z}$ (see Example 10.8). Let $a, b \in \mathbb{Z}$ with $a < 0$, $b > 0$ and let

$$\begin{aligned}&\tau_a = \inf\{t \ge 0 : X_t = a\}, \qquad \tau_b = \inf\{t \ge 0 : X_t = b\} \quad \text{and} \\ &\tau_{a,b} = \tau_a \wedge \tau_b.\end{aligned}$$

$\tau_{a,b}$ is a stopping time by Lemma 9.18. Let $A = \{\tau_{a,b} = \tau_a\}$ be the event where X hits a before hitting b. We want to compute $\mathbf{P}[A]$. By Exercise 2.3.1, almost surely $\limsup_{n\to\infty} X_n = \infty$ and $\liminf_{n\to\infty} X_n = -\infty$. Therefore, almost surely $\tau_a < \infty$ and $\tau_b < \infty$. By the optional stopping theorem, $X^{\tau_{a,b}}$ is a martingale. Since $\tau_{a,b} \wedge n \overset{n\to\infty}{\longrightarrow} \tau_{a,b}$ almost surely, we get $X_n^{\tau_{a,b}} \overset{n\to\infty}{\longrightarrow} X_{\tau_{a,b}}$ almost surely. As $|X_n^{\tau_{a,b}}|$ is bounded by $b-a$, we can infer that $X_n^{\tau_{a,b}} \overset{n\to\infty}{\longrightarrow} X_{\tau_{a,b}}$ also in L^1. Thus

$$\begin{aligned}0 &= \lim_{n\to\infty} \mathbf{E}\big[X_n^{\tau_{a,b}}\big] = \mathbf{E}[X_{\tau_{a,b}}] = a \cdot \mathbf{P}[\tau_{a,b} = \tau_a] + b \cdot \mathbf{P}[\tau_{a,b} = \tau_b] \\ &= b + (a-b)\mathbf{P}[\tau_{a,b} = \tau_a].\end{aligned}$$

We conclude that $\mathbf{P}[\tau_{a,b} = \tau_a] = \frac{b}{b-a}$. ◊

Example 10.17 Finally, we use our machinery in order to compute $\mathbf{E}[\tau_{a,b}]$ and $\mathbf{E}[\tau_a]$. The square variation process $\langle X\rangle$ (compare Definition 10.3) is given by

$$\langle X\rangle_n = \sum_{i=1}^{n} \mathbf{E}\big[(X_i - X_{i-1})^2 \,\big|\, \mathcal{F}_{i-1}\big] = n;$$

hence $(X_n^2 - n)_{n\in\mathbb{N}_0}$ is a martingale. By the optional stopping theorem,

$$0 = \mathbf{E}\big[X^2_{\tau_{a,b}\wedge n} - (\tau_{a,b}\wedge n)\big] \quad \text{for all } n \in \mathbb{N}_0.$$

Monotone convergence yields

$$\mathbf{E}[\tau_{a,b}] = \mathbf{E}\big[X^2_{\tau_{a,b}}\big] = a^2\mathbf{P}[\tau_{a,b} = \tau_a] + b^2\mathbf{P}[\tau_{a,b} = \tau_b] = |a|\cdot b.$$

In order to compute $\mathbf{E}[\tau_a]$, note that $\tau_{a,b} \uparrow \tau_a$ almost surely if $b \to \infty$. The monotone convergence theorem thus yields $\mathbf{E}[\tau_a] = \lim_{b\to\infty} \mathbf{E}[\tau_{a,b}] = \infty$. ◊

Remark 10.18 Evidently, $X_{\tau_b} = b > 0$. Hence $X_0 < \mathbf{E}[X_{\tau_b} \mid \mathcal{F}_0] = b$. The claim of the optional sampling theorem may thus fail, in general, if the stopping time is unbounded. ◊

Example 10.19 (Gambler's ruin problem) Consider a game of two persons, A and B. In each round, a coin is tossed. Depending on the outcome, either A gets a euro from B or vice versa. The game endures until one of the players is ruined. For simplicity, we assume that in the beginning A has $k_A \in \mathbb{N}$ euros while B has $k_B = N - k_A$ euros, where $N \in \mathbb{N}$, $N \geq k_A$. We want to know the probability of B's ruin. In Example 10.16 we saw that for a fair coin this probability is k_A/N. Now we allow the coin to be unfair.

Hence, let $Y_1, Y_2, \ldots$ be i.i.d. and $\sim\mathrm{Rad}_p$ (that is, $\mathbf{P}[Y_i = 1] = 1 - \mathbf{P}[Y_i = -1] = p$) for some $p \in (0,1) \setminus \{\frac{1}{2}\}$. Denote by $X_n := k_B + \sum_{i=1}^n Y_i$ the running total for B after n rounds, where formally we assume that the game continues even after one player is ruined. We define as above τ_0, τ_N and $\tau_{0,N}$ as the times of first entrance of X into $\{0\}$, $\{N\}$ and $\{0, N\}$, respectively. The ruin probability of B thus is $p_B^N := \mathbf{P}[\tau_{0,N} = \tau_0]$. Since X is not a martingale (except for the case $p = \frac{1}{2}$ that was excluded), we use a trick to construct a martingale. Define a new process Z by $Z_n := r^{X_n} = r^{k_B} \prod_{i=1}^n r^{Y_i}$, where $r > 0$ has to be chosen so that Z becomes a martingale. By Example 9.31, this is the case if and only if

$$\mathbf{E}\big[r^{Y_1}\big] = pr + (1-p)r^{-1} = 1;$$

hence, if $r = 1$ or $r = \frac{1-p}{p}$. Evidently, the choice $r = 1$ is useless (as Z does not yield any information on X); hence we assume $r = \frac{1-p}{p}$. We thus get

$$\tau_0 = \inf\{n \in \mathbb{N}_0 : Z_n = 1\} \quad \text{and} \quad \tau_N = \inf\big\{n \in \mathbb{N}_0 : Z_n = r^N\big\}.$$

(Note that here we cannot argue as above in order to show that $\tau_0 < \infty$ and $\tau_N < \infty$ almost surely. In fact, for $p \neq \frac{1}{2}$, *only one* of the statements holds. However, using, for example, the strong law of large numbers, we obtain that $\liminf_{n\to\infty} X_n = \infty$ (and thus $\tau_N < \infty$) almost surely if $p > \frac{1}{2}$. Similarly, $\tau_0 < \infty$ almost surely if $p < \frac{1}{2}$.) As in Example 10.16, the optional stopping theorem yields $r^{k_B} = Z_0 = \mathbf{E}[Z_{\tau_{0,N}}] = p_B^N + (1 - p_B^N)r^N$. Therefore, the probability of B's ruin is

$$p_B^N = \frac{r^{k_B} - r^N}{1 - r^N}. \tag{10.5}$$

If the game is advantageous for B (that is, $p > \frac{1}{2}$), then $r < 1$. In this case, in the limit $N \to \infty$ (with constant k_B),

$$p_B^\infty := \lim_{N\to\infty} p_B^N = r^{k_B}. \tag{10.6}$$

◊

Exercise 10.2.1 Let X be a square integrable martingale with square variation process $\langle X\rangle$. Let τ be a finite stopping time. Show the following:

(i) If $\mathbf{E}[\langle X\rangle_\tau] < \infty$, then

$$\mathbf{E}\big[(X_\tau - X_0)^2\big] = \mathbf{E}\big[\langle X\rangle_\tau\big] \quad \text{and} \quad \mathbf{E}[X_\tau] = \mathbf{E}[X_0]. \tag{10.7}$$

(ii) If $\mathbf{E}[\langle X\rangle_\tau] = \infty$, then both equalities in (10.7) may fail.

Exercise 10.2.2 We consider a situation that is more general than the one in the preceding example by assuming only that $Y_1, Y_2, \ldots$ are i.i.d. integrable random variables that are not almost surely constant (and $X_n = Y_1 + \ldots + Y_n$). We further assume that there is a $\delta > 0$ such that $\mathbf{E}[\exp(\theta Y_1)] < \infty$ for all $\theta \in (-\delta, \delta)$. Define a map $\psi : (-\delta, \delta) \to \mathbb{R}$ by $\theta \mapsto \log(\mathbf{E}[\exp(\theta Y_1)])$ and the process Z^θ by $Z_n^\theta := \exp(\theta X_n - n\psi(\theta))$ for $n \in \mathbb{N}_0$. Show the following:

(i) Z^θ is a martingale for all $\theta \in (-\delta, \delta)$.
(ii) ψ is strictly convex.
(iii) $\mathbf{E}[\sqrt{Z_n^\theta}] \overset{n\to\infty}{\longrightarrow} 0$ for $\theta \neq 0$.
(iv) $Z_n^\theta \overset{n\to\infty}{\longrightarrow} 0$ almost surely.

We may interpret Y_n as the difference between the premiums and the payments of an insurance company at time n. If the initial capital of the company is $k_0 > 0$, then $k_0 + X_n$ is the account balance at time n. We are interested in the ruin probability

$$p(k_0) = \mathbf{P}\big[\inf\{X_n + k_0 : n \in \mathbb{N}_0\} < 0\big]$$

depending on the initial capital.

It can be assumed that the premiums are calculated such that $\mathbf{E}[Y_1] > 0$. Show that if the equation $\psi(\theta) = 0$ has a solution $\theta^* \neq 0$, then $\theta^* < 0$. Show further that in this case, the *Cramér–Lundberg inequality* holds:

$$p(k_0) \le \exp\big(\theta^* k_0\big). \tag{10.8}$$

Equality holds if $k_0 \in \mathbb{N}$ and if Y_i assumes only the values -1 and 1. In this case, we get (10.6) with $r = \exp(\theta^*)$.

10.3 Uniform Integrability and Optional Sampling

We extend the optional sampling theorem to unbounded stopping times. We will see that this is possible if the underlying martingale is uniformly integrable (compare Definition 6.16).

Lemma 10.20 *Let $(X_n)_{n\in\mathbb{N}_0}$ be a uniformly integrable martingale. Then the family $(X_\tau : \tau$ is a finite stopping time$)$ is uniformly integrable.*

Proof By Theorem 6.19, there exists a monotone increasing, convex function $f : [0,\infty) \to [0,\infty)$ with

$$\liminf_{x\to\infty} f(x)/x = \infty \quad \text{and} \quad L := \sup_{n\in\mathbb{N}_0} \mathbf{E}\big[f\big(|X_n|\big)\big] < \infty.$$

If $\tau < \infty$ is a finite stopping time, then by the optional sampling theorem for bounded stopping times (Theorem 10.11 with $\tau = n$ and $\sigma = \tau \wedge n$), $\mathbf{E}[X_n \mid \mathcal{F}_{\tau\wedge n}] = X_{\tau\wedge n}$. Since $\{\tau \le n\} \in \mathcal{F}_{\tau\wedge n}$, Jensen's inequality yields

$$\begin{aligned}
\mathbf{E}\big[f\big(|X_\tau|\big)\mathbb{1}_{\{\tau\le n\}}\big] &= \mathbf{E}\big[f\big(|X_{\tau\wedge n}|\big)\mathbb{1}_{\{\tau\le n\}}\big] \\
&\le \mathbf{E}\big[\mathbf{E}\big[f\big(|X_n|\big)\bigm| \mathcal{F}_{\tau\wedge n}\big]\mathbb{1}_{\{\tau\le n\}}\big] \\
&= \mathbf{E}\big[f\big(|X_n|\big)\mathbb{1}_{\{\tau\le n\}}\big] \le L.
\end{aligned}$$

Hence $\mathbf{E}[f(|X_\tau|)] \le L$. By Theorem 6.19, the family

$$\{X_\tau, \tau \text{ is a finite stopping time}\}$$

is uniformly integrable. □

Theorem 10.21 (Optional sampling and uniform integrability) *Let $(X_n, n \in \mathbb{N}_0)$ be a uniformly integrable martingale (respectively supermartingale) and let $\sigma \le \tau$ be finite stopping times. Then $\mathbf{E}[|X_\tau|] < \infty$ and $X_\sigma = \mathbf{E}[X_\tau \mid \mathcal{F}_\sigma]$ (respectively $X_\sigma \ge \mathbf{E}[X_\tau \mid \mathcal{F}_\sigma]$).*

Proof First let X be a martingale. We have $\{\sigma \le n\} \cap F \in \mathcal{F}_{\sigma\wedge n}$ for all $F \in \mathcal{F}_\sigma$. Hence, by the optional sampling theorem (Theorem 10.11),

$$\mathbf{E}[X_{\tau\wedge n}\mathbb{1}_{\{\sigma\le n\}\cap F}] = \mathbf{E}[X_{\sigma\wedge n}\mathbb{1}_{\{\sigma\le n\}\cap F}].$$

By Lemma 10.20, $(X_{\sigma\wedge n}, n \in \mathbb{N}_0)$ and thus $(X_{\sigma\wedge n}\mathbb{1}_{\{\sigma\le n\}\cap F}, n \in \mathbb{N}_0)$ are uniformly integrable. Similarly, this holds for X_τ. Therefore, by Theorem 6.25,

$$\begin{aligned}\mathbf{E}[X_\tau \mathbb{1}_F] &= \lim_{n\to\infty} \mathbf{E}[X_{\tau\wedge n}\mathbb{1}_{\{\sigma\le n\}\cap F}] = \lim_{n\to\infty} \mathbf{E}[X_{\sigma\wedge n}\mathbb{1}_{\{\sigma\le n\}\cap F}] \\ &= \mathbf{E}[X_\sigma \mathbb{1}_F].\end{aligned}$$

We conclude that $\mathbf{E}[X_\tau \mid \mathcal{F}_\sigma] = X_\sigma$.

Now let X be a supermartingale and let $X = M + A$ be its Doob decomposition; that is, M is a martingale and $A \le 0$ is predictable and decreasing. Since

$$\mathbf{E}\big[|A_n|\big] = \mathbf{E}[-A_n] \le \mathbf{E}\big[|X_n - X_0|\big] \le \mathbf{E}\big[|X_0|\big] + \sup_{m\in\mathbb{N}_0} \mathbf{E}\big[|X_m|\big] < \infty,$$

we have $A_n \downarrow A_\infty$ for some $A_\infty \le 0$ with $\mathbf{E}[-A_\infty] < \infty$ (by the monotone convergence theorem). Hence A and thus $M = X - A$ are uniformly integrable (Theorem 6.18(ii)). Therefore,

$$\mathbf{E}\big[|X_\tau|\big] \le \mathbf{E}[-A_\tau] + \mathbf{E}\big[|M_\tau|\big] \le \mathbf{E}[-A_\infty] + \mathbf{E}\big[|M_\tau|\big] < \infty.$$

Furthermore,

$$\begin{aligned}\mathbf{E}[X_\tau \mid \mathcal{F}_\sigma] &= \mathbf{E}[M_\tau \mid \mathcal{F}_\sigma] + \mathbf{E}[A_\tau \mid \mathcal{F}_\sigma] \\ &= M_\sigma + A_\sigma + \mathbf{E}\big[(A_\tau - A_\sigma)\,\big|\,\mathcal{F}_\sigma\big] \\ &\le M_\sigma + A_\sigma = X_\sigma.\end{aligned}$$

□

Corollary 10.22 *Let X be a uniformly integrable martingale (respectively supermartingale) and let $\tau_1 \le \tau_2 \le \ldots$ be finite stopping times. Then $(X_{\tau_n})_{n\in\mathbb{N}}$ is a martingale (respectively supermartingale).*