Chapter 3
Electrons with Spin

In this chapter, we describe a simple experiment about the spin property of an electron and use it to determine what is a *physical observable* in quantum mechanics. We give a general description of physical systems with two distinct states. We conclude this chapter with a brief description how spin can be used in magnetic resonance imaging, a technique that is used in hospitals to reveal what is going on inside your brain.

3.1 The Stern-Gerlach Experiment

In this section we will follow an approach similar to the previous two chapters: we will describe a simple experiment that will reveal an internal property of electrons, called spin. We will then study various modifications of this experiment and deduce the behaviour of electron spin. This will lead us to describe the spin of the electron in a similar way to the description of a photon in an interferometer, based on states. We use the mathematical theory we developed in the previous chapter to predict measurement outcomes of the electron spin experiment.

In a Stern–Gerlach experiment (Gerlach and Stern 1922), shown schematically in Fig. 3.1, electrons are sent from a source to a fluorescent screen. A bright dot will appear at the position where the electron hits the screen. The electrons travel through a region with a strong (non-uniform) magnetic field, generated by two magnets aligned along the vertical axis (we call it the z-axis). In the absence of the magnetic field, the electrons travel along a central axis connecting the source with the screen (we call this the y-axis, which is directed *towards* the source). When the magnetic field is

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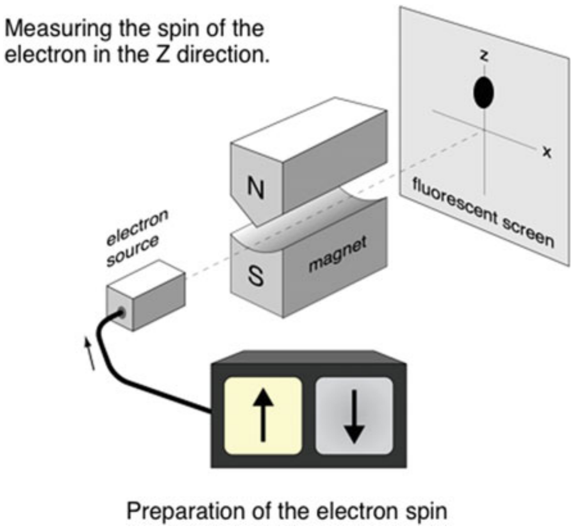

Fig. 3.1 A Stern–Gerlach apparatus. The "up" and "down" regions on the screen where the electron is detected define the electron spin states $|\uparrow\rangle$ and $|\downarrow\rangle$. The interactive figure is available online (see supplementary material 1)

turned on, the dots will appear in two regions: one slightly up from the central axis, and one slightly down from the axis, shown in Fig. 3.1. This means that the electrons have some kind of property that causes a deflection of the path in the presence of a magnetic field. We call this property the *spin* of the electron.

Electrons are particles with electric charge e, which means that they feel a sideways force $F = evB$ when they move at speed v through a magnetic field B. For simplicity, we assume that we compensated for this effect.

Since there are only two positions where the electron can hit the screen (the "up" region or the "down" region), we can again describe the state of the electron spin as $|\text{up}\rangle$ or $|\text{down}\rangle$. Following standard notation in quantum mechanics, we will write this as $|\uparrow\rangle$ and $|\downarrow\rangle$. An arbitrary spin state $|\text{spin}\rangle$ can then be written as a superposition of these two states:

$$|\text{spin}\rangle = a|\uparrow\rangle + b|\downarrow\rangle \,. \tag{3.1}$$

This is similar to the way we constructed the state of a photon inside a Mach–Zehnder interferometer in the previous chapter. In vector notation we have

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.2}$$

and the arbitrary spin state in Eq. (3.1) can be written as

$$|\text{spin}\rangle = \begin{pmatrix} a \\ b \end{pmatrix} . \tag{3.3}$$

To ensure that the vector associated with $|\text{spin}\rangle$ has length one, we calculate the length of a vector by taking the scalar product with itself:

$$\langle \text{spin}|\text{spin}\rangle = \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = |a|^2 + |b|^2 = 1 . \tag{3.4}$$

This leads to the normalisation $|a|^2 + |b|^2 = 1$, and is again similar to the results we derived in the previous chapter, but now with electron spin instead of photon path. In the next chapter we will provide yet another construction of such a quantum state, but then for the energy levels of an atom. For every physical system the procedure is the same: identify the distinct states of a system, give them labels inside a ket, and associate with each distinct state an orthogonal vector.

The same mathematics we developed in the previous chapter can now be used to describe the spin of an electron. The fluorescent screen plays the role of the detectors D_1 and D_2, and the probability of finding a dot at position "up" is given by

$$p_\uparrow = |\langle\uparrow|\text{spin}\rangle|^2 = |a|^2 , \tag{3.5}$$

and the probability of finding a dot at position "down" is given by

$$p_\downarrow = |\langle\downarrow|\text{spin}\rangle|^2 = |b|^2 . \tag{3.6}$$

So we see that $p_\uparrow + p_\downarrow = 1$ is again automatically satisfied via the normalisation condition $|a|^2 + |b|^2 = 1$.

When the source produces electrons in the state $|\uparrow\rangle$, we have $a = 1$ and $b = 0$, and therefore $p_\uparrow = 1$ and $p_\downarrow = 0$. The electron always hits the screen at the position "up". Similarly, if the source produces electrons with spin state $|\downarrow\rangle$ the electron will always hit the screen at position "down". This is how we defined the states $|\uparrow\rangle$ and $|\downarrow\rangle$ in the first place. When we create electrons in the spin state

$$|+\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle , \tag{3.7}$$

or

$$|-\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle , \tag{3.8}$$

the electrons have a 50:50 chance of hitting positions "up" and "down" on the screen. We can verify this statement by calculating the probabilities:

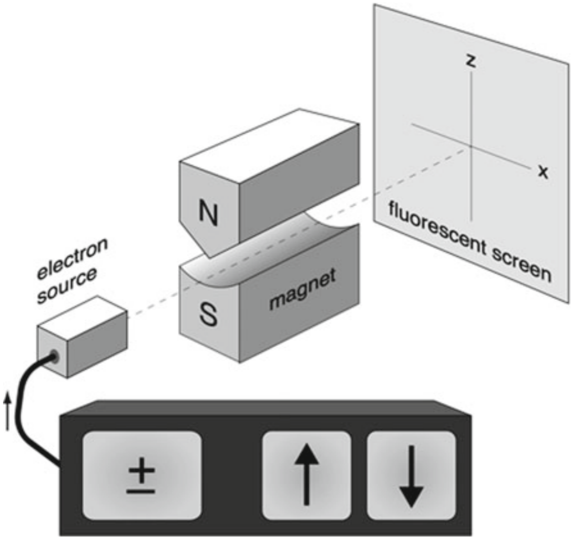

Fig. 3.2 Creating electrons in spin states $|\pm\rangle$ will give different probabilities of finding the electron in the "up" and "down" regions. The interactive figure is available online (see supplementary material 2)

$$p_{\uparrow} = |\langle\uparrow|+\rangle|^2 = \left|\frac{\langle\uparrow|\uparrow\rangle}{\sqrt{2}} + \frac{\langle\uparrow|\downarrow\rangle}{\sqrt{2}}\right|^2 = \frac{1}{2}\,, \tag{3.9}$$

and

$$p_{\downarrow} = |\langle\downarrow|+\rangle|^2 = \left|\frac{\langle\downarrow|\uparrow\rangle}{\sqrt{2}} + \frac{\langle\downarrow|\downarrow\rangle}{\sqrt{2}}\right|^2 = \frac{1}{2}\,, \tag{3.10}$$

where we used $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$ and $\langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 0$. You should check these probabilities yourself using the vector form for $|\uparrow\rangle$ and $|\downarrow\rangle$. Similarly, you can calculate the probabilities $p_{\uparrow}$ and $p_{\downarrow}$ given the electron state $|-\rangle$. The behaviour of electrons with spin $|+\rangle$ and $|-\rangle$ is shown in Fig. 3.2.

Looking again at the Stern–Gerlach experiment, you see that we can rotate the two magnets around the axis that connects the source and the screen (the dashed line). Let us investigate what happens when we rotate the two magnets over 90° (counterclockwise when facing the screen), so the magnets are aligned along the horizontal axis, called the x-axis. When we do the experiment, we find the results shown in Fig. 3.3.

If we prepare the electron in the spin state $|\uparrow\rangle$ we will find the fluorescent spots randomly on the left or on the right of the central vertical axis on the screen. Each electron has a 50:50 chance of going to the left and going to the right. The same behaviour occurs when we create the electron in the spin state $|\downarrow\rangle$.

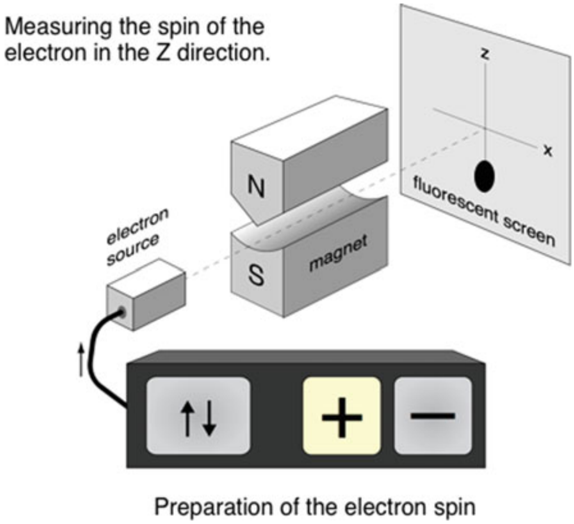

Fig. 3.3 A Stern–Gerlach apparatus with input states $|\pm\rangle$. The interactive figure is available online (see supplementary material 3)

However, when we create the electron in the state $|+\rangle$ and send it through the magnets oriented in the x-direction, we find that the electron always ends up on the left. Similarly, when we create the electron in the spin state $|-\rangle$, it will always create a fluorescent spot on the right. In other words, the spin states $|+\rangle$ and $|-\rangle$ act the same in the x-direction as the states $|\uparrow\rangle$ and $|\downarrow\rangle$ do in the z-direction. We can say that $|\uparrow\rangle$ is the state of an electron spin in the positive z-direction, and $|+\rangle$ is the state of an electron spin in the positive x-direction. In other words, spin must be a quantity with a directional character, which means it is a *vector*:

$$\mathbf{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} . \tag{3.11}$$

This kind of vector is different from the vectors $|\uparrow\rangle$ or $|+\rangle$. The quantum states are vectors in an abstract space that we created for the convenience of calculating probabilities, while the spin vector $\mathbf{S}$ is a vector in our own real three-dimensional space. It is crucial that you remember the difference between these two types of vectors.

You may be wondering about the y-direction of the electron spin. We cannot orient the magnets in the y-direction because they would block the path of the electron (unless we drill holes in the magnets), but we can create an electron with a spin state oriented in the y-direction—let's not worry about how we achieve this right now—and send it through the magnets oriented in the x- or z-direction. We expect that these electron states will give 50:50 distributions of fluorescent spots in the up and down region, as well as in the left and right region. Indeed, when we create electrons in the spin state

$$|\circlearrowleft\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{i}{\sqrt{2}}|\downarrow\rangle\,, \tag{3.12}$$

and

$$|\circlearrowright\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{i}{\sqrt{2}}|\downarrow\rangle\,, \tag{3.13}$$

we find that the probabilities of measuring spin $|\uparrow\rangle$ and $|\downarrow\rangle$ are

$$\begin{aligned} p_\uparrow &= |\langle\uparrow|\circlearrowleft\rangle|^2 = \frac{1}{2} \quad \text{and} \quad p_\downarrow = |\langle\downarrow|\circlearrowleft\rangle|^2 = \frac{1}{2} \\ p_\uparrow &= |\langle\uparrow|\circlearrowright\rangle|^2 = \frac{1}{2} \quad \text{and} \quad p_\downarrow = |\langle\downarrow|\circlearrowright\rangle|^2 = \frac{1}{2}\,, \end{aligned} \tag{3.14}$$

The probabilities of measuring spin $|+\rangle$ and $|-\rangle$ are

$$\begin{aligned} p_+ &= |\langle+|\circlearrowleft\rangle|^2 = \frac{1}{2} \quad \text{and} \quad p_- = |\langle-|\circlearrowleft\rangle|^2 = \frac{1}{2} \\ p_+ &= |\langle+|\circlearrowright\rangle|^2 = \frac{1}{2} \quad \text{and} \quad p_- = |\langle-|\circlearrowright\rangle|^2 = \frac{1}{2}\,, \end{aligned} \tag{3.15}$$

You should check these results both in Dirac notation and the vector notation. We will generalise these results to spins in any spatial direction in Sect. 3.3.

Finally, we should ask the question: What exactly is it that we are measuring in the Stern–Gerlach experiment? We have seen that the orientation of the magnets is linked to the components of the spin vector. When the magnets are oriented in the z-direction we are measuring the z-component of $\mathbf{S}$, or S_z. Similarly, when the magnets are oriented in the x-direction we are measuring S_x. We can also set the magnets at an angle θ from the z-axis. However, *we can measure only one spin component at a time*, because there is only one orientation of the magnets we can choose at any time. We cannot measure the entire vector $\mathbf{S}$ in a single measurement. This will have some interesting consequences, which we will return to later.

3.2 The Spin Observable

We have seen that the spin of an electron is described by a vector. We should also note that spin is a physical property with units, just like velocity (e.g., metres per second or miles per hour), or energy (Joules, ergs, or electron volts). In the case of spin the units are that of angular momentum: J s, or kg m^2 s^{-1}. This is because the spin of an electron is the quantum mechanical analog of the rotation around its own axis, which was discovered by Samuel Goudsmit and George Uhlenbeck (1925). Just like a spinning top, we have the rotation velocity (the magnitude) and the direction of the rotation axis. This is why spin is a vector: it has a magnitude and a direction. For electrons it is convenient to express the magnitude of the spin in terms of Dirac's constant

$$\hbar = \frac{h}{2\pi} = 1.054572 \times 10^{-34}\ \text{J s}. \tag{3.16}$$

You may recognise h as Planck's constant, and $\hbar$ is also called the reduced Planck constant. The value of the spin in the z-direction S_z of an electron in the state $|\uparrow\rangle$ is $+\hbar/2$, and the value of spin in the z-direction of an electron in the state $|\downarrow\rangle$ is $-\hbar/2$.

Every time you see h or $\hbar$, you are are dealing with a quantum mechanical situation.

Classically, one would expect that the spin can take any magnitude (the rotation velocity of the electron). After all, it is easy to imagine that the electron can rotate slower or faster around its own axis. But the Stern–Gerlach experiment tells us that this is not the case. If we measure the spin in the z-direction we find only two possible values, "up" or "down". This tells us that the electron spin is a fundamentally quantum mechanical quantity: its spin does not take a continuous range of values, but is restricted to two distinct values. We say that the spin of the electron is quantised. This happens not only for the spin, but for the energy as well (which we will explore in the next chapter), and it is the origin of the terms "quantum" mechanics. The quantisation of spin is an experimental fact of nature that we have to accept, just like we had to accept in the previous chapters that light is quantised (i.e., the quantum of light is the photon).

Next, we will give a more complete mathematical description of the so-called spin observable based on the findings of the Stern–Gerlach experiment. We know how an electron with spin state $|+\rangle$ behaves in a Stern–Gerlach apparatus oriented in the x-direction: the electron will create a fluorescent dot in the "+" region with probability $p_+ = 1$. This is the definition of the state $|+\rangle$. The numerical value of the spin in the x-direction S_x for the electron in state $|+\rangle$ is $+\hbar/2$, but the numerical value of S_z is different: when we actually measure it we find that the electron ends up in the "up" or "down" region with equal probability of $1/2$. On average, the value of S_z is therefore given by

$$\langle S_z \rangle = \frac{1}{2}\left(+\frac{\hbar}{2}\right) + \frac{1}{2}\left(-\frac{\hbar}{2}\right) = \frac{\hbar}{4} - \frac{\hbar}{4} = 0\,, \tag{3.17}$$

where we used the notation $\langle\cdot\rangle$ to denote the average. So while the individual measurement outcomes can be only $\pm\hbar/2$, the *average* spin in the z-direction over may measurements can take *any* value between $+\hbar/2$ and $-\hbar/2$.

For an electron with some arbitrary spin state $|\psi\rangle$, the average can be written as

$$\langle S_z \rangle = p_\uparrow \left(+\frac{\hbar}{2} \right) + p_\downarrow \left(-\frac{\hbar}{2} \right). \tag{3.18}$$

This is called the *expectation value* of S_z. We will now derive the mathematical form of the physical observable S_z from Eq. (3.18).

Using the Born rule for the measurement outcomes of the spin $p_\uparrow = |\langle \uparrow | \psi \rangle|^2$ and $p_\downarrow = |\langle \downarrow | \psi \rangle|^2$ for an electron in spin state $|\psi\rangle$, we can manipulate Eq. (3.18) as follows:

$$\begin{aligned} \langle S_z \rangle &= \frac{\hbar}{2} \left(p_\uparrow - p_\downarrow \right) \\ &= \frac{\hbar}{2} \left(|\langle \uparrow | \psi \rangle|^2 - |\langle \downarrow | \psi \rangle|^2 \right) \\ &= \frac{\hbar}{2} \langle \psi | \uparrow \rangle \langle \uparrow | \psi \rangle - \frac{\hbar}{2} \langle \psi | \downarrow \rangle \langle \downarrow | \psi \rangle . \end{aligned} \tag{3.19}$$

Note that both terms on the right-hand side of the last line have a $\langle \psi |$ on the left and a $|\psi\rangle$ on the right. We can take these out as common factors and write

$$\langle S_z \rangle = \langle \psi | \left(\frac{\hbar}{2} |\uparrow\rangle\langle\uparrow| - \frac{\hbar}{2} |\downarrow\rangle\langle\downarrow| \right) |\psi\rangle = \langle \psi | S_z | \psi \rangle , \tag{3.20}$$

where we defined

$$S_z = \frac{\hbar}{2} |\uparrow\rangle\langle\uparrow| - \frac{\hbar}{2} |\downarrow\rangle\langle\downarrow| . \tag{3.21}$$

You may wonder what the expression for S_z in Eq. (3.21) means. We can figure it out by using the vector notation for $|\uparrow\rangle$ and $\langle\uparrow|$:

$$\begin{aligned} |\uparrow\rangle\langle\uparrow| &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ |\downarrow\rangle\langle\downarrow| &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.22}$$

These "ket-bra" expressions are matrices! By substituting the matrices back into the expression for S_z, this leads to the matrix representation

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.23}$$

Note that $|\uparrow\rangle\langle\uparrow|$ is different from $\langle\uparrow|\uparrow\rangle$. The first is a 2×2 matrix, while the second is a single number (namely 1 in this case). If you are careful about the order of the vectors and remember that $|\cdot\rangle$ is a column vector and $\langle\cdot|$ is a row vector (and complex conjugate), matrix multiplication of the two will automatically give the correct result.

We can also derive the matrix representation for S_x. The expectation value $\langle S_x \rangle$ can be written as

$$\begin{aligned}
\langle S_x \rangle &= \frac{\hbar}{2}(p_+ - p_-) \\
&= \frac{\hbar}{2}\left(|\langle +|\psi\rangle|^2 - |\langle -|\psi\rangle|^2\right) \\
&= \frac{\hbar}{2}\langle\psi|+\rangle\langle +|\psi\rangle - \frac{\hbar}{2}\langle\psi|-\rangle\langle -|\psi\rangle \\
&= \langle\psi|\left(\frac{\hbar}{2}|+\rangle\langle +| - \frac{\hbar}{2}|-\rangle\langle -|\right)|\psi\rangle\,,
\end{aligned} \tag{3.24}$$

and we therefore have

$$S_x = \frac{\hbar}{2}|+\rangle\langle +| - \frac{\hbar}{2}|-\rangle\langle -|\,. \tag{3.25}$$

This has the same form as S_z, but now with $|+\rangle$ and $|-\rangle$ instead of $|\uparrow\rangle$ and $|\downarrow\rangle$. When we use the vector notation of $|+\rangle$ and $|-\rangle$ we obtain

$$\begin{aligned}
|+\rangle\langle +| &= \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}, \\
|-\rangle\langle -| &= \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}.
\end{aligned} \tag{3.26}$$

This leads to the matrix representation of S_x:

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}. \tag{3.27}$$

Following the exact same procedure again with the states $|\circlearrowleft\rangle$ and $|\circlearrowright\rangle$ we find the matrix representation of S_y:

$$S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}. \tag{3.28}$$

The average value of the spin in the y-direction for an electron in the state $|\psi\rangle$ is then given by the expectation value $\langle S_y\rangle = \langle\psi|S_y|\psi\rangle$.

The matrices in S_x, S_y, and S_z are used so much in quantum mechanics that they have their own name. They are the Pauli matrices:

$$\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix},\quad \sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix},\quad \sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}. \tag{3.29}$$

We have now found a remarkable aspect of quantum mechanics: A physical property such as the spin of an electron in the z-direction does not just have a simple value, but must be represented by a matrix, or operator. Operators that represent physical properties are observables. We can determine the value of a physical property only when we combine the state and the observable together. The measurement we perform determines the observable (for example, we choose S_x or S_z by setting the orientation of the magnets in the Stern–Gerlach experiment), while the spin state is determined by the preparation procedure of the electron spin before the measurement (we have not explored this preparation procedure here, but we assume that there is a way to achieve this). We calculate the probabilities of individual measurement outcomes, as well as the average value of physical observables using the state $|\psi\rangle$.

Previously, we encountered operators as a means to describe the transformation of a state. In particular, we have seen in Chap. 2 that the beam splitter is described by an operator that transforms the input state of a photon into the output state. However, the spin operators introduced here are fundamentally different, since they have physical units (for spin it is J s), while transformation matrices—and the quantum states themselves—must be dimensionless. In general, operators such as the beam splitter have mathematical properties that are different from the properties of observables. We will return to this distinction in Chap. 5. For now, it is important that the matrices associated with observables are of a different kind than matrices associated with state transformations.

3.3 The Bloch Sphere

We return now to the question how we can describe the spin of an electron in an arbitrary direction. This leads to the idea of a *state space*, in which each point represents a state. Classically, the spin state of the electron is the three-dimensional vector **S** in Eq. (3.11), so the state space is three-dimensional real space: each point in space uniquely defines a vector as an arrow from the origin to that point, where the direction of the arrow is the rotation axis, and the length of the arrow is the size of the spin.

How does this translate to the quantum mechanical state space? We immediately notice a problem: the state vectors of electron spin are two-dimensional (the column vectors have two entries, corresponding to two distinguishable measurement outcomes), while the classical state space for spin is three-dimensional. However, we have also seen that the state vectors can have complex numbers (which are composed of two real numbers), and it turns out that you can describe three-dimensional real spin vectors in terms of two-dimensional complex state vectors. We will now see how this is done.

We first write $|\uparrow\rangle$ and $|+\rangle$ in vector notation:

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} . \tag{3.30}$$

Notice that the vector $|+\rangle$ is rotated over 45° with respect to the vector $|\uparrow\rangle$. Therefore, a 90° rotation in real space (rotating the spin direction from the z-axis to the x-axis) corresponds to a 45° rotation in the state space of the spin. The angle in state space is half the angle in real space. If we rotate the spin source over an angle θ in real space (around the central axis in the Stern–Gerlach experiment), the spin state will be given by a vector that is rotated over an angle $\theta/2$:

$$|\psi(\theta)\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix} = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)|\downarrow\rangle\,. \tag{3.31}$$

We can check that this is true by rotating the magnets over an angle θ in real space, so that we measure the spin component

$$S_\theta = \cos\theta\, S_z + \sin\theta\, S_x = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{3.32}$$

We should find that the spin in that direction is $+\hbar/2$. We can calculate the expectation value (average) of the rotated spin operator, and if it is $+\hbar/2$ we know that all the spins must contribute $+\hbar/2$ to this average, and no spins contribute $-\hbar/2$ (otherwise the expectation value would be less than $\hbar/2$). Therefore the probability of getting the $+\hbar/2$ measurement result is 1, and the spin state in Eq. (3.31) is indeed the spin along the direction θ.

We calculate $\langle S_\theta\rangle$ using Eqs. (3.31) and (3.32) in vector and matrix form using matrix multiplication:

$$\begin{aligned}
\langle S_\theta\rangle &= \frac{\hbar}{2}\left(\cos\left(\tfrac{\theta}{2}\right)\ \sin\left(\tfrac{\theta}{2}\right)\right)\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix} \\
&= \frac{\hbar}{2}\left(\cos\left(\tfrac{\theta}{2}\right)\ \sin\left(\tfrac{\theta}{2}\right)\right)\begin{pmatrix} \cos\theta\, \cos\left(\frac{\theta}{2}\right) + \sin\theta\, \sin\left(\frac{\theta}{2}\right) \\ \sin\theta\, \cos\left(\frac{\theta}{2}\right) - \cos\theta\, \sin\left(\frac{\theta}{2}\right) \end{pmatrix} \\
&= \frac{\hbar}{2}\left(\cos\left(\tfrac{\theta}{2}\right)\ \sin\left(\tfrac{\theta}{2}\right)\right)\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix} \\
&= \frac{\hbar}{2}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) = \frac{\hbar}{2}\,,
\end{aligned} \tag{3.33}$$

where we used the trigonometric identities

$$\begin{aligned}
\cos\theta\cos\left(\frac{\theta}{2}\right) + \sin\theta\sin\left(\frac{\theta}{2}\right) &= \cos\left(\frac{\theta}{2}\right)\,, \\
\sin\theta\cos\left(\frac{\theta}{2}\right) - \cos\theta\sin\left(\frac{\theta}{2}\right) &= \sin\left(\frac{\theta}{2}\right)\,.
\end{aligned} \tag{3.34}$$

Since we find that $\langle S_\theta\rangle = \hbar/2$, the state $|\psi(\theta)\rangle$ is indeed the spin state in the direction θ, just like $|\uparrow\rangle$ is the spin state in the positive z-direction ($\theta = 0$) and $|+\rangle$ is the spin state in the positive x-direction ($\theta = 90°$). When $\theta = 180°$, we have rotated the spin vector upside-down, and the spin state $|\uparrow\rangle$ has become the orthogonal state $|\downarrow\rangle$.

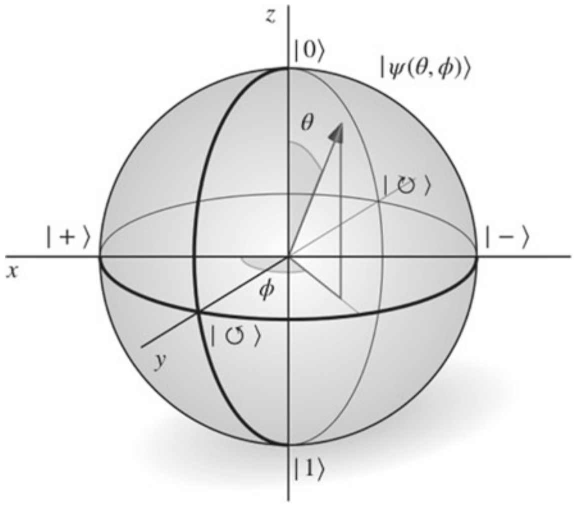

Fig. 3.4 The Bloch sphere

What about the y-direction? Since all possible values of θ, from 0° to 360°, gives us all possible directions in the xz-plane, we need another angle ϕ to measure the rotation around, say, the vertical (z) axis. We choose ϕ such that $\phi = 0$ puts the spin vector in the xz-plane. A convenient way to work out the ϕ-dependence of a spin state is to consider the spin in the positive x- and y-directions in vector notation:

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad |\circlearrowleft\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \tag{3.35}$$

where now the complex numbers come into play. When we trace out the path from $|+\rangle$ to $|\circlearrowleft\rangle$ to $|-\rangle$ to $|\circlearrowright\rangle$, the lower vector component in Eq. (3.35) goes from $+1$ to i to -1 to $-i$, and back to $+1$. In the previous chapter we saw that this is the typical behaviour of a phase factor $e^{i\phi}$. We therefore expect that the spin state in the xy-plane takes the form

$$|\psi(\phi)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\e^{i\phi}\end{pmatrix} = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{e^{i\phi}}{\sqrt{2}}|\downarrow\rangle. \tag{3.36}$$

Indeed, we can calculate the expectation value with respect to $|\psi(\phi)\rangle$ of the spin component

$$S_\phi = \cos\phi\ S_x + \sin\phi\ S_y\,, \tag{3.37}$$

and we find that $\langle S_\phi\rangle = \hbar/2$ for the state in Eq. (3.36). This proves that $|\psi(\phi)\rangle$ is the state of an electron spin in the direction ϕ.

We combine the xz-plane and the xy-plane to obtain a general expression for a spin state

$$|\psi(\theta,\phi)\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle\,. \qquad (3.38)$$

This expression contains two angles, and each value for (θ, ϕ) can therefore be identified with a point on the surface of a sphere. This is the so-called Bloch sphere, shown in Fig. 3.4.

The Bloch sphere is a convenient way of collecting all the possible spin states into a single picture. However, there are two important things to keep in mind:

1. Orthogonal state vectors correspond to vectors at an angle of 180° in the Bloch sphere (antipodal points on the sphere);
2. the identification of the spin direction with the direction in the Bloch sphere is a happy accident.

We need to elaborate on this second point. Every quantum system that has two distinct states, such as the photon states $|\text{upper}\rangle$ and $|\text{lower}\rangle$ inside the arms of a Mach–Zehnder interferometer, can be described by a collection of vectors that fit perfectly on the Bloch sphere. To see this, any normalised complex superposition of $|\text{upper}\rangle$ and $|\text{lower}\rangle$ is a valid quantum state:

$$|\text{photon}\rangle = a|\text{upper}\rangle + b|\text{lower}\rangle\,, \qquad (3.39)$$

with $|a|^2 + |b|^2 = 1$. We can put a phase shift $e^{i\phi}$ in the lower arm of the interferometer after a beam splitter with reflectivity $R = \sin^2(\theta/2)$, so we can have the state

$$|\text{photon}(\theta,\phi)\rangle = \cos\left(\frac{\theta}{2}\right)|\text{upper}\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\text{lower}\rangle\,. \qquad (3.40)$$

Since $|\text{upper}\rangle$ and $|\text{lower}\rangle$ are not states of a vector observable like S_z, but just two possibilities for the photon to go through the interferometer, there is no intrinsic meaning to the angles θ and ϕ in our real three-dimensional space, and the interpretation of the direction of the state vector as the spin direction in space is accidental: It does not hold in general.

In real space, the most general spin direction can be constructed from the spin in the z-direction as a rotation over θ around the y-axis followed by a rotation over ϕ around the z-axis. The classical spin vector in three dimensions is then given by

$$\mathbf{S}(\theta,\phi) = S\begin{pmatrix}\cos\phi\,\sin\theta\\ \sin\phi\,\sin\theta\\ \cos\theta\end{pmatrix}, \qquad (3.41)$$

where S is the classical spin magnitude. We can imagine that the angles θ and ϕ depend on time, and that the state of the electron spin evolves over time, tracing out a path on the Bloch sphere. We will consider this in more detail in the next chapter.

Finally, we make the connection between spin observables and points in the Bloch sphere. We have seen that we can measure the spin component in a particular spatial direction. If the spin vector is pointing in the direction given by Eq. (3.41), then we obtain the spin in the opposite direction by making the substitution $\theta \to \theta + \pi$ and leaving ϕ unchanged. This will take $\mathbf{S}(\theta, \phi)$ to $-\mathbf{S}(\theta, \phi)$. Since these are opposite spin directions, a measurement in a Stern–Gerlach experiment along this direction can perfectly distinguish between these two spins, and the corresponding quantum states should be orthogonal (since perfectly distinguishable states correspond to orthogonal states). We define $|\psi_1\rangle = |\psi(\theta, \phi)\rangle$ and $|\psi_2\rangle = |\psi(-\theta, \phi)\rangle$, and the scalar product $\langle\psi_1|\psi_2\rangle$ should be equal to zero. First, we calculate $\langle\psi_1|$:

$$\langle\psi_1| = \cos\left(\frac{\theta}{2}\right)\langle\uparrow| + e^{-i\phi}\sin\left(\frac{\theta}{2}\right)\langle\downarrow| \, . \tag{3.42}$$

Notice the sign change in $i\phi$ due to the complex conjugate. The state $|\psi_2\rangle$ with $\theta \to \theta + \pi$ becomes

$$|\psi_2\rangle = -\sin\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\phi}\cos\left(\frac{\theta}{2}\right)|\downarrow\rangle \, . \tag{3.43}$$

The scalar product then becomes

$$\langle\psi_1|\psi_2\rangle = -\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) + e^{-i\phi}e^{i\phi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) = 0 \, , \tag{3.44}$$

as expected (we used $e^{-i\phi}e^{i\phi} = e^{-i\phi+i\phi} = e^0 = 1$). This means that antipodal states in the Bloch sphere are orthogonal. Moreover, each axis passing through the origin in the Bloch sphere connects two antipodal points, and therefore each axis through the origin can be seen as a spin component observable because it connects an “up” and “down” state in that particular direction.

Having defined two angles, θ and ϕ, to cover the entire Bloch sphere, we must be careful not to cover the sphere more than once. If we let both θ and ϕ take values between zero and 2π, we end up covering the Bloch sphere twice (you should verify this!). The standard solution is to adjust the domain of θ, so we have

$$0 \leq \theta \leq \pi \qquad \text{and} \qquad 0 \leq \phi < 2\pi \, , \tag{3.45}$$

where we made sure to exclude $\phi = 2\pi$, since it is the same as $\phi = 0$.

Now suppose that we prepare the electron spin in a state described by a vector from the origin to a point on the surface of the Bloch sphere. The spin observable we wish to measure is given by an axis through the origin of the Bloch sphere. The probabilities of the two measurement outcomes are now determined entirely by the *projection of the state vector onto the observable axis*.

As an example, consider the general state

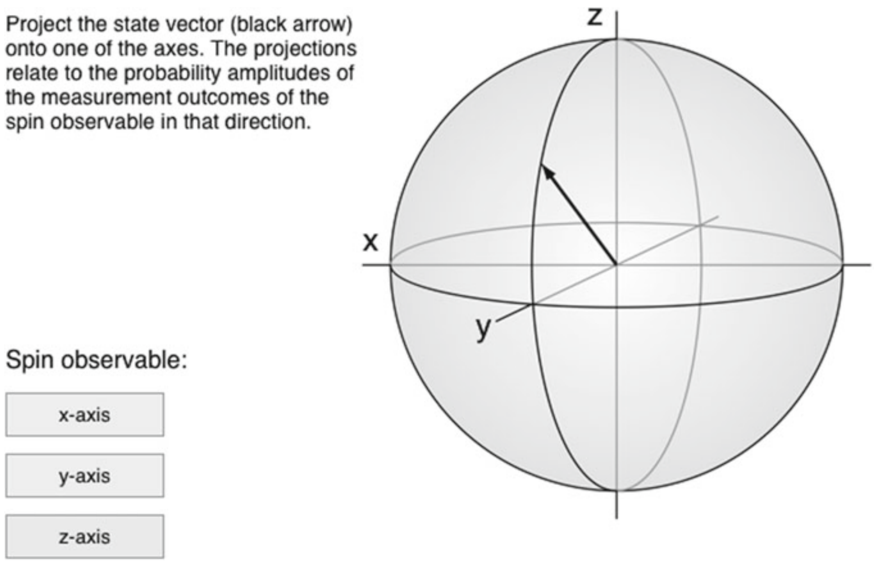

Fig. 3.5 Projections in the Bloch sphere. The interactive figure is available online (see supplementary material 4)

$$|\psi(\theta,\phi)\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle\,, \tag{3.46}$$

and a measurement of the S_z observable. The projection K of the state vector onto the z-axis is given by the cosine of the angle between them (since the length of the state vector is 1). Since this angle is just θ, we have

$$K = \cos\theta\,. \tag{3.47}$$

Compare this with the classical spin value in the z-direction in Eq. (3.41)! At the same time, we can calculate the probabilities for finding measurement outcomes $\uparrow$ and $\downarrow$:

$$p_\uparrow = \cos^2\left(\frac{\theta}{2}\right) \quad \text{and} \quad p_\downarrow = \sin^2\left(\frac{\theta}{2}\right)\,. \tag{3.48}$$

Using the double angle formula

$$\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = \cos\theta\,, \tag{3.49}$$

we relate the outcome probabilities to K via

$$K = p_\uparrow - p_\downarrow\,. \tag{3.50}$$

By using $p_\uparrow + p_\downarrow = 1$ we can derive the values of $p_\uparrow$ and $p_\downarrow$ from K alone:

$$p_\uparrow = \frac{1+K}{2} \quad \text{and} \quad p_\downarrow = \frac{1-K}{2}\,. \tag{3.51}$$

We can repeat this procedure for projections along any other direction as well (there is nothing intrinsically special about the z-direction). A few more examples of projections in the Bloch sphere are shown in Fig. 3.5.

3.4 The Uncertainty Principle

Quantum mechanics is all about calculating probabilities, but there is a bit more to this than just calculating the probabilities of measurement outcomes and the average value of physical observables using the expectation value. Sometimes we are interested in the uncertainty we have about certain measurement outcomes. For example, when we prepare an electron in the spin state $|\uparrow\rangle$ and send it through a Stern–Gerlach apparatus aligned in the z-direction, the probability that we will find a fluorescent dot in the "up" region is $p_\uparrow = 1$. In other words, there is no uncertainty about the spin in the z-direction. Similarly, a measurement along the x-direction will give $p_+ = p_- = 1/2$. This means that we have *maximum* uncertainty because no measurement outcome is more likely than the other. When we measure at a small angle θ from the z-axis we have only a little bit of uncertainty. We want to quantify this.

We see that the uncertainty of the spin in the direction θ is related to the spin observable S_θ and the spin state $|\psi\rangle$ of the electron. Instead of the expectation value of the spin we can calculate how much the measurement outcomes deviate from the mean. We can accomplish this by taking the difference between S_θ and its mean using the operator $S_\theta - \langle S_\theta\rangle\mathbb{I}$. However, if we calculate the expectation value of this new operator we find that it is zero:

$$\langle (S_\theta - \langle S_\theta\rangle\mathbb{I})\rangle = \langle S_\theta\rangle - \langle S_\theta\rangle\langle\psi|\psi\rangle = \langle S_\theta\rangle - \langle S_\theta\rangle = 0\,, \tag{3.52}$$

because the deviation can be both positive and negative, and averages out. We need to force it to be positive, which we can do by taking the square. The uncertainty ΔS_θ can then be defined as

$$\Delta S_\theta = \sqrt{\langle\psi|(S_\theta - \langle S_\theta\rangle)^2|\psi\rangle}\,, \tag{3.53}$$

where we take the square root in order to preserve the units of S_θ. Another way to calculate this is

$$(\Delta S_\theta)^2 = \langle\psi|S_\theta^2|\psi\rangle - \langle\psi|S_\theta|\psi\rangle^2\,. \tag{3.54}$$

The quantity $(\Delta S_\theta)^2$ is called the variance of the operator S_θ given the spin state $|\psi\rangle$, and ΔS_θ is the standard deviation.

Suppose that an electron is created in the spin state $|\uparrow\rangle$. The expectation value of S_z is then

$$\langle S_z\rangle = \langle\uparrow|S_z|\uparrow\rangle = \langle\uparrow|\left(\frac{\hbar}{2}|\uparrow\rangle\right) = \frac{\hbar}{2}\,, \tag{3.55}$$

and using

$$S_z^2 = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{3.56}$$

the expectation value of S_z^2 is calculated as

$$\langle S_z^2\rangle = \langle\uparrow|S_z^2|\uparrow\rangle = \frac{\hbar^2}{4}\,. \tag{3.57}$$

The variance is therefore

$$(\Delta S_z)^2 = \frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\right)^2 = 0\,. \tag{3.58}$$

In other words, there is no uncertainty about the outcome of a spin S_z measurement of an electron in state $|\uparrow\rangle$, as we determined earlier.

However, if we measure the spin of the same electron in the x-direction we obtain entirely different behaviour. The expectation value of S_x is zero:

$$\langle S_x\rangle = \langle\uparrow|S_x|\uparrow\rangle = \frac{\hbar}{2}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = 0\,, \tag{3.59}$$

but the variance of S_x will not be zero. The matrix form of S_x^2 is

$$S_x^2 = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{3.60}$$

We therefore calculate

$$\langle S_x^2\rangle = \langle\uparrow|S_x^2|\uparrow\rangle = \frac{\hbar^2}{4}\,, \tag{3.61}$$

so

$$(\Delta S_x)^2 = \langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}\,. \tag{3.62}$$

There is therefore some uncertainty about the spin components in the x-direction: $\Delta S_x = \hbar/2$. Can you show that $\Delta S_y = \hbar/2$?

We can measure only one component of the electron spin at any given time in a Stern–Gerlach experiment, due to our choice of a particular orientation of the magnets. As the above example shows, this leaves us uncertain about the other spin components of the electron. Classically, measurements reveal the values of the spin components, and we would expect that we can just perform another spin measurement on the same electron, this time in the y-direction. So instead of recording the electron as a fluorescent dot on the screen, we can send it into a second Stern–Gerlach experiment and measure another spin component of our choice. If we choose to measure the spin in the x-direction we find that the probabilities of measuring "+" or "−" are $p_+ = p_- = 1/2$.

Suppose that our second measurement reveals that the x-component of the spin is $+$. Can we say that the spin of the electron before it entered the first Stern–Gerlach experiment had spin $+\hbar/2$ in both the z- and x-direction? If this were true, then a third Stern–Gerlach experiment measuring the z component again should give the value $+\hbar/2$. But this is not what we find experimentally. We find instead that the probabilities of the measurement outcomes $\uparrow$ and $\downarrow$ are $p_\uparrow = p_\downarrow = 1/2$. So we conclude that the measurement of S_x must have *disturbed* the spin state of the electron.

Measuring a particular spin component of an electron *creates* maximum uncertainty of the spin components in the orthogonal directions. Repeating this experiment for another spin component will reduce that uncertainty for the measured component but again create uncertainty in the original spin component.

We cannot measure the three orthogonal spin components of the original spin state with arbitrary precision.

This is a consequence of the fact that we can measure only one of the spin components of **S** at a time. It is also closely related to the example of the QND measurement in Chap. 2, which destroyed the interference in the output beams of a Mach–Zehnder interferometer. We will return to quantum uncertainty in more detail in Chap. 9.

3.5 ❧ Magnetic Resonance Imaging

Electrons are not the only particles with spin. For example, protons also have spin with maximum value $\hbar/2$, and we can describe proton spins in the same way as electron spins. In this section we will see how proton spins can be used in an important practical application called magnetic resonance imaging, which among other things allows doctors to take a look inside the brain of a living patient.

Spin is essentially a magnetic moment. In other words, a particle with spin behaves as a little magnet that wants to align itself along an external magnetic field. If you have ever played with little magnets you are familiar with this effect: try to push two magnets together north pole to north pole, and you feel the magnets push back. If you do not hold the magnets tight, they will slip out of your fingers and flip so they are lined up north pole to south pole. It's a fun game until you get a piece of skin caught between the magnets!

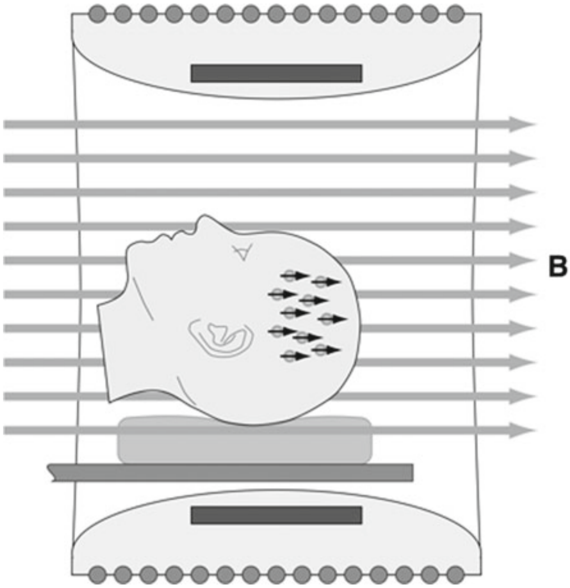

Fig. 3.6 Spins aligned to a magnetic field for magnetic resonance imaging. The interactive figure is available online (see supplementary material 5)

We can describe this in terms of the potential energy of the magnets: the potential energy for the north-north alignment is much higher that the potential energy of the north-south alignment of the two magnets. If the magnets are not held in place firmly they will reconfigure themselves in the state of lowest potential energy, just like a ball that is released at the top of a tower will move towards the point of lowest potential energy, i.e., the ground. The potential energy of a proton with spin in a magnetic field is given by

$$E = -\frac{e}{mc}\mathbf{S} \cdot \mathbf{B}\,, \tag{3.63}$$

where **B** is the vector that indicates the magnetic field (with its magnitude measured in units of Tesla), **S** is the spin vector, e is the charge of the proton, m is the mass of the proton, and c is the speed of light in vacuum. The proton has the highest potential energy when its spin is anti-aligned with the magnetic field (due to the minus sign), and the lowest potential energy when the spin is aligned with the magnetic field. The spin will therefore want to align itself with the magnetic field. Let's suppose that the magnetic field points in the positive z-direction. The proton spin will then relax into the state $|\uparrow\rangle$. Note that $\uparrow$ and $\downarrow$ denote spin directions, and *not* energy. The energy of $|\uparrow\rangle$ is *lower* than that of $|\downarrow\rangle$.

Just like a ball that gets kicked in the air, we can give the proton spin a kick with a small burst of radiation (a photon). If we want to kick the spin into the higher energy state $|\downarrow\rangle$ that is anti-aligned with the magnetic field, the energy of the radiation must overcome the potential energy difference of the two states:

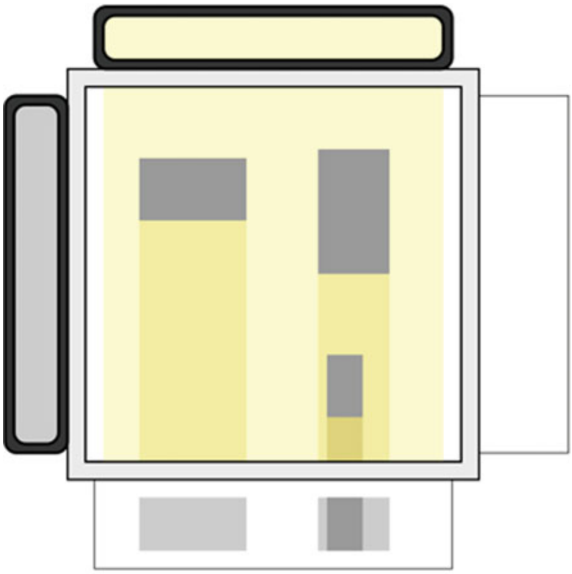

Fig. 3.7 The principle of tomography. The interactive figure is available online (see supplementary material 6)

$$\Delta E = \frac{e\hbar B}{mc} \,. \tag{3.64}$$

Suppose that the magnetic field has a strength of 3.0 T. The frequency f of the photon must then be at least

$$f = \frac{\Delta E}{h} = 46 \text{ MHz}. \tag{3.65}$$

This is a Very High Frequency (VHF) radio wave, similar to that used in old-fashioned analog FM radio and television broadcasting. We call the pulse of radiation that flips the spin of the proton an "RF" pulse since it has Radio Frequency.

Once the proton spin is in a state of higher potential energy $|\downarrow\rangle$ it will not stay there, just like a kicked ball will not remain hovering in the air. The proton will spontaneously emit a photon of frequency f and relax back to the ground state $|\uparrow\rangle$. We can describe the RF pulse as a transformation U_{RF} on the spin of the proton as

$$U_{\text{RF}}|\uparrow\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle \,. \tag{3.66}$$

Similarly, the spontaneous emission transformation can also be described by a transformation U_{RF}, with $U_{\text{RF}}|\downarrow\rangle = |\uparrow\rangle$. This is using the matrix U_{RF} as a transformation (instead of an observable), much like the beam splitter transformation in the previous chapter. If the RF pulse has exactly the right frequency, the pulse is said to be "on resonance" with the transition $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$. Since a proton is the nucleus of a hydrogen atom, we call the above process Nuclear Magnetic Resonance, or NMR for short. It is used in Magnetic Resonance Imaging (MRI) as shown in Fig. 3.6.

Let's consider how MRI works for a brain scan. The brain consists of different tissues that have a varying amount of hydrogen. In an NMR process, regions with a large amount of hydrogen will return a brighter RF pulse than regions with lower amounts of hydrogen. The MRI scanner measures how much RF radiation is coming from a certain direction, but it cannot measure the depth of the source in the brain. To find this out, MRI scanners use a mathematical technique called "tomography". By looking at the RF radiation coming from many different directions we can work out in great detail where the bright and dark spots are.

Tomography is like solving a puzzle. For a very simple example, consider the interactive tomogram in Fig. 3.7. We can send in light from two perpendicular directions onto a two-dimensional shape, which casts two different shadows. The puzzle is to deduce the 2D shape from the shadows. We record the shadows, and use them as input in a computer programme, which deduces the image from the shadows. The image extraction in MRI works in a similar way.

Exercises

1. The state of an electron spin is given by

$$|\psi\rangle = \frac{1}{\sqrt{3}}|\uparrow\rangle + \sqrt{\frac{2}{3}}|\downarrow\rangle\,.$$

   What is the probability of finding spin $\downarrow$ in a measurement outcome? What is the probability of finding measurement outcome "+", with $|+\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$?
2. Normalise the state $2|\uparrow\rangle + 4|\downarrow\rangle$.
3. An electron is prepared in the spin state $2|\uparrow\rangle - 3i|\downarrow\rangle$. Normalise this state and calculate the probability of finding spin "up" and spin "+", corresponding to $|\uparrow\rangle$ and $|+\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$, respectively. What is the expectation value of the z-component of the spin?
4. Construct the matrix form of S_y similar to Eq. (3.23).
5. Give the matrix representation of the observable $S_\phi = \cos\phi\; S_x + \sin\phi\; S_y$. Calculate the expectation value of S_ϕ given the spin state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|\uparrow\rangle + \frac{\sqrt{2}\,e^{i\pi/6}}{\sqrt{3}}|\downarrow\rangle\,,$$

   with $|\uparrow\rangle$ and $|\downarrow\rangle$ the spin up and down states in the z-direction. What is the uncertainty ΔS_ϕ?

6. For the state in Exercise 5, calculate the probabilities of finding spin "up" and spin "down" in the z-direction.
7. Determine the angles θ and ϕ for which the state in Exercise 5 is the quantum state associated with the "up" direction.
8. Calculate the uncertainty ΔS_θ for the operator $S_\theta = \cos\theta\ S_z + \sin\theta\ S_x$ given the spin state $|\uparrow\rangle$. Does your result conform to your expectation?
9. In Eq. (3.45) we have covered the Bloch sphere using two angles. However, we have covered two special points on the sphere too many times. Can you tell which points? Does it matter in this case? Hint: remember that a global phase is unobservable.
10. Construct a matrix form of the observable associated with the path in a Mach–Zehnder interferometer where we find the photon. You will need to take special care in choosing the measurement values.
11. We prepare an electron spin state in the direction (θ, ϕ), which can be written as

$$\frac{\hbar}{2}\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0.433 \\ 0.750 \\ 0.500 \end{pmatrix}.$$

    What is the quantum state of the electron?
12. For the electron spin state of the previous question, relate the probabilities of finding outcomes $+$ and $-$ in the x-direction to the projection onto the x-axis.
13. An electron with spin state

$$|\psi\rangle = \frac{3}{5}|\uparrow\rangle + \frac{4}{5}|\downarrow\rangle$$

    has its spin measured in the x-direction. What is the expectation value $\langle S_x\rangle$? What is the uncertainty ΔS_x?
14. An electron with spin state

$$|\psi\rangle = \frac{3}{5}|\uparrow\rangle + \frac{4}{5}|\downarrow\rangle$$

    has its spin measured in the y-direction. What is the expectation value $\langle S_y\rangle$? What is the uncertainty ΔS_y?
15. An electron with spin state

$$|\psi\rangle = \frac{3}{5}|\uparrow\rangle + \frac{4}{5}|\downarrow\rangle$$

    has its spin measured in the z-direction. What is the expectation value $\langle S_z\rangle$? What is the uncertainty ΔS_z?

16. The state of an electron spin is given by

$$|\psi\rangle = \frac{2}{\sqrt{13}}|\uparrow\rangle + \frac{3i}{\sqrt{13}}|\downarrow\rangle \, .$$

Calculate $\langle S_\theta \rangle$ and ΔS_θ.

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