Chapter 25
The Itô Integral

The Itô integral allows us to integrate stochastic processes with respect to the increments of a Brownian motion or a somewhat more general stochastic process. We develop the Itô integral first for Brownian motion and then for generalized diffusion processes (so called Itô processes). In the third section, we derive the celebrated Itô formula. This is the chain rule for the Itô integral that enables us to do explicit calculations with the Itô integral. In the fourth section, we use the Itô formula to obtain a stochastic solution of the classical Dirichlet problem. This in turn is used in the fifth section in order to show that like symmetric simple random walk, Brownian motion is recurrent in low dimensions and transient in high dimensions.

25.1 Itô Integral with Respect to Brownian Motion

Let $W = (W_t)_{t\geq 0}$ be a Brownian motion on the space $(\Omega, \mathcal{F}, \mathbf{P})$ with respect to the filtration $\mathbb{F}$ that satisfies the usual conditions (see Definition 21.22). That is, W is a Brownian motion and an $\mathbb{F}$-martingale. The aim of this section is to construct an integral

$$I_t^W(H) = \int_0^t H_s\, dW_s$$

for a large class of integrands $H : \Omega \times [0,\infty) \to \mathbb{R}$, $(\omega, t) \mapsto H_t(\omega)$ in such a way that $(I_t^W(H))_{t\geq 0}$ is a continuous $\mathbb{F}$-martingale. Since almost all paths $s \mapsto W_s(\omega)$ of Brownian motion are of locally infinite variation, $W(\omega)$ is not the distribution function of a signed Lebesgue–Stieltjes measure on $[0,\infty)$. Hence $I_t^W(H)$ cannot be defined in the framework of classical integration theory. The basic new idea is to establish the integral as an L^2-limit. We start with an elementary example to illustrate this.

Example 25.1 Assume that $X_1, X_2, \ldots$ are i.i.d. $\mathrm{Rad}_{1/2}$ random variables; that is, $\mathbf{P}[X_n = 1] = \mathbf{P}[X_n = -1] = \frac{1}{2}$. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Under

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which assumptions on $(h_n)_{n\in\mathbb{N}}$ is the series

$$R := \sum_{n\in\mathbb{N}} h_n X_n \tag{25.1}$$

well-defined? If $\sum_{n\in\mathbb{N}} |h_n| < \infty$, then the series converges absolutely for *every* ω. In this case, there is no problem. Now assume that only the weaker condition $\sum_{n\in\mathbb{N}} h_n^2 < \infty$ holds. In this case, the series (25.1) does not necessarily converge any more for every ω. However, we have $\mathbf{E}[h_n X_n] = 0$ for each $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \mathbf{Var}[h_n X_n] = \sum_{n=1}^{\infty} h_n^2 < \infty$. Hence $R_N := \sum_{k=1}^{N} h_k X_k$ converges in L^2 (for $N \to \infty$). We can thus define the series R in (25.1) as the L^2-limit of the partial sums R_N. Note that (at least formally) for the approximating sums the order of summation matters. In a sense, we have constructed $\sum_{n=1}^{\infty}$ instead of $\sum_{n\in\mathbb{N}}$.

An equivalent formulation that gives a flavor of what is to come is the following. Denote by ℓ^2 the Hilbert space of square summable sequences of real numbers with inner product $\langle h, g\rangle = \sum_{n=1}^{\infty} h_n g_n$ and norm $\|g\| = \langle g, g\rangle^{1/2}$. Let ℓ^f be the subspace of those sequences with only finitely many nonzero entries. Then $R(h) = \sum_{n\in\mathbb{N}} h_n X_n$ for $h \in \ell^f$ is well-defined (since it is a finite sum). Since

$$\mathbf{E}\big[R(h)^2\big] = \mathbf{Var}\big[R(h)\big] = \sum_{n\in\mathbb{N}} \mathbf{Var}[h_n X_n] = \sum_{n\in\mathbb{N}} h_n^2 = \|h\|^2,$$

the map $R : \ell^f \to \mathcal{L}^2(\mathbf{P})$ is an isometry. As $\ell^f \subset \ell^2$ is dense, there is a unique continuous extension of R to ℓ^2. Hence, if $h \in \ell^2$ and $(h^N)_{N\in\mathbb{N}}$ is a sequence in ℓ^f with $\|h^N - h\| \overset{N\to\infty}{\longrightarrow} 0$, then $R(h^N) \overset{N\to\infty}{\longrightarrow} R(h)$ in the L^2 sense. In particular, $h_n^N := h_n \mathbb{1}_{\{n\leq N\}}$, $n \in \mathbb{N}$, $N \in \mathbb{N}$, is an approximating sequence for h, and we have $R(h^N) = \sum_{n=1}^{N} h_n X_n$. Thus the approximation of R with the partial sums R_N that we described above is a special case of this construction. ◊

The programme for the construction of the Itô integral $I_t^W(H)$ is the following. First consider *simple functions* as integrands H; that is, the map $t \mapsto H_t(\omega)$ is a step function. For these H, the integral can easily be defined as a finite sum. The next step is to extend the integral, as in Example 25.1, to integrands that can be approximated in a certain L^2-space by simple integrands.

Definition 25.2 Denote by $\mathcal{E}$ the vector space of maps $H : \Omega \times [0,\infty) \to \mathbb{R}$ of the form

$$H_t(\omega) = \sum_{i=1}^{n} h_{i-1}(\omega) \mathbb{1}_{(t_{i-1}, t_i]},$$

where $n \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_n$ and h_{i-1} is bounded and $\mathcal{F}_{t_{i-1}}$-measurable for every $i = 1, \ldots, n$. $\mathcal{E}$ is called the vector space of predictable simple processes.

We equip $\mathcal{E}$ with a (pseudo) norm $\|\cdot\|_{\mathcal{E}}$ by defining

$$\|H\|_{\mathcal{E}}^2 = \sum_{i=1}^n \mathbf{E}\big[h_{i-1}^2\big](t_i - t_{i-1}) = \mathbf{E}\bigg[\int_0^\infty H_s^2\, ds\bigg].$$

Definition 25.3 For $H \in \mathcal{E}$ and $t \geq 0$, define

$$I_t^W(H) = \sum_{i=1}^n h_{i-1}(W_{t_i \wedge t} - W_{t_{i-1} \wedge t})$$

and

$$I_\infty^W(H) = \sum_{i=1}^n h_{i-1}(W_{t_i} - W_{t_{i-1}}).$$

Clearly, for every bounded stopping time τ,

$$\begin{aligned}\mathbf{E}\big[I_\tau^W(H)\big] &= \sum_{i=1}^n \mathbf{E}\big[h_{i-1}\big(W_{t_i}^\tau - W_{t_{i-1}}^\tau\big)\big] \\ &= \sum_{i=1}^n \mathbf{E}\big[h_{i-1}\mathbf{E}\big[W_{t_i}^\tau - W_{t_{i-1}}^\tau \,\big|\, \mathcal{F}_{t_{i-1}}\big]\big] = 0\end{aligned}$$

since, by the optional stopping theorem (OST), the stopped Brownian motion W^τ is an $\mathbb{F}$-martingale. Hence (again by the OST) $(I_t^W(H))_{t \geq 0}$ is an $\mathbb{F}$-martingale. In particular, we have $\mathbf{E}[(I_{t_{i+1}}^W(H) - I_{t_i}^W(H))(I_{t_{j+1}}^W(H) - I_{t_j}^W(H))] = 0$ for $i \neq j$. Therefore,

$$\begin{aligned}\mathbf{E}\big[I_\infty^W(H)^2\big] &= \sum_{i=1}^n \mathbf{E}\big[\big(I_{t_i}^W(H) - I_{t_{i-1}}^W(H)\big)^2\big] \\ &= \sum_{i=1}^n \mathbf{E}\big[h_{i-1}^2(W_{t_i} - W_{t_{i-1}})^2\big] \\ &= \sum_{i=1}^n \mathbf{E}\big[h_{i-1}^2\big](t_i - t_{i-1}) = \|H\|_{\mathcal{E}}^2. \end{aligned} \tag{25.2}$$

From these considerations, the following statement is immediate.

Theorem 25.4

(i) *The map $I_\infty^W : \mathcal{E} \to \mathcal{L}^2(\Omega, \mathcal{F}, \mathbf{P})$ is an isometric linear map* (*with respect to* $\|\cdot\|_{\mathcal{E}}$ *and* $\|\cdot\|_2$).

(ii) *The process $(I_t^W(H))_{t \geq 0}$ is an L^2-bounded continuous $\mathbb{F}$-martingale.*

Proof Only the linearity remains to be shown. However, this is trivial. □

The idea is to extend the map I_∞^W continuously from $\mathcal{E}$ to a suitable closure $\overline{\mathcal{E}}$ of $\mathcal{E}$. Now as a subspace of what space should we close $\mathcal{E}$? A minimal requirement is that $(\omega, t) \mapsto H_t(\omega)$ be measurable (with respect to $\mathcal{F} \otimes \mathcal{B}([0,\infty))$) and that H be adapted.

Definition 25.5 A stochastic process $X = (X_t)_{t\geq 0}$ with values in a Polish space E is called

(i) *product measurable* if $(\omega, t) \mapsto X_t(\omega)$ is measurable with respect to $\mathcal{F} \otimes \mathcal{B}([0,\infty)) - \mathcal{B}(E)$,
(ii) *progressively measurable* if, for every $t \geq 0$, the map $\Omega \times [0,t] \to E$, $(\omega, s) \mapsto X_s(\omega)$ is measurable with respect to $\mathcal{F}_t \otimes \mathcal{B}([0,t]) - \mathcal{B}(E)$,
(iii) *predictable* (or *previsible*) if $(\omega, t) \mapsto X_t(\omega)$ is measurable with respect to the predictable σ-algebra $\mathcal{P}$ on $\Omega \times [0,\infty)$:

$$\mathcal{P} := \sigma(X : X \text{ is a left continuous adapted process}).$$

Remark 25.6 Any $H \in \mathcal{E}$ is predictable. This property ensures that $I^M(H)$ is a martingale for every (even discontinuous) martingale M. The notion of predictability is important only for integration with respect to discontinuous martingales. As we will not develop that calculus in this book, predictability will not be central for us. ◊

Remark 25.7 If H is progressively measurable, then H is evidently also product measurable and adapted. With a little work, the converse can also be shown: If H is adapted and product measurable, then there is a progressively measurable modification of H (see, e.g., [115, pp. 68ff]). ◊

Theorem 25.8 *If H is adapted and right continuous or left continuous, then H is progressively measurable. If H is adapted and a.s. right continuous or left continuous, then there exists a version of H that is progressively measurable.*

In particular, every predictable process is progressively measurable.

Proof See Exercise 21.1.4. □

We consider $\mathcal{E}$ as a subspace of

$$\mathcal{E}_0 := \left\{ H : \text{ product measurable, adapted and } \|H\|^2 := \mathbf{E}\left[\int_0^\infty H_t^2\,dt\right] < \infty \right\}.$$

Let $\overline{\mathcal{E}}$ denote the closure of $\mathcal{E}$ in $\mathcal{E}_0$.

Theorem 25.9 *If H is progressively measurable (for instance, left continuous or right continuous and adapted) and $\mathbf{E}[\int_0^\infty H_t^2\,dt] < \infty$, then $H \in \overline{\mathcal{E}}$.*

Proof Let H be progressively measurable and $\mathbf{E}[\int_0^\infty H_t^2\,dt] < \infty$. It is enough to show that, for any $T > 0$, there exists a sequence $(H^n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that

$$\mathbf{E}\left[\int_0^T \left(H_s - H_s^n\right)^2 ds\right] \overset{n\to\infty}{\longrightarrow} 0. \tag{25.3}$$

Step 1. First assume that H is continuous and bounded. Define $H_0^n = 0$ and

$$H_t^n = H_{i2^{-n}T} \quad \text{if } i2^{-n}T < t \le (i+1)2^{-n}T \text{ for some } i = 0, \dots, 2^n - 1$$

and $H_t^n = 0$ for $t > T$. Then $H^n \in \mathcal{E}$, and we have $H_t^n(\omega) \overset{n\to\infty}{\longrightarrow} H_t(\omega)$ for all $t > 0$ and $\omega \in \Omega$. By the dominated convergence theorem, we get (25.3).

Step 2. Now let H be progressively measurable and bounded. It is enough to show that there exist continuous adapted processes H^n, $n \in \mathbb{N}$, for which (25.3) holds. Let

$$H_t^n := n \int_{(t-1/n)\vee 0}^{t\wedge T} H_s\,ds \quad \text{for } t \ge 0, n \in \mathbb{N}.$$

Then H^n is continuous, adapted and bounded by $\|H\|_\infty$. By the fundamental theorem of calculus (see Exercise 13.1.7), we have

$$H_t^n(\omega) \overset{n\to\infty}{\longrightarrow} H_t(\omega) \quad \text{for } \lambda\text{-almost all } t \in [0, T] \text{ and for all } \omega \in \Omega. \tag{25.4}$$

By Fubini's theorem and the dominated convergence theorem, we thus conclude that

$$\mathbf{E}\left[\int_0^T \left(H_s - H_s^n\right)^2 ds\right] = \int_{\Omega\times[0,T]} \left(H_s(\omega) - H_s^n(\omega)\right)^2 (\mathbf{P}\otimes\lambda)\big(d(\omega, s)\big) \overset{n\to\infty}{\longrightarrow} 0.$$

Step 3. Now let H be progressively measurable, and assume $\mathbf{E}[\int_0^\infty H_t^2 dt] < \infty$. It is enough to show that there exists a sequence $(H^n)_{n\in\mathbb{N}}$ of bounded, progressively measurable processes such that (25.3) holds. Manifestly, we can choose $H_t^n = H_t \mathbb{1}_{\{|H_t|<n\}}$. □

Definition 25.10 (Itô integral) For $H \in \overline{\mathcal{E}}$, define the *Itô integral*

$$\int_0^\infty H_s\,dW_s := I_\infty^W(H)$$

as the continuous extension of the map $I_\infty^W : \mathcal{E} \to \mathcal{L}^2(\mathbf{P})$ to the closure $\overline{\mathcal{E}}$ of $\mathcal{E}$. In other words, if $(H^n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{E}$ with $\|H - H^n\| \overset{n\to\infty}{\longrightarrow} 0$, then we define $I_\infty^W(H)$ by

$$I_\infty^W(H) := \lim_{n\to\infty} I_\infty^W\left(H^n\right) \quad \text{in } L^2.$$

If τ is a stopping time, then in the following we use the abbreviation

$$H_t^{(\tau)} := H_t \mathbb{1}_{\{t \le \tau\}} \quad \text{for } t \ge 0.$$

(Note that this is not the stopped process $H_t^\tau = H_{\tau \wedge t}$.)

Theorem 25.11

(i) *The map* $I_\infty^W : \overline{\mathcal{E}} \to \mathcal{L}^2(\Omega, \mathcal{F}, \mathbf{P})$ *is linear and*

$$\mathbf{E}\big[I_\infty^W(H)^2\big] = \mathbf{E}\bigg[\int_0^\infty H_s^2\, ds\bigg].$$

(ii) *For every* $H \in \overline{\mathcal{E}}$, *the process* $\tilde{I}^W(H)$ *defined by* $\tilde{I}_t^W(H) := I_\infty^W(H^{(t)})$ *is an* L^2*-bounded* $\mathbb{F}$*-martingale that has a continuous modification* $I^W(H)$.

Definition 25.12 (Itô integral as a process) Let $I^W(H)$ be the continuous version of the martingale $(I_\infty^W(H^{(t)}))_{t \ge 0}$ (see Theorem 25.11(ii)). Denote by

$$\int_s^t H_r\, dW_r := I_t^W(H) - I_s^W(H) \quad \text{for } 0 \le s \le t \le \infty$$

the Itô integral of H with respect to Brownian motion W on the interval $[s, t]$.

Proof of Theorem 25.11 (i) This is a direct consequence of the definition of $I_\infty^W(H)$.

(ii) Let $(H^n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{E}$ with $\|H^n - H\| \overset{n\to\infty}{\longrightarrow} 0$. By Theorem 25.4(ii), we have

$$I_\infty^W\big((H^n)^{(t)}\big) = I_t^W\big(H^n\big) = \mathbf{E}\big[I_\infty^W\big(H^n\big) \,\big|\, \mathcal{F}_t\big] \quad \text{for all } t \ge 0, n \in \mathbb{N}.$$

Since $\|(H^n)^{(t)} - H^{(t)}\| \le \|H^n - H\| \overset{n\to\infty}{\longrightarrow} 0$, this implies (using Corollary 8.21)

$$\tilde{I}_t^W(H) = \lim_{n\to\infty} I_t^W\big(H^n\big) = \lim_{n\to\infty} \mathbf{E}\big[I_\infty^W\big(H^n\big) \,\big|\, \mathcal{F}_t\big] = \mathbf{E}\big[I_\infty^W(H) \,\big|\, \mathcal{F}_t\big].$$

Hence $\tilde{I}^W(H)$ is an L^2-bounded martingale and $I_t^W(H^n) \overset{n\to\infty}{\longrightarrow} \tilde{I}_t^W(H)$ in L^2 for every $t \ge 0$. By Theorem 25.4(ii), $I^W(H^n)$ is continuous for every $n \in \mathbb{N}$. Thus, by Exercise 21.4.3, there exists a continuous modification $I^W(H)$ of $\tilde{I}^W(H)$. □

The last step in the construction of the Itô integral is to weaken the strong integrability condition $\mathbf{E}[\int_0^\infty H_s^2\, ds] < \infty$. We start with a simple observation.

Let τ be a stopping time and recall that $\int_0^\tau H_s\, dW_s$ denotes the random variable that for any ω assumes the value $(\int_0^{\tau(\omega)} H_s\, dW_s)(\omega)$.

Lemma 25.13 *Let* τ *be a stopping time and let* $H \in \overline{\mathcal{E}}$.

(i) *We have*

$$\int_0^\tau H_s\,dW_s = \int_0^\infty H_s^{(\tau)}\,dW_s := \int_0^\infty H_s \mathbb{1}_{\{s\le\tau\}}\,dW_s \quad a.s.$$

(ii) *In particular, for any $t \ge 0$, on the event $\{\tau \ge t\}$ we have*

$$\int_0^t H_s\,dW_s = \int_0^t H_s^{(\tau)}\,dW_s \quad a.s.$$

(iii) *Let $G \in \overline{\mathcal{E}}$ be such that $H_s = G_s$ for all $s \le \tau$. Then*

$$\int_0^\tau H_s\,dW_s = \int_0^\tau G_s\,dW_s \quad a.s.$$

Proof (i) Assume first that τ takes values in $\{k/2^n : k \in \mathbb{N}_0\} \cup \{\infty\}$ for some $n \in \mathbb{N}$. Then $\mathbb{1}_{\{k/2^n\le\tau\}}\mathbb{1}_{\{t\in((k-1)/2^n,k/2^n]\}} \in \mathcal{E}$ for all $k \in \mathbb{N}$. If, in addition, $H \in \mathcal{E}$, then also $H^{(\tau)} \in \mathcal{E}$ and the claim follows directly from the definition of the Itô integral (Definition 25.3). Now let $H \in \overline{\mathcal{E}}$ and let $(H^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{E}$ such that $\|H^k - H\|_{\mathcal{E}} \overset{k\to\infty}{\longrightarrow} 0$. Writing $H_t^{k,(\tau)} := H_t^k \mathbb{1}_{\{t\le\tau\}}$ we get that $\|H^{k,(\tau)} - H^{(\tau)}\|_{\mathcal{E}} \overset{k\to\infty}{\longrightarrow} 0$. By choosing a suitable sequence $k_m \uparrow \infty$, we obtain

$$\begin{aligned}\int_0^\tau H_s\,dW_s &= \lim_{m\to\infty} \int_0^\tau H_s^{k_m}\,dW_s \\ &= \lim_{m\to\infty} \int_0^\infty H_s^{k_m,(\tau)}\,dW_s = \int_0^\infty H_s^{(\tau)}\,dW_s \quad \text{a.s.}\end{aligned}$$

Finally, assume that τ is an arbitrary stopping time and define $\tau_n := 2^{-n}\lceil 2^n\tau\rceil$ for $n \in \mathbb{N}$. Then (τ_n) is a sequence of stopping times with $\tau_n \downarrow \tau$. Recall that $I^W(H)$ is continuous and note that $\|H^{(\tau_n)} - H^{(\tau)}\|_{\mathcal{E}} \overset{n\to\infty}{\longrightarrow} 0$. Hence by taking a suitable sequence $n(m) \uparrow \infty$, we get

$$\begin{aligned}\int_0^\tau H_s\,dW_s &= \lim_{m\to\infty} \int_0^{\tau_{n(m)}} H_s\,dW_s \\ &= \lim_{m\to\infty} \int_0^\infty H_s^{(\tau_{n(m)})}\,dW_s = \int_0^\infty H_s^{(\tau)}\,dW_s \quad \text{a.s.}\end{aligned}$$

(ii), (iii) These statements are direct consequences of (i). □

Definition 25.14 Let $\mathcal{E}_{\text{loc}}$ be the space of progressively measurable stochastic processes H with

$$\int_0^T H_s^2\,ds < \infty \quad \text{a.s.} \quad \text{for all } T > 0.$$

Lemma 25.15 *For every $H \in \mathcal{E}_{\text{loc}}$, there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times with $\tau_n \uparrow \infty$ almost surely and $\mathbf{E}[\int_0^{\tau_n} H_s^2\,ds] < \infty$ and hence such that $H^{(\tau_n)} \in \overline{\mathcal{E}}$ for every $n \in \mathbb{N}$.*

Proof Define

$$\tau_n := \inf\left\{t \geq 0 : \int_0^t H_s^2\,ds \geq n\right\}.$$

By the definition of $\mathcal{E}_{\text{loc}}$, we have $\tau_n \uparrow \infty$ almost surely. By construction, we have $\|H^{(\tau_n)}\|^2 = \mathbf{E}[\int_0^{\tau_n} H_s^2\,ds] \leq n$. □

Definition 25.16 Let $H \in \mathcal{E}_{\text{loc}}$ and let $(\tau_n)_{n\in\mathbb{N}}$ be as in Lemma 25.15. For $t \geq 0$, define the Itô integral as the almost sure limit

$$\int_0^t H_s\,dW_s := \lim_{n\to\infty} \int_0^t H_s^{(\tau_n)}\,dW_s. \tag{25.5}$$

Theorem 25.17 *Let $H \in \mathcal{E}_{\text{loc}}$.*

(i) *The limit in (25.5) is well-defined and continuous at t. Up to a.s. equality, it is independent of the choice of the sequence $(\tau_n)_{n\in\mathbb{N}}$.*
(ii) *If τ is a stopping time with $\mathbf{E}[\int_0^{\tau} H_s^2\,ds] < \infty$, then the stopped Itô integral $(\int_0^{\tau\wedge t} H_s\,dW_s)_{t\geq 0}$ is an L^2-bounded, continuous martingale.*
(iii) *If $\mathbf{E}[\int_0^T H_s^2\,ds] < \infty$ for all $T > 0$, then $(\int_0^t H_s\,dW_s)_{t\geq 0}$ is a square integrable continuous martingale.*

Proof (i) By Lemma 25.13(ii), on the event $\{\tau_n \geq t\}$, we have

$$\int_0^t H_s\,dW_s = \int_0^t H_s^{(\tau_n)}\,dW_s.$$

Hence the limit exists, is continuous and is independent of the choice of the sequence $(\tau_n)_{n\in\mathbb{N}}$.

(ii) This is immediate by Theorem 25.11.

(iii) As we can choose $\tau_n = n$, this follows from (ii). □

Theorem 25.18 *Let H be progressively measurable and $\mathbf{E}[\int_0^T H_s^2\,ds] < \infty$ for all $T > 0$. Then*

$$M_t := \int_0^t H_s\,dW_s, \quad t \geq 0,$$

defines a square integrable continuous martingale, and

$$(N_t)_{t\geq 0} := \left(M_t^2 - \int_0^t H_s^2\,ds\right)_{t\geq 0}$$

is a continuous martingale with $N_0 = 0$.

Proof It is enough to show that N is a martingale. Clearly, N is adapted. Let τ be a bounded stopping time. Then

$$\begin{aligned}\mathbf{E}[N_\tau] &= \mathbf{E}\left[M_\tau^2 - \int_0^\tau H_s^2\,ds\right]\\ &= \mathbf{E}\left[\left(\int_0^\infty H_s^{(\tau)}\,dW_s\right)^2\right] - \mathbf{E}\left[\int_0^\infty \big(H_s^{(\tau)}\big)^2\,ds\right] = 0.\end{aligned}$$

Thus, by the optional stopping theorem (see Exercise 21.1.3(iii)), N is a martingale. □

Recall the notions of local martingales and square variation from Section 21.10.

Corollary 25.19 *If $H \in \mathcal{E}_{\text{loc}}$, then the Itô integral $M_t = \int_0^t H_s\,dW_s$ is a continuous local martingale with square variation process $\langle M\rangle_t = \int_0^t H_s^2\,ds$.*

Example 25.20

(i) $W_t = \int_0^t 1\,dW_s$ is a square integrable martingale, and $(W_t^2 - t)_{t\geq 0}$ is a continuous martingale.
(ii) Since $\mathbf{E}[\int_0^T W_s^2\,ds] = \frac{T^2}{2} < \infty$ for all $T \geq 0$, $M_t := \int_0^t W_s\,dW_s$ is a continuous, square integrable martingale, and $(M_t^2 - \int_0^t W_s^2\,ds)_{t\geq 0}$ is a continuous martingale.
(iii) Assume that H is progressively measurable and bounded, and let $M_t := \int_0^t H_s\,dW_s$. Then M is progressively measurable (since it is continuous and adapted) and

$$\mathbf{E}\left[\int_0^T M_s^2\,ds\right] = \int_0^T \left(\int_0^s \mathbf{E}\big[H_r^2\big]\,dr\right)^2 ds \leq \frac{T^2\|H\|_\infty^2}{2}.$$

Hence $\widetilde{M}_t := \int_0^t M_s\,dW_s$ is a square integrable, continuous martingale and $(\widetilde{M}_t^2 - \int_0^t M_s^2\,dW_s)_{t\geq 0}$ is a continuous martingale.

◊

25.2 Itô Integral with Respect to Diffusions

If

$$H = \sum_{i=1}^n h_{i-1}\mathbb{1}_{(t_{i-1},t_i]} \in \mathcal{E}, \tag{25.6}$$

then the elementary integral

$$I_t^M(H) = \sum_{i=1}^{n} h_{i-1}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t})$$

is a martingale (respectively local martingale) if M is a martingale (respectively local martingale). Furthermore,

$$\begin{aligned}\mathbf{E}\big[\big(I_\infty^M(H)\big)^2\big] &= \sum_{i=1}^{n} \mathbf{E}\big[h_{i-1}^2(M_{t_i} - M_{t_{i-1}})^2\big] \\ &= \sum_{i=1}^{n} \mathbf{E}\big[h_{i-1}^2\big(\langle M\rangle_{t_i} - \langle M\rangle_{t_{i-1}}\big)\big] \\ &= \mathbf{E}\left[\int_0^\infty H_t^2 d\langle M\rangle_t\right]\end{aligned}$$

if the expression on the right-hand side is finite. Roughly speaking, the procedure in Section 25.1 by which we defined the Itô integral for Brownian motion and integrands $H \in \overline{\mathcal{E}}$ can be repeated to construct a stochastic integral with respect to M for a large class of integrands H. Essentially, in the definition of the norm on $\mathcal{E}$ we have to replace dt (that is, the square variation of Brownian motion) by the square variation $d\langle M\rangle_t$ of M:

$$\|H\|_M^2 := \mathbf{E}\left[\int_0^\infty H_t^2 \, d\langle M\rangle_t\right].$$

Extending the integral to the closure $\overline{\mathcal{E}}$ works just as for Brownian motion. The tricky point is to check whether a given integrand is in $\overline{\mathcal{E}}$. For example, for discontinuous martingales M the integrands have to be predictable in order for the stochastic integral to be a martingale (not to mention the difficulty of establishing for such M, the existence of the square variation process). For the case of discrete time processes, we saw this in Section 9.3. Now if M is a continuous martingale with continuous square variation $\langle M\rangle$, then the following problem occurs. In the proof of Theorem 25.9 in Step 2, in order to show that progressively measurable processes H are in $\overline{E}$, we used the fact that $H_t^n(\omega) \overset{n\to\infty}{\longrightarrow} H_t(\omega)$ for Lebesgue-almost all t and all ω. Now if $d\langle M\rangle_t$ is not *absolutely* continuous with respect to the Lebesgue measure, then this is not sufficient to infer convergence of the integrals with respect to $d\langle M\rangle_t$. In the case of absolutely continuous square variation, however, that proof works without change. As in Section 25.1, we obtain the following theorem.

Theorem 25.21 *Let M be a continuous local martingale with absolutely continuous square variation $\langle M\rangle$ and let H be a progressively measurable process with $\int_0^T H_s^2 \, d\langle M\rangle_s < \infty$ a.s. for all $T \geq 0$. Then the Itô integral $N_t := \int_0^t H_s \, dM_s$ is well-defined and is a continuous local martingale with square variation $\langle N\rangle_t =$*

$\int_0^t H_s^2 \, d\langle M\rangle_s$. *For any sequence* $(\tau_n)_{n\in\mathbb{N}}$ *with* $\tau_n \uparrow \infty$ *and* $\|H^{(\tau_n)}\|_M < \infty$, *and for any family* $(H^{n,m}, n, m \in \mathbb{N}) \subset \mathcal{E}$ *with* $\|H^{n,m} - H^{(\tau_n)}\|_M \overset{m\to\infty}{\longrightarrow} 0$, *we have*

$$\int_0^t H_s \, dM_s = \lim_{n\to\infty} \lim_{m\to\infty} I_t^M\big(H^{m,n}\big) \quad \textit{in probability for all } t \ge 0.$$

The following theorem formulates a certain generalization.

Theorem 25.22 *Let* M^1 *and* M^2 *be continuous local martingales with absolutely continuous square variation. Let* H^i *be progressively measurable processes with* $\int_0^T (H_s^i)^2 \, d\langle M^i\rangle_s < \infty$ *for* $i = 1, 2$ *and* $T < \infty$. *Let* $N_t^i := \int_0^t H_s^i \, dM_s^i$ *for* $i = 1, 2$. *Then* N^1 *and* N^2 *are continuous local martingales with quadratic covariation* $\langle N^i, N^j\rangle_t = \int_0^t H_s^i H_s^j \, d\langle M^i, M^j\rangle_s$ *for* $i, j \in \{1, 2\}$. *If* M^1 *and* M^2 *are independent, then* $\langle N^1, N^2\rangle \equiv 0$.

Proof First assume $H^1, H^2 \in \mathcal{E}$. Then there are numbers $0 = t_0 < t_1 < \ldots < t_n$ and $\mathcal{F}_{t_k}$-measurable bounded maps h_k^i, $i = 1, 2$, $k = 0, \ldots, n-1$ such that

$$H_t^i(\omega) = \sum_{k=1}^n h_{k-1}^i(\omega) \mathbb{1}_{(t_{k-1}, t_k]}(t).$$

Therefore,

$$N_t^i N_t^j = \sum_{k,l=1}^n h_{k-1}^i h_{l-1}^j \big(M_{t_k\wedge t}^i - M_{t_{k-1}\wedge t}^i\big)\big(M_{t_l\wedge t}^j - M_{t_{l-1}\wedge t}^j\big).$$

Those summands with $k \ne l$ are local martingales. For any of the summands with $k = l$,

$$\Big(h_{k-1}^i h_{k-1}^j \Big(\big(M_{t_k\wedge t}^i - M_{t_{k-1}\wedge t}^i\big)\big(M_{t_k\wedge t}^j - M_{t_{k-1}\wedge t}^j\big) \\ - \Big(\big\langle M^i, M^j\big\rangle_{t_k\wedge t} - \big\langle M^i, M^j\big\rangle_{t_{k-1}\wedge t}\Big)\Big)\Big)_{t\ge 0}$$

is a local martingale. Since

$$\sum_{k=1}^n h_{k-1}^i h_{k-1}^j \Big(\big\langle M^i, M^j\big\rangle_{t_k\wedge t} - \big\langle M^i, M^j\big\rangle_{t_{k-1}\wedge t}\Big) = \int_0^t H_s^i H_s^j \, d\big\langle M^i, M^j\big\rangle_s,$$

$(N_t^i N_t^j - \int_0^t H_s^i H_s^j \, d\langle M^i, M^j\rangle_s)_{t\ge 0}$ is a continuous local martingale.

The case of general progressively measurable H^1, H^2 that satisfy an integrability condition follows by the usual L^2-approximation arguments.

If M^1 and M^2 are independent, then $\langle M^1, M^2\rangle \equiv 0$. □

In the following, we consider processes that can be expressed as Itô integrals with respect to a Brownian motion. For these processes, we give a different and more detailed proof of Theorem 25.21.

Definition 25.23 Let W be a Brownian motion and let σ and b be progressively measurable stochastic processes with $\int_0^t \sigma_s^2 + |b_s|\, ds < \infty$ almost surely for all $t \geq 0$. Then we say that the process X defined by

$$X_t = \int_0^t \sigma_s\, dW_s + \int_0^t b_s\, ds \quad \text{for } t \geq 0$$

is a generalized *diffusion process* (or, briefly, generalized diffusion) with *diffusion coefficient* σ and *drift* b. Often X is called an *Itô process*.

In particular, if σ and b are of the form $\sigma_s = \tilde{\sigma}(X_s)$ and $b_s = \tilde{b}(X_s)$ for certain maps $\tilde{\sigma} : \mathbb{R} \to [0, \infty)$ and $\tilde{b} : \mathbb{R} \to \mathbb{R}$, then X is called a diffusion (in the proper sense).

In contrast with generalized diffusions, we will see that under certain regularity assumptions on the coefficients, diffusions in the proper sense are Markov processes (compare Theorems 26.8, 26.10 and 26.26).

A diffusion X can always be decomposed as $X = M + A$, where $M_t = \int_0^t \sigma_s\, dW_s$ is a continuous local martingale with square variation $\langle M \rangle_t = \int_0^t \sigma_s^2\, ds$ (by Corollary 25.19) and $A_t = \int_0^t b_s\, ds$ is a continuous process of locally finite variation.

Clearly, for the H in (25.6), we have

$$\begin{aligned}\int_0^t H_s\, dM_s &= \sum_{i=1}^n h_{i-1}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}) \\ &= \sum_{i=1}^n h_{i-1} \int_{t_{i-1} \wedge t}^{t_i \wedge t} \sigma_s\, dW_s = \int_0^t (H_s \sigma_s)\, dW_s.\end{aligned}$$

For progressively measurable H with $\int_0^T H_s^2\, d\langle M \rangle_s = \int_0^T (H_s \sigma_s)^2\, ds < \infty$ for all $T \geq 0$, we thus define the Itô integral as

$$\int_0^t H_s\, dM_s := \int_0^t (H_s \sigma_s)\, dW_s.$$

Without further work, in particular, without relying on Theorem 25.21, we get the following theorem.

Theorem 25.24 *Let $X = M + A$ be a generalized diffusion with σ and let b be as in Definition 25.23. Let H be progressively measurable with*

$$\int_0^T H_s^2 \sigma_s^2\, ds < \infty \quad \textit{a.s.} \quad \textit{for all } T \geq 0 \tag{25.7}$$

and

$$\int_0^T |H_s b_s|\, ds < \infty \quad \textit{a.s.} \quad \textit{for all } T \geq 0. \tag{25.8}$$

Then the process Y defined by

$$Y_t := \int_0^t H_s\,dX_s := \int_0^t H_s\,dM_s + \int_0^t H_s\,dA_s := \int_0^t H_s\sigma_s\,dW_s + \int_0^t H_s b_s\,ds$$

is a generalized diffusion with diffusion coefficient $(H_s\sigma_s)_{s\geq 0}$ *and drift* $(H_s b_s)_{s\geq 0}$. *In particular,* $N_t := \int_0^t H_s\,dM_s$ *is a continuous local martingale with square variation process* $\langle N\rangle_t = \int_0^t H_s^2\,d\langle M\rangle_s = \int_0^t H_s^2\sigma_s^2\,ds$.

Exercise 25.2.1 Let M be a continuous local martingale with absolutely continuous square variation $\langle M\rangle$ (e.g., a generalized diffusion), and let H be progressively measurable and continuous with $\int_0^T H_s^2\,d\langle M\rangle_s < \infty$ for all $T \geq 0$. Further, assume that $\mathcal{P} = (\mathcal{P}^{(n)})_{n\in\mathbb{N}}$ is an admissible sequence of partitions (see Definition 21.56).

(i) Show that for all $T \geq 0$, in the sense of stochastic convergence, we have

$$\int_0^T H_s\,dM_s = \lim_{n\to\infty} \sum_{t\in\mathcal{P}_T^n} H_t(M_{t'} - M_t). \tag{25.9}$$

(ii) Show that there exists a subsequence of $\mathcal{P}$ such that almost surely, we have (25.9) for all $T \geq 0$.

25.3 The Itô Formula

This and the following two sections are based on lecture notes of Hans Föllmer.

If $t \mapsto X_t$ is a differentiable map with derivative X' and $F \in C^1(\mathbb{R})$ with derivative F', then we have the classical substitution rule

$$F(X_t) - F(X_0) = \int_0^t F'(X_s)\,dX_s = \int_0^t F'(X_s)X_s'\,ds. \tag{25.10}$$

This remains true even if X is continuous and has locally finite variation (see Section 21.10); that is, if X is the distribution function of an absolutely continuous signed measure on $[0,\infty)$. In this case, the derivative X' exists as a Radon–Nikodym derivative almost everywhere, and it is easy to show that (25.10) also holds in this case.

The paths of Brownian motion W are nowhere differentiable (Theorem 21.17 due to Paley, Wiener and Zygmund) and thus have everywhere locally infinite variation. Hence a substitution rule as simple as (25.10) cannot be expected. Indeed, it is easy to see that such a rule *must* be false: Choose $F(x) = x^2$. Then the right-hand side in (25.10) (with X replaced by W) is $\int_0^t 2W_s\,dW_s$ and is hence a martingale. The left-hand side, however, equals W_t^2, which is a submartingale that only becomes a martingale by subtracting t. Indeed, this t is the additional term that shows up in the substitution rule for Itô integrals, the so-called Itô formula. A somewhat bold heuristic puts us on the right track: For small t, W_t is of order $\sqrt{t}$. If we formally write $dW_t = \sqrt{dt}$ and carry out a Taylor expansion of $F \in C^2(\mathbb{R})$ up to second order, then we obtain

$$dF(W_t) = F'(W_t)\,dW_t + \frac{1}{2}F''(W_t)(dW_t)^2 = F'(W_t)\,dW_t + \frac{1}{2}F''(W_t)\,dt.$$

Rewriting this as an integral yields

$$F(W_t) - F(W_0) = \int_0^t F'(W_s)\,dW_s + \int_0^t \frac{1}{2}F''(W_s)\,ds. \tag{25.11}$$

(For certain discrete martingales, we derived a similar formula in Example 10.9.) The main goal of this section is to show that this so-called *Itô formula* is indeed correct.

The subsequent discussion in this section does not explicitly rely on the assumption that we integrate with respect to Brownian motion. All that is needed is that the function with respect to which we integrate have continuous square variation (along a suitable admissible sequence of partitions $\mathcal{P} = (\mathcal{P}^n)_{n\in\mathbb{N}}$)). In particular, for Brownian motion, $\langle W\rangle_t = t$.

In the following, let $\mathcal{P} = (\mathcal{P}^n)_{n\in\mathbb{N}}$ be an admissible sequence of partitions (recall the definition of $\mathcal{C}_{\mathrm{qv}} = \mathcal{C}_{\mathrm{qv}}^{\mathcal{P}}$, $\mathcal{P}_T^n$, $\mathcal{P}_{S,T}^n$, t' and so on from Definitions 21.56 and 21.58). Let $X \in C([0,\infty))$ with continuous square variation (along $\mathcal{P}$)

$$T \mapsto \langle X\rangle_T = V_T^2(X) = \lim_{n\to\infty} \sum_{t\in\mathcal{P}_T} (X_{t'} - X_t)^2.$$

For Brownian motion, we have $W \in \mathcal{C}_{\mathrm{qv}}^{\mathcal{P}}$ almost surely for any admissible sequence of partitions (Theorem 21.64) and $\langle W\rangle_T = T$. For continuous local martingales M passing to a suitable subsequence $\mathcal{P}'$ of $\mathcal{P}$ ensures that $M \in \mathcal{C}_{\mathrm{qv}}^{\mathcal{P}'}$ almost surely (Theorem 21.70).

Now fix $\mathcal{P}$ and let $X \in \mathcal{C}_{\mathrm{qv}}$ be a (deterministic) function.

Theorem 25.25 (Pathwise Itô formula) *Let $X \in \mathcal{C}_{\mathrm{qv}}$ and $F \in C^2(\mathbb{R})$. Then, for all $T \geq 0$, there exists the limit*

$$\int_0^T F'(X_s)\,dX_s := \lim_{n\to\infty} \sum_{t\in\mathcal{P}_T^n} F'(X_t)(X_{t'} - X_t). \tag{25.12}$$

Furthermore, the Itô formula holds:

$$F(X_T) - F(X_0) = \int_0^T F'(X_s)\,dX_s + \frac{1}{2}\int_0^T F''(X_s)\,d\langle X\rangle_s. \tag{25.13}$$

Here the right integral in (25.13) is understood as a classical (Lebesgue–Stieltjes) integral.

Remark 25.26 If M is a continuous local martingale, then, by Exercise 25.2.1, the Itô integral $\int_0^T F'(M_s)\,dM_s$ is the stochastic limit of $\sum_{t\in\mathcal{P}_T^n} F'(M_t)(M_{t'} - M_t)$ as $n \to \infty$. Thus, in fact, for $X = M(\omega)$, the pathwise integral in (25.12) coincides with the Itô integral (a.s.). In particular, for the Itô integral of Brownian motion, the Itô formula (25.11) holds. $\Diamond$

Proof of Theorem 25.25 We have to show that the limit in (25.12) exists and that (25.13) holds.

For $n \in \mathbb{N}$ and $t \in \mathcal{P}_T^n$ (with successor $t' \in \mathcal{P}_T^n$), the Taylor formula yields

$$F(X_{t'}) - F(X_t) = F'(X_t)(X_{t'} - X_t) + \frac{1}{2}F''(X_t)\cdot(X_{t'} - X_t)^2 + R_t^n, \qquad (25.14)$$

where the remainder

$$R_t^n = \big(F''(\xi) - F''(X_t)\big)\cdot\frac{1}{2}(X_{t'} - X_t)^2$$

(for a suitable ξ between X_t and $X_{t'}$) can be bounded as follows. As X is continuous, $C := \{X_t : t \in [0,T]\}$ is compact and $F''\big|_C$ is uniformly continuous. Thus, for every $\varepsilon > 0$, there exists a $\delta > 0$ with

$$\big|F''(X_r) - F''(X_s)\big| < \varepsilon \quad \text{for all } r, s \in [0,T] \text{ with } |X_r - X_s| < \delta.$$

Since X is uniformly continuous on $[0,T]$ and since the mesh size $|\mathcal{P}^n|$ of the partition goes to 0 as $n \to \infty$, for every $\delta > 0$, there exists an N_δ such that

$$\sup_{n\ge N_\delta}\ \sup_{t\in\mathcal{P}_T^n} |X_{t'} - X_t| < \delta.$$

Hence, for $n \ge N_\delta$ and $t \in \mathcal{P}_T^n$,

$$\big|R_t^n\big| \le \frac{1}{2}\varepsilon(X_{t'} - X_t)^2.$$

Summing over $t \in \mathcal{P}_T^n$ in (25.14) yields

$$\sum_{t\in\mathcal{P}_T^n}\big(F(X_{t'}) - F(X_t)\big) = F(X_t) - F(X_0)$$

and

$$\sum_{t\in\mathcal{P}_T^n}\big|R_t^n\big| \le \varepsilon \sum_{t\in\mathcal{P}_T^n}(X_{t'} - X_t)^2 \overset{n\to\infty}{\longrightarrow} \varepsilon\langle X\rangle_t < \infty.$$

As $\varepsilon > 0$ was arbitrary, we get $\sum_{t\in\mathcal{P}_T^n}|R_t^n| \overset{n\to\infty}{\longrightarrow} 0$. We have (see Exercise 21.10.2)

$$\sum_{t\in\mathcal{P}_T^n}\frac{1}{2}F''(X_t)(X_{t'} - X_t)^2 \overset{n\to\infty}{\longrightarrow} \frac{1}{2}\int_0^T F''(X_s)\,d\langle X\rangle_s.$$

Hence, in (25.14) the sum of the remaining terms also has to converge. That is, the limit in (25.12) exists. □

As a direct consequence, we obtain the Itô formula for the Itô integral with respect to diffusions.

Theorem 25.27 (Itô formula for diffusions) *Let $Y = M + A$ be a (generalized) diffusion (see Definition 25.23), where $M_t = \int_0^t \sigma_s \, dW_s$ and $A_t = \int_0^t b_s \, ds$. Let $F \in C^2(\mathbb{R})$. Then we have the Itô formula*

$$\begin{aligned} F(Y_t) - F(Y_0) &= \int_0^t F'(Y_s) \, dM_s + \int_0^t F'(Y_s) \, dA_s + \frac{1}{2} \int_0^t F''(Y_s) \, d\langle M \rangle_s \\ &= \int_0^t F'(Y_s) \sigma_s \, dW_s + \int_0^t \left(F'(Y_s) b_s + \frac{1}{2} F''(Y_s) \sigma_s^2 \right) ds. \end{aligned} \tag{25.15}$$

In particular, for Brownian motion,

$$F(W_t) - F(W_0) = \int_0^t F'(W_s) \, dW_s + \frac{1}{2} \int_0^t F''(W_s) \, ds. \tag{25.16}$$

As an application of the Itô formula, we characterize Brownian motion as a continuous local martingale with a certain square variation process.

Theorem 25.28 (Lévy's characterization of Brownian motion) *Let $X \in \mathcal{M}_{\mathrm{loc},c}$ with $X_0 = 0$. Then the following are equivalent.*

(i) $(X_t^2 - t)_{t \ge 0}$ *is a local martingale.*
(ii) $\langle X \rangle_t = t$ *for all* $t \ge 0$.
(iii) X *is a Brownian motion.*

Proof (iii) $\Longrightarrow$ (i) This is obvious.

(i) $\Longleftrightarrow$ (ii) This is clear since the square variation process is unique.

(ii) $\Longrightarrow$ (iii) It is enough to show that $X_t - X_s \sim \mathcal{N}_{0,t-s}$ given $\mathcal{F}_s$ for $t > s \ge 0$. Employing the uniqueness theorem for characteristic functions, it is enough to show that (with $i = \sqrt{-1}$) for $A \in \mathcal{F}_s$ and $\lambda \in \mathbb{R}$, we have

$$\varphi_{A,\lambda}(t) := \mathbf{E}\big[e^{i\lambda(X_t - X_s)} \mathbb{1}_A\big] = \mathbf{P}[A] e^{-\lambda^2 (t-s)/2}.$$

Applying Itô's formula separately to the real and the imaginary parts, we obtain

$$e^{i\lambda X_t} - e^{i\lambda X_s} = \int_s^t i\lambda e^{i\lambda X_r} \, dX_r - \frac{1}{2} \int_s^t \lambda^2 e^{i\lambda X_r} \, dr.$$

Therefore,

$$\begin{aligned}&\mathbf{E}\big[e^{i\lambda(X_t-X_s)}\,\big|\,\mathcal{F}_s\big]-1\\&\quad=\mathbf{E}\bigg[\int_s^t i\lambda e^{i\lambda(X_r-X_s)}\,dX_r\,\bigg|\,\mathcal{F}_s\bigg]-\frac{1}{2}\lambda^2\mathbf{E}\bigg[\int_s^t e^{i\lambda(X_r-X_s)}\,dr\,\bigg|\,\mathcal{F}_s\bigg].\end{aligned}$$

Now $M_t := \operatorname{Re}\int_s^t i\lambda e^{i\lambda(X_r-X_s)}\,dX_r$ and $N_t := \operatorname{Im}\int_s^t i\lambda e^{i\lambda(X_r-X_s)}\,dX_r$, $t\ge s$ are continuous local martingales with

$$\langle M\rangle_t=\int_s^t \lambda^2\sin\big(\lambda(X_r-X_s)\big)^2\,dr\le\lambda^2(t-s)$$

and

$$\langle N\rangle_t=\int_s^t \lambda^2\cos\big(\lambda(X_r-X_s)\big)^2\,dr\le\lambda^2(t-s).$$

Thus, by Corollary 21.76, M and N are martingales. Hence we have

$$\mathbf{E}\bigg[\int_s^t i\lambda e^{i\lambda(X_r-X_s)}\,dX_r\,\bigg|\,\mathcal{F}_s\bigg]=0.$$

Since $A\in\mathcal{F}_s$, Fubini's theorem yields

$$\begin{aligned}\varphi_{A,\lambda}(t)-\varphi_{A,\lambda}(s)&=\mathbf{E}\big[e^{i\lambda(X_t-X_s)}\mathbb{1}_A\big]-\mathbf{P}[A]\\&=-\frac{1}{2}\lambda^2\int_s^t\mathbf{E}\big[e^{i\lambda(X_r-X_s)}\mathbb{1}_A\big]\,dr=-\frac{1}{2}\lambda^2\int_s^t\varphi_{A,\lambda}(r)\,dr.\end{aligned}$$

That is, $\varphi_{A,\lambda}$ is the solution of the linear differential equation

$$\varphi_{A,\lambda}(s)=\mathbf{P}[A]\quad\text{and}\quad\frac{d}{dt}\varphi_{A,\lambda}(t)=-\frac{1}{2}\lambda^2\varphi_{A,\lambda}(t).$$

The unique solution is $\varphi_{A,\lambda}(t)=\mathbf{P}[A]e^{-\lambda^2(t-s)/2}$. □

As a consequence of this theorem, we get that any continuous local martingale whose square variation process is absolutely continuous (as a function of time) can be expressed as an Itô integral with respect to some Brownian motion.

Theorem 25.29 (Itô's martingale representation theorem) *Let M be a continuous local martingale with $M_0=0$ and absolutely continuous square variation $t\mapsto\langle M\rangle_t$. Then, on a suitable extension of the underlying probability space, there exists a Brownian motion W with*

$$M_t=\int_0^t\sqrt{\frac{d\langle M\rangle_s}{ds}}\,dW_s\quad\textit{for all } t\ge0.$$

Proof Assume that the probability space is rich enough to carry a Brownian motion $\widetilde{W}$ that is independent of M. Let

$$f_t := \lim_{n\to\infty} n\big(\langle M\rangle_t - \langle M\rangle_{t-1/n}\big) \quad \text{for } t > 0.$$

Then f is a progressively measurable version of the Radon–Nikodym derivative $\frac{d\langle M\rangle_t}{dt}$. Clearly, $\int_0^T \mathbb{1}_{\{f_t>0\}} f_t^{-1}\, d\langle M\rangle_t \le T < \infty$ for all $T > 0$. Hence the following integrals are well-defined, and furthermore, as a sum of continuous martingales,

$$W_t := \int_0^t \mathbb{1}_{\{f_s>0\}} f_s^{-1/2}\, dM_s + \int_0^t \mathbb{1}_{\{f_s=0\}}\, d\widetilde{W}_s$$

is a continuous local martingale. By Theorem 25.22, we have

$$\begin{aligned}\langle W\rangle_t &= \int_0^t \mathbb{1}_{\{f_s>0\}} f_s^{-1}\, d\langle M\rangle_s + \int_0^t \mathbb{1}_{\{f_s=0\}}\, ds\\ &= \int_0^t \mathbb{1}_{\{f_s>0\}} f_s^{-1} f_s\, ds + \int_0^t \mathbb{1}_{\{f_s=0\}}\, ds = t.\end{aligned}$$

Hence, by Theorem 25.28, W is a Brownian motion. On the other hand, we have

$$\begin{aligned}\int_0^t f_s^{1/2}\, dW_s &= \int_0^t \mathbb{1}_{\{f_s>0\}} f_s^{1/2} f_s^{-1/2}\, dM_s + \int_0^t \mathbb{1}_{\{f_s=0\}} f_s^{1/2}\, d\widetilde{W}_s\\ &= \int_0^t \mathbb{1}_{\{f_s>0\}}\, dM_s.\end{aligned}$$

However,

$$M_t - \int_0^t \mathbb{1}_{\{f_s>0\}}\, dM_s = \int_0^t \mathbb{1}_{\{f_s=0\}}\, dM_s$$

is a continuous local martingale with square variation $\int_0^t \mathbb{1}_{\{f_s=0\}}\, d\langle M\rangle_s = 0$ and hence it almost surely equals zero. Therefore, we have $M_t = \int_0^t f_s^{1/2}\, dW_s$, as claimed. $\square$

We come next to a multidimensional version of the pathwise Itô formula. To this end, let $\mathcal{C}_{\mathrm{qv}}^d$ be the space of continuous maps $X : [0,\infty) \to \mathbb{R}^d$, $t \mapsto X_t = (X_t^1, \ldots, X_t^d)$ such that, for $k, l = 1, \ldots, d$, the quadratic covariation (see Definition 21.58) $\langle X^k, X^l\rangle$ exists and is continuous. Further, let $C^2(\mathbb{R}^d)$ be the space of twice continuously differentiable functions F on $\mathbb{R}^d$ with partial derivatives $\partial_k F$ and $\partial_k\partial_l F$, $k, l = 1, \ldots, d$. Denote by ∇ the gradient and by $\Delta = (\partial_1^2 + \ldots + \partial_d^2)$ the *Laplace operator*.

Theorem 25.30 (Multidimensional pathwise Itô formula) *Let* $X \in \mathcal{C}_{\mathrm{qv}}^d$ *and* $F \in C^2(\mathbb{R}^d)$. *Then*

$$F(X_T) - F(X_0) = \int_0^T \nabla F \, dX_s + \frac{1}{2} \sum_{k,l=1}^d \int_0^T \partial_k \partial_l F(X_s) \, d\langle X^k, X^l \rangle_s,$$

where

$$\int_0^T \nabla F(X_s) \, dX_s := \lim_{n\to\infty} \sum_{t\in\mathcal{P}_T^n} \sum_{k=1}^d \partial_k F(X_t)\big(X_{t'}^k - X_t^k\big).$$

Proof This works just as in the one-dimensional case. We leave the details as an exercise. □

Remark 25.31 If each of the integrals $\int_0^T \partial_k F(X_s) \, dX_s^k$ exists, then

$$\int_0^T \nabla F(X_s) \, dX_s = \sum_{k=1}^d \int_0^T \partial_k F(X_s) \, dX_s^k.$$

Note that existence of the individual integrals does not follow from the existence of the integral $\int_0^T \nabla F(X_s) \, dX_s$. ◊

Corollary 25.32 (Product rule) *If* $X, Y, X - Y, X + Y \in \mathcal{C}_{\mathrm{qv}}$, *then*

$$X_T Y_T = X_0 Y_0 + \int_0^T Y_s \, dX_s + \int_0^T X_s \, dY_s + \langle X, Y \rangle_T \quad \textit{for all } T \geq 0$$

if both integrals exist. In particular, the product rule holds if X and Y are continuous local martingales.

Proof By assumption (and using the polarization formula), the covariation $\langle X, Y \rangle$ exists. Applying Theorem 25.30 with $F(x, y) = xy$, the claim follows.

For continuous local martingales, by Exercise 25.2.1, there exists a suitable sequence of partitions $\mathcal{P}$ such that the integrals exist (pathwise). □

Now let $Y = M + A$ be a d-dimensional generalized diffusion. Hence

$$M_t^k = \sum_{l=1}^d \int_0^t \sigma_s^{k,l} \, dW_s^l \quad \text{and} \quad A_t^k = \int_0^t b_s^k \, ds \quad \text{for } t \geq 0,\ k = 1, \ldots, d.$$

Here $W = (W^1, \ldots, W^d)$ is a d-dimensional Brownian motion and $\sigma^{k,l}$ (respectively b^k) are progressively measurable, locally square integrable (respectively locally integrable) stochastic processes for every $k, l = 1, \ldots, d$. Since $\langle W^k, W^l \rangle_t = t \cdot \mathbb{1}_{\{k=l\}}$, we have $\langle Y^k, Y^l \rangle_t = \langle M^k, M^l \rangle_t = \int_0^t a_s^{k,l} \, ds$, where

$$a_s^{k,l} := \sum_{i=1}^d \sigma_s^{k,i} \sigma_s^{i,l}$$

is the covariance matrix of the diffusion M. In particular, we have $M \in \mathcal{C}^d_{\mathrm{qv}}$ almost surely. Note that (by Exercise 25.2.1), there exists a partition sequence $\mathcal{P}$ such that the integrals $\int_0^t \sigma_s^{k,l} \partial_k F(Y_s)\, dW_s^l$ in (25.17) exist in the pathwise sense. As a corollary of the multidimensional pathwise Itô formula (Theorem 25.30 and Remark 25.31), we thus get the following theorem.

Theorem 25.33 (Multidimensional Itô formula) *Let Y be as above and let $F \in C^2(\mathbb{R}^d)$. Then*

$$\begin{aligned} F(Y_T) - F(Y_0) &= \int_0^T \nabla F(Y_s)\, dY_s + \frac{1}{2}\sum_{k,l=1}^d \int_0^T \partial_k \partial_l F(Y_s)\, d\langle M^k, M^l\rangle_s \\ &= \sum_{k,l=1}^d \int_0^t \sigma_s^{k,l} \partial_k F(Y_s)\, dW_s^l + \sum_{k=1}^d \int_0^t b_s^k \partial_k F(Y_s)\, ds \\ &\quad + \frac{1}{2}\sum_{k,l=1}^d \int_0^t a_s^{k,l} \partial_k \partial_l F(Y_s)\, ds. \end{aligned} \tag{25.17}$$

In particular, for Brownian motion, we have

$$F(W_t) - F(W_0) = \sum_{k=1}^d \int_0^t \partial_k F(W_s)\, dW_s^k + \frac{1}{2}\int_0^t \Delta F(W_s)\, ds. \tag{25.18}$$

Corollary 25.34 *The process $(F(W_t))_{t\geq 0}$ is a continuous local martingale if and only if F is harmonic (that is, $\Delta F \equiv 0$).*

Proof If F is harmonic, then as a sum of Itô integrals, $F(W_t) = F(W_0) + \sum_{k=1}^d \int_0^t \partial_k F(W_s)\, dW_s^k$ is a continuous local martingale.

On the other hand, if F is a continuous local martingale, then as a difference of continuous local martingales, $\int_0^t \Delta F(W_s)\, ds$ is also a continuous local martingale. As $t \mapsto \int_0^t \Delta F(W_s)\, ds$ has finite variation, we have $\int_0^t \Delta F(W_s)\, ds = 0$ for all $t \geq 0$ almost surely (by Corollary 21.72). Hence $\Delta F \equiv 0$. □

Corollary 25.35 (Time-dependent Itô formula) *If $F \in C^{2,1}(\mathbb{R}^d \times \mathbb{R})$, then*

$$\begin{aligned} &F(W_T, T) - F(W_0, 0) \\ &\quad = \sum_{k=1}^d \int_0^T \partial_k F(W_s, s)\, dW_s^k + \int_0^T \left(\partial_{d+1} + \frac{1}{2}(\partial_1^2 + \ldots + \partial_d^2)\right) F(W_s, s)\, ds. \end{aligned}$$

Proof Apply Theorem 25.33 to $Y = (W_t^1, \ldots, W_t^d, t)_{t\geq 0}$. □

Exercise 25.3.1 (Fubini's theorem for Itô integrals) Let $X \in \mathcal{C}_{\mathrm{qv}}$ and assume that $g : [0, \infty)^2 \to \mathbb{R}$ is continuous and (in the interior) twice continuously differentiable in the second coordinate with derivative $\partial_2 g$. Use the product rule (Corollary 25.32) to show that

$$\int_0^s \left(\int_0^t g(u, v)\, du \right) dX_v = \int_0^t \left(\int_0^s g(u, v)\, dX_v \right) du$$

and

$$\int_0^s \left(\int_0^v g(u, v) du \right) dX_v = \int_0^s \left(\int_u^s g(u, v)\, dX_v \right) du.$$

Exercise 25.3.2 (Stratonovich integral) Let $\mathcal{P}$ be an admissible sequence of partitions, $X \in \mathcal{C}_{\mathrm{qv}}^{\mathcal{P}}$ and $F \in C^2(\mathbb{R})$ with derivative $f = F'$. Show that, for every $t \geq 0$, the *Stratonovich integral*

$$\int_0^T f(X_t) \circ dX_t := \lim_{n\to\infty} \sum_{t \in \mathcal{P}_T^n} f\left(\frac{X_{t'} + X_t}{2} \right) (X_{t'} - X_t)$$

is well-defined, and that the classical substitution rule

$$F(X_T) - F(X_0) = \int_0^T F'(X_t) \circ dX_t$$

holds. Show that, in contrast with the Itô integral, the Stratonovich integral with respect to a continuous local martingale is, in general, not a local martingale.

25.4 Dirichlet Problem and Brownian Motion

As for discrete Markov chains (compare Section 19.1), the solutions of the Dirichlet problem in a domain $G \subset \mathbb{R}^d$ can be expressed in terms of a d-dimensional Brownian motion that is stopped upon hitting the boundary G.

In the following, let $G \subset \mathbb{R}^d$ be a bounded open set.

Definition 25.36 (Dirichlet problem) Let $f : \partial G \to \mathbb{R}$ be continuous. A function $u : \overline{G} \to \mathbb{R}$ is called a solution of the Dirichlet problem on G with boundary value f if u is continuous, twice differentiable in G and

$$\begin{aligned} \Delta u(x) &= 0 \qquad \text{for } x \in G, \\ u(x) &= f(x) \qquad \text{for } x \in \partial G. \end{aligned} \tag{25.19}$$

For sufficiently smooth domains, there always exists a solution of the Dirichlet problem (see, e.g., [79, Section 4.4]). If there is a solution, then as a consequence of Theorem 25.38, it is unique.

In the following, let $W = (W^1, \ldots, W^d)$ be a d-dimensional Brownian motion with respect to a filtration $\mathbb{F}$ that satisfies the usual conditions. We write $\mathbf{P}_x$ and $\mathbf{E}_x$ for probabilities and expectations if W is started at $W_0 = x = (x^1, \ldots, x^d) \in \mathbb{R}^d$. If $A \subset \mathbb{R}^d$ is open, then

$$\tau_{A^c} := \inf\{t > 0 : W_t \in A^c\}$$

is an $\mathbb{F}$-stopping time (see Exercise 21.4.4). Since G is bounded, we have $G \subset (-a, a) \times \mathbb{R}^{d-1}$ for some $a > 0$. Thus $\tau_{G^c} \le \tau_{((-a,a)\times\mathbb{R}^{d-1})^c}$. By Exercise 21.2.4 (applied to W^1), for $x \in G$,

$$\mathbf{E}_x[\tau_{G^c}] \le \mathbf{E}_x[\tau_{((-a,a)\times\mathbb{R}^{d-1})^c}] = \big(a - x^1\big)\big(a + x^1\big) < \infty. \tag{25.20}$$

In particular, $\tau_{G^c} < \infty$ $\mathbf{P}_x$-almost surely. Hence $W_{\tau_{G^c}}$ is a $\mathbf{P}_x$-almost surely well-defined random variable with values in ∂G.

Definition 25.37 For $x \in G$, denote by

$$\mu_{x,G} = \mathbf{P}_x \circ W_{\tau_{G^c}}^{-1}$$

the *harmonic measure* on ∂G.

Theorem 25.38 *If u is a solution of the Dirichlet problem on G with boundary value f, then*

$$u(x) = \mathbf{E}_x\big[f(W_{\tau_{G^c}})\big] = \int_{\partial G} f(y)\mu_{x,G}(dy) \quad \textit{for } x \in G. \tag{25.21}$$

In particular, the solution of the Dirichlet problem is always unique.

Proof Let $G_1 \subset G_2 \subset \ldots$ be a sequence of open sets with $x \in G_1$, $G_n \uparrow G$ and $\overline{G}_n \subset G$ for every $n \in \mathbb{N}$. Hence, in particular, every $\overline{G}_n$ is compact and thus ∇u is bounded on $\overline{G}_n$. We abbreviate $\tau := \tau_{G^c}$ and $\tau_n := \tau_{G_n^c}$.

As u is harmonic (that is, $\Delta u = 0$), by the Itô formula, for $t < \tau$,

$$u(W_t) = u(W_0) + \int_0^t \nabla u(W_s)\, dW_s = u(W_0) + \sum_{k=1}^d \int_0^t \partial_k u(W_s)\, dW_s^k. \tag{25.22}$$

In particular, $M := (u(W_t))_{t\in[0,\tau)}$ is a local martingale up to τ (however, in general, it is not a martingale). For $t < \tau_n$, we have

$$\big(\partial_k u(W_s)\big)^2 \le C_n := \sup_{y\in\overline{G}_n} \big\|\nabla u(y)\big\|_2^2 < \infty \quad \text{for all } k = 1, \ldots, d.$$

Hence, by (25.20),

$$\mathbf{E}\bigg[\int_0^{\tau_n} \big(\partial_k u(W_s)\big)^2\, ds\bigg] \le C_n \mathbf{E}_x[\tau_n] \le C_n \mathbf{E}[\tau] < \infty.$$

Thus, by Theorem 25.17(ii), for every $n \in \mathbb{N}$, the stopped process M^{τ_n} is a martingale. Therefore,

$$\mathbf{E}_x\big[u(W_{\tau_n})\big] = \mathbf{E}_x[M_{\tau_n}] = \mathbf{E}_x[M_0] = u(x). \tag{25.23}$$

As W is continuous and $\tau_n \uparrow \tau$, we have $W_{\tau_n} \stackrel{n\to\infty}{\longrightarrow} W_\tau \in \partial G$. Since u is continuous, we get

$$u(W_{\tau_n}) \stackrel{n\to\infty}{\longrightarrow} u(W_\tau) = f(W_\tau). \tag{25.24}$$

Again, since u is continuous and $\overline{G}$ is compact, u is bounded. By the dominated convergence theorem, (25.24) implies convergence of the expectations; that is (also using (25.23)),

$$u(x) = \lim_{n\to\infty} \mathbf{E}_x\big[u(W_{\tau_n})\big] = \mathbf{E}_x\big[f(W_\tau)\big]. \qquad \square$$

Exercise 25.4.1 Let $G = \mathbb{R} \times (0, \infty)$ be the open upper half plane of $\mathbb{R}^2$ and $x = (x_1, x_2) \in G$. Show that $\tau_{G^c} < \infty$ almost surely and that the harmonic measure $\mu_{x,G}$ on $\mathbb{R} \cong \partial G$ is the Cauchy distribution with scale parameter x_2 that is shifted by x_1: $\mu_{x,G} = \delta_{x_1} * \mathrm{Cau}_{x_2}$.

Exercise 25.4.2 Let $d \geq 3$ and let $G = \mathbb{R}^{d-1} \times (0, \infty)$ be an open half space of $\mathbb{R}^d$. Let $x = (x_1, \ldots, x_d) \in G$. Show that $\tau_{G^c} < \infty$ almost surely and that the harmonic measure $\mu_{x,G}$ on $\mathbb{R}^{d-1} \cong \partial G$ has the density

$$\frac{\mu_{x,G}(dy)}{dy} = \frac{\Gamma(d/2)}{\pi^{d/2}} \frac{x_d}{\sqrt{(x_1 - y_1)^2 + \ldots + (x_{d-1} - y_{d-1})^2 + x_d^2}}.$$

Exercise 25.4.3 Let $r > 0$ and let $B_r(0) \subset \mathbb{R}^d$ be the open ball with radius r centered at the origin. For $x \in B_r(0)$, determine the harmonic measure $\mu_{x,B_r(0)}$.

25.5 Recurrence and Transience of Brownian Motion

By Pólya's theorem (Theorem 17.39), symmetric simple random walk $(X_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}^d$ is recurrent (that is, it visits every point infinitely often) if and only if $d \leq 2$. If $d > 2$, then the random walk is transient and eventually leaves every bounded set $A \subset \mathbb{Z}^d$. To give a slightly different (though equivalent) formulation of this,

$$\liminf_{n\to\infty} \|X_n\| = 0 \quad \text{a.s.} \qquad \Longleftrightarrow \qquad d \leq 2$$

and

$$\lim_{n\to\infty} \|X_n\| = \infty \quad \text{a.s.} \qquad \Longleftrightarrow \qquad d > 2.$$

The main result of this section is that a similar dichotomy also holds for Brownian motion.

Theorem 25.39 *Let $W = (W^1, \dots, W^d)$ be a d-dimensional Brownian motion.*

(i) *If $d \leq 2$, then W is recurrent in the sense that*

$$\liminf_{t\to\infty} \|W_t - y\| = 0 \quad a.s. \quad \text{for every } y \in \mathbb{R}^d.$$

In particular, almost surely the path $\{W_t : t \geq 0\}$ is dense in $\mathbb{R}^d$.

(ii) *If $d > 2$, then W is transient in the sense that*

$$\lim_{t\to\infty} \|W_t\| = \infty \quad a.s.,$$

and for any $y \in \mathbb{R}^d \setminus \{0\}$, we have $\inf\{\|W_t - y\| : t \geq 0\} > 0$ almost surely.

The basic idea of the proof is to use a suitable Dirichlet problem (and the result of Section 25.4) to compute the probabilities for W to hit certain balls,

$$B_R(x) := \left\{y \in \mathbb{R}^d : \|x - y\| < R\right\}.$$

Let $0 < r < R < \infty$ and let $G_{r,R}$ be the annulus

$$G_{r,R} := B_R(0) \setminus \overline{B}_r(0) = \left\{x \in \mathbb{R}^d : r < \|x\| < R\right\}.$$

Recall that, for closed $A \subset \mathbb{R}^d$, we write $\tau_A = \inf\{t > 0 : W_t \in A\}$ for the stopping time of first entrance into A. We further write

$$\tau_s := \inf\left\{t > 0 : \|W_t\| = s\right\} \quad \text{and} \quad \tau_{r,R} = \inf\{t > 0 : W_t \notin G_{r,R}\}.$$

If we start W at $W_0 \in G_{r,R}$, then clearly $\tau_{r,R} = \tau_r \wedge \tau_R$. On the boundary of $G_{r,R}$, define the function f by

$$f(x) = \begin{cases} 1, & \text{if } \|x\| = r, \\ 0, & \text{if } \|x\| = R. \end{cases} \tag{25.25}$$

Define $u_{r,R} : \overline{G}_{r,R} \to \mathbb{R}$ by

$$u_{r,R}(x) = \frac{V(\|x\|) - V(R)}{V(r) - V(R)},$$

where $V : (0, \infty) \to \mathbb{R}$ is Newton's potential function

$$V(s) = V_d(s) = \begin{cases} s, & \text{if } d = 1, \\ \log(s), & \text{if } d = 2, \\ -s^{2-d}, & \text{if } d > 2. \end{cases} \tag{25.26}$$

It is easy to check that $\varphi : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}, x \mapsto V_d(\|x\|)$ is harmonic (that is, $\Delta\varphi \equiv 0$). Hence $u_{r,R}$ is the solution of the Dirichlet problem on $G_{r,R}$ with boundary value f.

By Theorem 25.38, for $x \in G_{r,R}$,

$$\mathbf{P}_x[\tau_{r,R} = \tau_r] = \mathbf{P}_x\big[\|W_{\tau_{r,R}}\| = r\big] = \mathbf{E}_x\big[f(W_{\tau_{r,R}})\big] = u_{r,R}(x). \tag{25.27}$$

Theorem 25.40 *For $r > 0$ and $x, y \in \mathbb{R}^d$ with $\|x - y\| > r$, we have*

$$\mathbf{P}_x\big[W_t \in B_r(y) \text{ for some } t > 0\big] = \begin{cases} 1, & \text{if } d \le 2, \\ \left(\frac{\|x-y\|}{r}\right)^{2-d}, & \text{if } d > 2. \end{cases}$$

Proof Without loss of generality, assume $y = 0$. Then

$$\mathbf{P}_x[\tau_r < \infty] = \lim_{R\to\infty} \mathbf{P}_x[\tau_{r,R} = \tau_r] = \lim_{R\to\infty} \frac{V(\|x\|) - V(R)}{V(r) - V(R)}$$

$$= \begin{cases} 1, & \text{if } d = 2, \\ \frac{V_d(\|x\|)}{V_d(r)}, & \text{if } d > 2, \end{cases}$$

since $\lim_{R\to\infty} V_d(R) = \infty$ if $d \le 2$ and $= 0$ if $d > 2$. □

Proof of Theorem 25.39 Using the strong Markov property of Brownian motion, we get for $r > 0$

$$\mathbf{P}_x\Big[\liminf_{t\to\infty} \|W_t\| < s\Big] = \mathbf{P}_x\bigg[\bigcup_{s\in(0,r)} \bigcap_{R>\|x\|} \big\{\|W_t\| < r \text{ for some } t > \tau_R\big\}\bigg]$$

$$= \sup_{s\in(0,r)} \inf_{R>\|x\|} \mathbf{P}_x\big[\|W_t\| \le s \text{ for some } t > \tau_R\big]$$

$$= \sup_{s\in(0,r)} \inf_{R>\|x\|} \mathbf{P}_x\big[\mathbf{P}_{W_{\tau_R}}[\tau_s < \infty]\big].$$

However, by Theorem 25.40 (since $\|W_{\tau_R}\| = R$ for $R > \|x\|$), we have

$$\mathbf{P}_{W_{\tau_R}}[\tau_s < \infty] = \begin{cases} 1, & \text{if } d \le 2, \\ (s/R)^{d-2}, & \text{if } d > 2. \end{cases}$$

Therefore,

$$\mathbf{P}\Big[\liminf_{t\to\infty} \|W_t\| < r\Big] = \begin{cases} 1, & \text{if } d \le 2, \\ 0, & \text{if } d > 2. \end{cases}$$

This implies the claim. □

Definition 25.41 (Polar set) A set $A \subset \mathbb{R}^d$ is called *polar* if

$$\mathbf{P}_x[W_t \notin A \text{ for all } t > 0] = 1 \quad \text{for all } x \in \mathbb{R}^d.$$

Theorem 25.42 *If $d = 1$, then only the empty set is polar. If $d \geq 2$, then $\{y\}$ is polar for every $y \in \mathbb{R}^d$.*

Proof For $d = 1$, the statement is obvious since

$$\limsup_{t\to\infty} W_t = \infty \quad \text{and} \quad \liminf_{t\to\infty} W_t = -\infty \quad \text{a.s.}$$

Hence, due to the continuity of W, every point $y \in \mathbb{R}$ will be hit (infinitely often).

Now let $d \geq 2$. Without loss of generality, assume $y = 0$. If $x \neq 0$, then

$$\begin{aligned}\mathbf{P}_x[\tau_{\{0\}} < \infty] &= \lim_{R\to\infty} \mathbf{P}_x[\tau_{\{0\}} < \tau_R] \\ &= \lim_{R\to\infty} \inf_{r>0} \mathbf{P}_x[\tau_{r,R} = \tau_r] \\ &= \lim_{R\to\infty} \inf_{r>0} u_{r,R}(x) = 0 \end{aligned} \tag{25.28}$$

since $V_d(r) \stackrel{r\to 0}{\longrightarrow} -\infty$ if $d \geq 2$.

On the other hand, if $x = 0$, then the strong Markov property of Brownian motion (and the fact that $\mathbf{P}_0[W_t = 0] = 0$ for all $t > 0$) implies

$$\begin{aligned}\mathbf{P}_0[\tau_{\{0\}} < \infty] &= \sup_{t>0} \mathbf{P}_0[W_s = 0 \text{ for some } s \geq t] \\ &= \sup_{t>0} \mathbf{P}_0\big[\mathbf{P}_{W_t}[\tau_{\{0\}} < \infty]\big] = 0.\end{aligned}$$

Note that in the last step, we used (25.28). □