Chapter 2
Photons and Interference

In this chapter, we will give the mathematical description of the previous chapter's experiments. This will show us how we can calculate the probabilities of measurement outcomes in quantum mechanics. We need to introduce simple matrices and basic complex numbers.

2.1 Photon Paths and Superpositions

In physics, we often refer to the *state* of a physical system as the collection of all its physical properties. In quantum mechanics, we have to take a little more care when talking about properties, because as we have seen, they are not independent of the measurements or "observations" of the system. Nevertheless, we do refer to the state of a quantum system as the description of the system that allows us to determine its physical behaviour. We give an example of classical and quantum states in Fig. 2.1.

To explore the quantum state of a photon, consider the second experiment of the previous chapter, where the photon enters the beam splitter from the left. The other way in which a photon can enter the beam splitter and trigger the detectors is from the top. We say that the photon has two distinct quantum states, indicated by the labels "left" or "top". We write this as

$$|\text{left}\rangle \qquad \text{or} \qquad |\text{top}\rangle\,. \tag{2.1}$$

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Fig. 2.1 The state of a system. The interactive figure is available online (see supplementary material 1)

The symbol "$|\cdot\rangle$" is called a "ket", and is often used to indicate a quantum state. This is called "Dirac notation", after one of the founders of quantum mechanics, Paul Dirac (1988). The label inside the ket may not have a mathematical meaning, and is mainly an indicator as to which physical state the ket refers to.

We must distinguish the state of the photon before the beam splitter from the state after the beam splitter, because the beam splitter will change the state of the photon. After the beam splitter, there are again two paths that the photon can take, namely "right" and "down". We also associate quantum states with the photon in these paths, denoted by

$$|\text{right}\rangle \quad \text{or} \quad |\text{down}\rangle\,. \tag{2.2}$$

To distinguish clearly the state of the photon after the beam splitter (let's say at time t_2) from the state of the photon before the beam splitter (at time $t_1 < t_2$) we can add an extra label "2" as a subscript to the ket. We do not have to do this, but it is a convenient reminder for now.

A photon entering the beam splitter from the left in state $|\text{left}\rangle_1$ will typically not be in the exact state $|\text{right}\rangle_2$ or $|\text{down}\rangle_2$ after the beam splitter. To see why, consider again the experiment with a single beam splitter. If the photon was always in the state $|\text{right}\rangle_2$, we would expect detector D_1 to fire every time. Similarly, if the photon was always in the state $|\text{down}\rangle_2$, we would expect detector D_2 to fire every time. However, we saw that the photons are detected randomly in detector D_1 and D_2, so the state of the photon must be some combination of $|\text{right}\rangle_2$ and $|\text{down}\rangle_2$.

Next, we note that the third experiment of the previous chapter (the Mach–Zehnder interferometer) gives rise to interference of the laser light, a phenomenon that is explained by the wave nature of light. We obtain interference by taking a *superposition* of two waves (see Fig. 2.2). A similar effect was observed using the photon, which always triggered detector D_2, and never D_1. It is therefore natural to suppose that the states $|\text{right}\rangle_2$ and $|\text{down}\rangle_2$ can be combined in a superposition. If we denote the state of the photon after the beam splitter by $|\text{photon}\rangle_2$, the superposition becomes

$$|\text{photon}\rangle_2 = a|\text{right}\rangle_2 + b|\text{down}\rangle_2\,, \tag{2.3}$$

where a and b are numbers whose meaning we will return to later. The photon is neither in state $|\text{right}\rangle_2$, nor in the state $|\text{down}\rangle_2$, but in an entirely different state, namely $|\text{photon}\rangle_2$.

Fig. 2.2 Interference of water waves

We could continue to develop the theory in terms of kets, but at this point we note a great mathematical convenience. The state of the photon behaves as a *vector*. We can add vectors to make another vector, and we can identify two special vectors that correspond to the distinct states $|\text{right}\rangle_2$ and $|\text{down}\rangle_2$. These vectors point in perpendicular directions, and they are called *orthogonal* vectors:

$$|\text{right}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\text{down}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{2.4}$$

Consequently, the state of the photon after a beam splitter can be written as

$$|\text{photon}\rangle = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} . \tag{2.5}$$

Equation (2.5) is equivalent to Eq. (2.3) and can be represented graphically as in Fig. 2.3. Note that the states $|\text{right}\rangle$ and $|\text{down}\rangle$ are orthogonal vectors of length 1. This means that any conceivable superposition of the photon paths can be decomposed into these two vectors.

It is now the *direction* of the vector (with components a and b) that determines the superposition, and we assume that the length of the vector remains unchanged. This places the restriction $a^2 + b^2 = 1$ on the state $|\text{photon}\rangle$. In the special case of the beam splitters in the previous chapter we have $a = b = 1/\sqrt{2}$, and the state of the photon is

$$|\text{photon}\rangle = \frac{1}{\sqrt{2}}|\text{right}\rangle + \frac{1}{\sqrt{2}}|\text{down}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.6}$$

after the beam splitter.

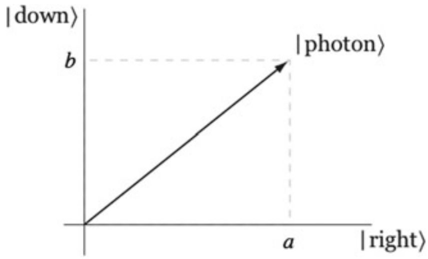


Fig. 2.3 The state of the photon as a vector

If the state of the photon after a beam splitter can be in a superposition, then it makes sense that the state of the photon before the beam splitter can also be written as an arbitrary superposition of $|\text{left}\rangle_1$ and $|\text{top}\rangle_1$. Indeed, when we look at the second beam splitter of the Mach–Zehnder interferometer this is exactly what happens, since the input of the beam splitter is in fact the output of a previous beam splitter. We are once more talking about a photon that can be in a superposition of two distinct states, and the mathematical description can again be given in terms of vectors:

$$|\text{left}\rangle_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\text{top}\rangle_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{2.7}$$

The label "1" on the ket determines again the stage in the experiment before the beam splitter. We could have added the same label to the vectors, but this will quickly become cumbersome and after a bit of practice it will be clear to which stage of the experiment a vector refers. If in doubt, you can always add the labels. The vectors $|\text{left}\rangle$ and $|\text{top}\rangle$ are called *basis vectors*, since they can be used to decompose any vector in a two-dimensional space.

An important point about these vectors is that they are not vectors in the space that you and I occupy. Rather, they live in an abstract "space of possibilities" for the photon, or "state space". The more possibilities there are for the photon (such as many paths), the more dimensions the space will have, because each distinct path has its own orthogonal vector. For now, we limit ourselves to two-dimensional spaces.

2.2 *Mathematical Intermezzo*: Matrix Multiplication

Quantum mechanics is formulated in terms of matrices, with which you may not yet be familiar. Luckily, we will only need a few basic rules that are easy to explain. First, a matrix is a rectangular array of numbers, for example

$$\begin{pmatrix} 8 & 3 & 2 \\ 2 & 5 & 7 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 12 & 0 \\ 1 & 13 \end{pmatrix} . \tag{2.8}$$

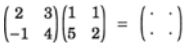


Fig. 2.4 Matrix multiplication. The interactive figure is available online (see supplementary material 2)

As you can see, a matrix has rows and columns. Sometimes a matrix has only one column or one row. Those matrices we refer to as vectors. For example:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & 4 & 3 \end{pmatrix}. \tag{2.9}$$

To understand the rest of this book, you need to know how to multiply two matrices. It is a very simple rule, based on the scalar product (also known as inner product, or dot product). Suppose that we have two vectors, $\mathbf{a} = (2, 4)$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. The scalar product is

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 2 \times 1 + 4 \times 3 = 14\,. \tag{2.10}$$

We have written the scalar product in matrix notation, with $\mathbf{a}$ a 1×2 row vector and $\mathbf{b}$ a 2×1 column vector. Note that the length of the row of $\mathbf{a}$ must be equal to the length of the column of $\mathbf{b}$ for the scalar product to work. The outcome of the salar product is a "matrix" with one row and one column, more commonly known as a number.

Matrix multiplication works in a very similar way to the scalar product (see Fig. 2.4). Let us try to multiply the two matrices in Eq. (2.8). There are two possibilities:

$$\begin{pmatrix} 8 & 3 & 2 \\ 2 & 5 & 7 \end{pmatrix} \begin{pmatrix} 12 & 0 \\ 1 & 13 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 12 & 0 \\ 1 & 13 \end{pmatrix} \begin{pmatrix} 8 & 3 & 2 \\ 2 & 5 & 7 \end{pmatrix}. \tag{2.11}$$

The length of the rows of the first matrix must equal the length of the columns of the second, and therefore only the matrix multiplication on the right makes sense. To calculate the product of these matrices we calculate the scalar product of each row of the left matrix with each column of the right matrix. The position of the scalar product in the resulting matrix is given by the row position of the first matrix and the column position of the second. Hence

$$\begin{array}{cc} & \begin{pmatrix} 8 & 3 & 2 \\ 2 & 5 & 7 \end{pmatrix} \\ \begin{pmatrix} 12 & 0 \\ 1 & 13 \end{pmatrix} & \begin{matrix} & \vdots & \\ \cdot\cdot & \mathbf{68} & \end{matrix} \end{array} = \begin{pmatrix} 96 & 36 & 24 \\ 34 & \mathbf{68} & 93 \end{pmatrix}. \tag{2.12}$$

The placement of the numbers is shown in detail in Fig. 2.4.

2.3 The Beam Splitter as a Matrix

We have identified the state of the photon before the beam splitter with a vector of length 1 in a two-dimensional plane spanned by the basis vectors $|\text{left}\rangle_1$ and $|\text{top}\rangle_1$, and the state of the photon after the beam splitter with a vector spanned by the basis vectors $|\text{right}\rangle_2$ and $|\text{down}\rangle_2$. The next question is how we can describe the action of the beam splitter.

Clearly, the beam splitter should turn the state of the photon before the beam splitter into the state of the photon after the beam splitter. This is called a *transformation*, and we can write it symbolically as

$$U_{\text{BS}}|\text{in}\rangle_1 = |\text{out}\rangle_2\,, \tag{2.13}$$

where we expressed the output state after the beam splitter $|\text{out}\rangle_2$ in terms of the input state before the beam splitter $|\text{in}\rangle_1$, and an *operator* U_{BS} that denotes the action of the beam splitter. For example, we can have

$$|\text{in}\rangle_1 = |\text{left}\rangle_1 \qquad \text{or} \qquad |\text{in}\rangle_1 = \frac{1}{\sqrt{2}}|\text{left}\rangle_1 + \frac{1}{\sqrt{2}}|\text{top}\rangle_1\,, \tag{2.14}$$

or any other combination of $|\text{left}\rangle_1$ and $|\text{top}\rangle_1$. The operator U_{BS} acts on the state (vector) that is written to its right.

We can't do much yet with Eq. (2.13) as it is written, but we can use the vector language developed previously to turn Eq. (2.13) into a form that we can do calculations with. The beam splitter operator U_{BS} turns one vector into another vector, and it should therefore be described by a matrix. Since the row length of U_{BS} should equal the column length of the vector it acts upon (from the matrix multiplication rule), the matrix representing U_{BS} must have row length 2 (i.e., two columns). Second, after matrix multiplication we want again a state vector of dimension 2, so the matrix representing U_{BS} have two rows. Therefore, U_{BS} must be a 2×2 matrix. If you have trouble following this reasoning, you should study Fig. 2.4 again. We can use the behaviour of the photon in the various experiments to figure out the numerical values of the matrix elements.

First, consider again the example of the beam splitter of experiment 2 in the previous chapter. Using Eqs. (2.6) and (2.7) in vector notation, we have

$$U_{\text{BS}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{2.15}$$

and we can write the 2×2 matrix form of U_{BS} as

$$U_{BS} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} , \tag{2.16}$$

where u_{11}, u_{12}, u_{21}, and u_{22} are the numbers we need to determine. We use matrix multiplication to find the first column of the matrix U_{BS}. Equation (2.15) becomes

$$U_{BS}|\text{left}\rangle_1 = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2.17}$$

From this, we deduce that

$$u_{11} = u_{21} = \frac{1}{\sqrt{2}} . \tag{2.18}$$

This gives us the first column of the matrix U_{BS}.

How do we obtain the second column of the matrix U_{BS}? For this we must know how U_{BS} acts on the state

$$|\text{top}\rangle_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{2.19}$$

We have not considered this situation before, and we should return momentarily to a beam splitter with a variable reflectivity. Suppose that the beam splitter transmits every photon. Then $|\text{left}\rangle_1$ will turn into $|\text{right}\rangle_2$ and $|\text{top}\rangle_1$ will turn into $|\text{bottom}\rangle_2$. In our vector notation it is clear that the orthogonal input vectors turn into orthogonal output vectors. The same is true when the beam splitter is completely reflective and $|\text{left}\rangle_1$ will turn into $|\text{bottom}\rangle_2$ and $|\text{top}\rangle_1$ will turn into $|\text{right}\rangle_2$. We therefore suppose that *the beam splitter transforms orthogonal input states into orthogonal output states*.

In the case of the beam splitter of experiment 2, the state $|\text{top}\rangle_1$ will be transformed into a state orthogonal to the right-hand side of Eq. (2.15). If we determine

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{perpendicular to} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.20}$$

then we obtain

$$U_{BS} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \tag{2.21}$$

The elements u_{12} and u_{22} can then be found in the same way as Eq. (2.17), and the matrix U_{BS} is determined to be

$$U_{BS} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} , \tag{2.22}$$

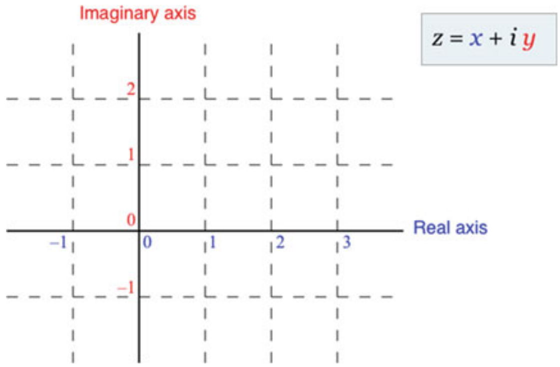


Fig. 2.5 The plane of complex numbers. The interactive figure is available online (see supplementary material 3)

and this is the full mathematical description of a beam splitter with a reflectivity of 50% operating on photons that can enter from the left or the top. To obtain Eq. (2.22) we have assumed that orthogonal vectors are turned into different orthogonal vectors, and we need to test wether this is a correct assumption via carefully designed experiments. In the next sections we will see that our choice indeed predicts the right experimental outcomes in the Mach–Zehnder interferometer.

Next, we can apply the operation U_{BS} not just to the input states $|\mathrm{left}\rangle_1$ and $|\mathrm{top}\rangle_1$, but also to any superposition of these basis states. For an arbitrary input state $|\mathrm{in}\rangle = a|\mathrm{left}\rangle + b|\mathrm{top}\rangle$ we calculate the output state $|\mathrm{out}\rangle$ as

$$|\mathrm{out}\rangle = U_{\mathrm{BS}}|\mathrm{in}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} a+b \\ a-b \end{pmatrix}. \tag{2.23}$$

This is a general property of matrix operators: they apply to any input state vector.

2.4 *Mathematical Intermezzo*: Complex Numbers

Quantum mechanics uses complex numbers, so it is important that you are familiar with the basics (that's all we need for now). The idea of a complex number is that, contrary to what you may have learned previously, $\sqrt{-1}$ is not nonsense. However, it is not a normal number either. We can call $\sqrt{-1} = i$, and treat it as a new kind of number, called an imaginary number. All the normal rules of arithmetic apply, so you can multiply i with any other number, or add any number to it. The question is then: what does $z = 4i + 3$ mean? Since the imaginary part ($4i$) is a different kind of number than 3, we can put the two kinds of numbers along perpendicular axes: The real numbers are plotted along the horizontal axis, and the imaginary numbers are plotted along the vertical axis. The combination of a real and an imaginary number is called a *complex number*, and can be represented by a point in a two-dimensional complex plane, as shown in Fig. 2.5.

When we add two complex numbers, we add the real part and the imaginary parts separately:

$$(x + iy) + (u + iv) = (x + u) + i(y + v)\,, \tag{2.24}$$

where x, y, u, and v are real numbers. Notice that the real and imaginary parts really do add separately, like they are the horizontal and vertical components of a two-dimensional space (a plane). Do not confuse this space with the two-dimensional state space for the quantum states $|\text{photon}\rangle$, though! These are completely different things.

Multiplying two complex numbers is again the same as multiplying two normal numbers, where we use $i \times i = -1$. So we have

$$\begin{aligned}(x + iy)(u + iv) &= xu + ixv + iyu - yv \\ &= (xu - yv) + i(xv + yu)\,. \end{aligned} \tag{2.25}$$

There is again a real part $xu - yv$ and an imaginary part $xv + yu$ to the resulting complex number. An important procedure is to take the *complex conjugate* of a complex number a, denoted by a^* :

$$(x + iy)^* = x - iy\,. \tag{2.26}$$

In other words, complex conjugation will change an i into a $-i$ and vice versa, wherever it is in your formula. The product of a complex number $z = x + iy$ with its complex conjugate $z^* = x - iy$ is always a positive real number $|z|^2$:

$$|z|^2 = (x + iy)^*(x + iy) = (x - iy)(x + iy) = x^2 + y^2\,. \tag{2.27}$$

This is the length of the vector from the origin $(0, 0)$ to the point $z = (x, y)$ in the complex plane, and that is why the complex conjugate will show up again and again. If you expect a real number from your calculation (for example a probability), and instead you get a complex number, chances are you have forgotten to take the complex conjugate somewhere along the way.

Numbers in the complex plane can also be represented in polar coordinates, which is particularly convenient for describing phases (which we will need in quantum mechanics). Imagine a line from the point in the complex plane to the origin. The length of that line is called r (we just saw that $r = \sqrt{|z|^2}$) and the angle of that line with the positive horizontal axis is ϕ. Every point in the complex plane can then be written as

$$z = re^{i\phi} = r\cos\phi + ir\sin\phi\,. \tag{2.28}$$

From this, you can show that we can write the sine and the cosine as

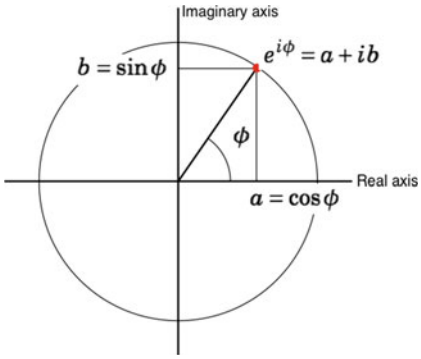


Fig. 2.6 The polar decomposition of complex numbers with radius $r = 1$. The interactive figure is available online (see supplementary material 4)

$$\sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i} \quad \text{and} \quad \cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}\,. \tag{2.29}$$

The argument ϕ is called the phase (since it returns the number to the starting position after it has increased by 2π), and r is called the magnitude. You can prove yourself that $r = \sqrt{|z|^2} = |z|$ and the product of two complex numbers becomes

$$z_1 \times z_2 = r_1\, e^{i\phi_1} \times r_2\, e^{i\phi_2} = r_1 r_2\, e^{i(\phi_1+\phi_2)}\,. \tag{2.30}$$

The polar decomposition of complex numbers is shown in Fig. 2.6.

2.5 The Phase in an Interferometer

We have to rectify an oversimplification. You may have assumed that the numbers a and b in Eq. (2.3) are real numbers. However, in quantum mechanics we must allow a and b to be complex numbers. Why do we need complex numbers? To answer this, we have to look at Eq. (2.21). We constructed the output state of the beam splitter when the photon enters from the top by requiring that it is orthogonal to the output state when the photon entered from the left. But what does the minus sign mean? We know that in experiment 3 we obtain interference in the Mach–Zehnder interferometer, which prevents one detector from registering photons. The minus sign in Eq. (2.21) is the origin of the opposite phase of the two waves entering detector D_1 (and leaving it dark due to destructive interference).

Let us perform a short calculation to show how the Mach–Zehnder interferometer and the interference works with our vectors. The photon enters the Mach–Zehnder from the left in the first beam splitter:

$$|\text{in}\rangle_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,. \tag{2.31}$$

We apply the first beam splitter operator to this state in order to obtain the state of the photon inside the interferometer:

$$|\text{inside}\rangle_2 = U_{\text{BS}}|\text{in}\rangle_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \tag{2.32}$$

followed by the second beam splitter

$$\begin{aligned}|\text{out}\rangle_3 = U_{\text{BS}}|\text{inside}\rangle_2 &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}1+1\\1-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = |D_2\rangle_3,\end{aligned} \tag{2.33}$$

where $|D_2\rangle_3$ is the state of the photon entering detector D_2 at some time t_3 after the second beam splitter. This is exactly what we wanted to achieve: a photon entering the interferometer from the left will not trigger detector D_1 because each time it will exit the beam splitter on the side that triggers D_2.

We can do the calculation in Eq. (2.33) slightly differently: we can first calculate the product of the two beam splitters:

$$U_{\text{BS}}U_{\text{BS}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbb{I}\,. \tag{2.34}$$

The matrix $\mathbb{I}$ is very special, because it always transforms a vector into itself. We call it the *identity*. For our calculation, it means that the expression

$$|\text{out}\rangle_3 = U_{\text{BS}}U_{\text{BS}}|\text{in}\rangle_1 \tag{2.35}$$

will take the form

$$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}. \tag{2.36}$$

Physically, we obtain the interference by carefully tuning the lengths of the two paths between the beam splitters. For classical light (a wave) we must make the path lengths equal in order to send all the light into detector D_2.

If we want all the light to end up in detector D_1 instead, the path lengths must differ by half a wavelength. As we saw in experiment 3, the photons follow the same pattern: they go where the classical signal goes, only probabilistically. How do we describe this change in path length mathematically?

First, applying a path length difference is again an operator, and in matrix notation it should be placed between the two beam splitter matrices. Second, it must not mix the two paths, because photons cannot jump from one path inside the interferometer to the other. This means that the matrix must be diagonal: only the matrix elements along the diagonal line from the top left to the bottom right of the matrix can be nonzero. It turns out that changing the relative path length by half a wavelength takes the following form:

$$U_{\lambda/2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.37}$$

where $\lambda/2$ denotes half a wavelength. Putting this operator together with the beam splitter operators, we use matrix multiplication to describe the successive operations of beam splitter, phase shift, and beam splitter again:

$$\begin{aligned} U_{\mathrm{BS}} U_{\lambda/2}\, U_{\mathrm{BS}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{2.38}$$

Now when we calculate the state of the photon leaving the interferometer $|\mathrm{out}\rangle_4 = U_{BS} U_{\lambda/2}\, U_{BS} |\mathrm{left}\rangle_1$ at time t_4, we find that

$$|\mathrm{out}\rangle_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |D_1\rangle_4 \,. \tag{2.39}$$

In other words, the photon now triggers detector D_1. This is exactly what we expect from classical physics when we change the path length of one arm by half a wavelength. By changing the phase of the wave in one path relative to the other (i.e., adding half a wavelength in one path), we switch the output from $|D_2\rangle$ to $|D_1\rangle$.

Next, consider how the operator $U_{\lambda/2}$ affects the state in Eq. (2.6):

$$U_{\lambda/2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{2.40}$$

The mysterious minus sign just appeared in the state vector! So clearly, the sign has something to do with the relative phase of the two photon states inside the interferometer. The state

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} |\mathrm{right}\rangle_3 + \frac{1}{\sqrt{2}} |\mathrm{down}\rangle_3 \tag{2.41}$$

is a superposition of two photon paths with the same phase, while

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} |\mathrm{right}\rangle_3 - \frac{1}{\sqrt{2}} |\mathrm{down}\rangle_3 \tag{2.42}$$

is a superposition of two photon paths with a relative phase difference of half a wavelength.

The path length is of course something that we can vary continuously, and this raises the question how we can include an arbitrary phase difference ϕ in the interferometer. As a first guess, we could say

$$U_\phi = \begin{pmatrix} 1 & 0 \\ 0 & \phi \end{pmatrix} ? \tag{2.43}$$

where ϕ varies continuously between $+1$ and -1. This will not work, however, because an arbitrary $\phi < 1$ will shorten the state vector, and we said that the state vector must have length 1 (the importance of this will become clear in the next section). Also, a phase must take values between 0 and 2π, rather than between $+1$ and -1.

What will work is making the phase *complex*:

$$U_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} . \tag{2.44}$$

This will change an arbitrary photon state as follows:

$$U_\phi \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ be^{i\phi} \end{pmatrix} , \tag{2.45}$$

which means that in general the numbers a and b must allow complex values. It also means that the unit length condition $a^2 + b^2 = 1$ must now be replaced by $|a|^2 + |b|^2 = 1$, where $|a|^2 = a^*a$ and $|b|^2 = b^*b$. We call a and b the *amplitudes* of the photon states $|\text{right}\rangle$ and $|\text{down}\rangle$. We can combine the two beam splitters and the relative phase shift into one overall transformation of the Mach–Zehnder interferometer U_{MZ} on the state of a photon entering it. Since the interferometer is nothing more than a beam splitter, followed by a phase shift, followed by the second beam splitter, we write

$$\begin{aligned} U_{\text{MZ}} = U_{\text{BS}}\, U_\phi\, U_{\text{BS}} &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 + e^{i\phi} & 1 - e^{i\phi} \\ 1 - e^{i\phi} & 1 + e^{i\phi} \end{pmatrix} . \end{aligned} \tag{2.46}$$

Note that the first beam splitter the photon encounters in the Mach–Zehnder interferometer is represented by the matrix on the far right in the second line of equation (2.46), since that matrix acts on the state vector first. When we apply U_{MZ} to the input state $|\text{left}\rangle_1$, the result is

$$|\text{out}\rangle = U_{\text{MZ}} |\text{left}\rangle_1 = \frac{1}{2} \begin{pmatrix} 1 + e^{i\phi} \\ 1 - e^{i\phi} \end{pmatrix} . \tag{2.47}$$

Clearly, when $\phi = 0$ the photon exits the interferometer into detector D_2, while for $\phi = \pi$ the photon ends up in detector D_1, as required.

2.6 *Mathematical Intermezzo*: Probabilities

In quantum mechanics, the basis of everything we can say about nature is our ability to calculate probabilities of measurement outcomes. We therefore need to introduce a little bit of probability theory. Luckily, we do not require anything more than a way to calculate averages at this point.

A probability is the likelihood of a certain event happening. If the probability of an event is zero, we know for a fact that it will not happen, while a probability of one means that it certainly will happen. So both 0 and 1 are expressions of certainty. Suppose that we have a coin with two sides, denoted "heads" and "tails". We can give a symbol p_h to the probability that we find heads in a coin toss, and we give the symbol p_t to the probability that we find tails. If the coin is sufficiently thin that it never lands on its side, we know that we are going to find either heads or tails. Therefore $p_\mathrm{h} + p_\mathrm{t} = 1$. If the coin is fair we expect no difference between heads and tails, so

$$p_\mathrm{h} = p_\mathrm{t} = \frac{1}{2}\,. \tag{2.48}$$

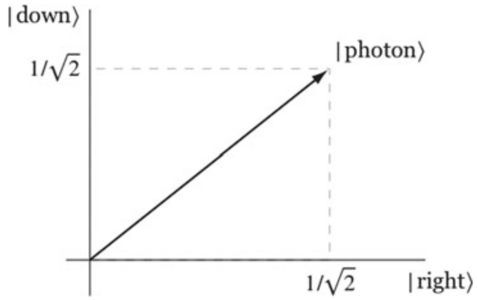
This is the definition of a fair coin. We measure the probabilities by making a very large number of coin tosses N, and the relative frequencies then approach the probabilities p_h and p_t:

$$p_\mathrm{h} = \lim_{N\to\infty} \frac{N_\mathrm{h}}{N} \qquad \text{and} \qquad p_\mathrm{t} = \lim_{N\to\infty} \frac{N_\mathrm{t}}{N}\,. \tag{2.49}$$

Note that the probabilities of every possible way in which a situation can play out must always sum to one, because you know with certainty that whatever happens, something happens. Even if that something is "nothing happens". For example, we may calculate (based on the way we throw and the dimensions of the coin) that $p_\mathrm{h} = 0.47$ and $p_\mathrm{t} = 0.48$. Then either we made a mistake in our calculation, or we have not taken into account all possibilities. For example, the probability for the coin to land on its side may be $p_\mathrm{s} = 0.05$ (or we decide not to toss the coin at all with probability $p_\mathrm{n} = 0.05$, or with probability $p_\mathrm{r} = 0.05$ the coin fell behind the radiator and could be retrieved but not read out; the possibilities are endless). Therefore, probabilities are always numbers between 0 and 1 without any units, and the probabilities of all possible outcomes must sum to 1.

Next, we can use probabilities to calculate averages. Suppose that we use the coin we described earlier to determine how much money you will win: If we throw "heads" you receive £50, and if you throw "tails" you pay £10. In the unlikely event that the coin lands on its side, you win nothing. What is the average amount of money W that you win or lose in this game? To determine this, we multiply each amount of money you can win with the probability of the corresponding outcome of the coin toss:

Fig. 2.7 Probability amplitudes



$$W = p_h \times £50 + p_t \times (-£10) + p_{side} \times 0$$
$$= 0.47 \times £50 - 0.48 \times £10 + 0.05 \times 0 = £18.70\,. \qquad (2.50)$$

So on average you will win money, and this is a game worth playing! (The real question is: who will play this game with you?) In the next section we will determine how we can calculate probabilities in quantum mechanics.

2.7 How to Calculate Probabilities

Quantum mechanics allows us to calculate the probabilities of various measurement outcomes in an experiment. In other words, we can calculate the probability of finding a photon in detector D_1 or D_2 of the Mach–Zehnder interferometer for any value of the phase difference ϕ due to a difference in the two path lengths between the beam splitters. This may seem a calculation of only modest interest, but it will turn out to underpin a very important experiment, detailed in the next section. We will first determine what the probabilities should be in a few special cases, and then extract from this the general formula.

Consider again the beam splitter with a reflectivity of one half and a photon entering from the left. According to Eq. (2.6), the state of the photon after the beam splitter can be written as

$$|\text{photon}\rangle = \frac{1}{\sqrt{2}}|\text{right}\rangle + \frac{1}{\sqrt{2}}|\text{down}\rangle\,. \qquad (2.51)$$

Furthermore, we know from the previous chapter that the probability of finding the photon in the right output beam is $p_{\text{right}} = 1/2$ and the probability of finding the photon in the down output beam is $p_{\text{down}} = 1/2$, obeying the rule that $p_{\text{right}} + p_{\text{down}} = 1$ (if you ever find that your probabilities do not sum to 1, you know you have made a mistake somewhere, or forgot to account for a possible measurement outcome). When we represent this graphically in Fig. 2.7, we see that the projections of the vector $|\text{photon}\rangle$ onto the axes defined by $|\text{right}\rangle$ and $|\text{down}\rangle$ are equal. This gives us an indication that the components of the vector $|\text{photon}\rangle$ have something to do with the probabilities.

However, the components do not sum to 1 (instead, they sum to $\sqrt{2} > 1$), so they cannot be straightforward probabilities. On the other hand, the squares of the components do sum up to 1 (this was our normalisation condition $|a|^2 + |b|^2 = 1$), and could therefore be interpreted as probabilities. Indeed, one of the fundamental rules in quantum mechanics says that the probability of a measurement outcome is the absolute square of the corresponding amplitude. For example, suppose that the state of a photon after a beam splitter is given by

$$|\text{photon}\rangle = a|\text{right}\rangle + b|\text{down}\rangle\,. \tag{2.52}$$

The probability of finding the photon emerging in the beam to the right of the beam splitter is then given by $|a|^2$. Similarly, the probability of finding the photon emerging in the beam downwards from the beam splitter is given by $|b|^2$.

This rule is simple enough, but we need a slightly more general way of calculating it. First of all, let $\langle\cdot|\cdot\rangle$ denote the scalar product between two vectors. The scalar product is sometimes also called a "bracket", and the vector $\langle\cdot|$ is called a "bra" vector. We construct a bra vector from a ket vector by turning the column into a row vector (the top number in the column vector becomes the left number in the row vector), and take the *complex conjugate* of all the vector elements.

Since the states $|\text{right}\rangle$ and $|\text{down}\rangle$ are orthogonal vectors with length 1, we have

$$\begin{aligned}\langle\text{right}|\text{right}\rangle &= \langle\text{down}|\text{down}\rangle = 1\\ \langle\text{right}|\text{down}\rangle &= \langle\text{down}|\text{right}\rangle = 0\,.\end{aligned} \tag{2.53}$$

This means that we can calculate the following:

$$\begin{aligned}\langle\text{right}|\text{photon}\rangle &= a\langle\text{right}|\text{right}\rangle + b\langle\text{right}|\text{down}\rangle\\ &= a \times 1 + b \times 0 = a\,.\end{aligned} \tag{2.54}$$

The probability of finding the photon on the right of the beam splitter is the absolute square of this, namely

$$p_{\text{right}} = |\langle\text{right}|\text{photon}\rangle|^2 = |a|^2\,. \tag{2.55}$$

Similarly, the probability of finding the photon propagating downwards from the beam splitter is

$$p_{\text{down}} = |\langle\text{down}|\text{photon}\rangle|^2 = |b|^2\,. \tag{2.56}$$

The general rule is that we identify a state with each measurement outcome $|\text{outcome}\rangle$ (such as $|\text{right}\rangle$ and $|\text{down}\rangle$). The probability of the measurement outcome is then the absolute square of the salar product between the state of the system and the state corresponding to the measurement outcome:

$$p(\text{outcome}) = |\langle\text{outcome}|\text{state}\rangle|^2\,. \tag{2.57}$$

This is called the Born rule, after Max Born Born (1926), who first stated it. It is the general rule for calculating probabilities of measurement outcomes in quantum mechanics.

When you calculate the scalar product of two complex vectors, you must remember that the entries of the bra vector are the complex conjugate of the ket vector. For example, we can have two vectors $|\psi\rangle$ and $|\phi\rangle$ (remember that the labels are fairly arbitrary; here we use the greek letters "psi": ψ, and "phi": ϕ):

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad |\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}, \tag{2.58}$$

where a, b, c, and d are complex numbers. Since the bra of $|\psi\rangle$ is

$$\langle\psi| = \begin{pmatrix} a^* & b^* \end{pmatrix}, \tag{2.59}$$

the scalar product between $|\psi\rangle$ and $|\phi\rangle$ becomes

$$\langle\psi|\phi\rangle = \begin{pmatrix} a^* \; b^* \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = a^*c + b^*d\,. \tag{2.60}$$

If $a = 2 + 3i$, $b = 1 - 4i$, $c = 8 - 3i$, and $d = 6 + i$ the scalar product $\langle\psi|\phi\rangle$ is

$$\begin{aligned} \langle\psi|\phi\rangle &= a^*c + b^*d \\ &= (2 - 3i)(8 - 3i) + (1 + 4i)(6 + i) \\ &= 16 - 6i - 24i - 9 + 6 + i + 24i - 4 = 9 - 5i\,. \end{aligned} \tag{2.61}$$

The vectors $|\psi\rangle$ and $|\phi\rangle$ are not normalised:

$$\begin{aligned} \langle\psi|\psi\rangle &= |a|^2 + |b|^2 = 30 \\ \langle\phi|\phi\rangle &= |a|^2 + |b|^2 = 110\,, \end{aligned} \tag{2.62}$$

which means that $|\psi\rangle$ and $|\phi\rangle$ cannot be proper quantum state vectors. We will encounter this situation in later parts of this book. In order to turn $|\psi\rangle$ and $|\phi\rangle$ into proper state vectors with length 1, we have to "normalise" them. This is easy, we just divide the vector by its length to get a new vector of length 1:

$$|\psi_{\text{norm}}\rangle = \frac{|\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}} \quad \text{and} \quad |\phi_{\text{norm}}\rangle = \frac{|\phi\rangle}{\sqrt{\langle\phi|\phi\rangle}}, \tag{2.63}$$

where $|\psi_{\text{norm}}\rangle$ and $|\phi_{\text{norm}}\rangle$ are now normalised. You should check that this works. Note that it won't work if you forget to take the complex conjugate when turning a ket into a bra!

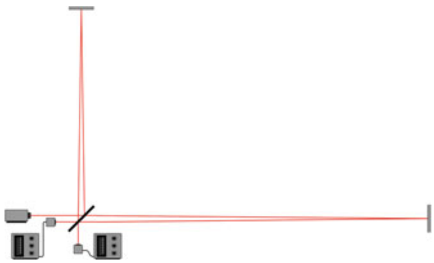


Fig. 2.8 A gravitational wave detector. The interactive figure is available online (see supplementary material 5)

2.8 ❧ Gravitational Wave Detection

The Mach–Zehnder interferometer we have discussed in this chapter is used in the measurement of gravitational waves. These are waves predicted by Einstein's theory of General Relativity, and occur when two very massive bodies (such as black holes or neutron stars) collide. You can think of them as "ripples" in the fabric of space and time, stretching and compressing space itself. On 14 September 2015, these waves were spotted for the first time by the LIGO collaboration Abbott et al. (2015), and it was inferred that two black holes of 29 and 36 solar masses spiralled into each other. In the last moments before the merger (around half a second), the binary system emitted about three solar masses worth of energy.[1] When a gravitational wave engulfs Earth, it will stretch space a little in one direction, and compress space in the perpendicular direction. It will therefore stretch one arm of an interferometer more than the other, and we can detect this as a phase shift. So the experiment tries to see a tiny phase shift (a thousandth of the width of a proton) by measuring a stream of photons arriving in detectors D_1 and D_2. Let's calculate how this works.

First, we need to modify the Mach–Zehnder interferometer to obtain two long arms. We do this by orienting the mirrors in Fig. 1.4 such that the beams are sent back to the first beam splitter, instead of a second. This is called a Michelson–Morley interferometer, (see Fig. 2.8). The mathematical description of the Michelson–Morley interferometer is exactly the same as the Mach–Zehnder interferometer: the photons don't care whether they encounter one beam splitter or another; a beam splitter does what a beam splitter does.

When we set up the interferometer as before, the output state of the photon is given by

$$|\text{out}\rangle_4 = U_{\text{BS}} U_\phi \, U_{\text{BS}} |\text{left}\rangle_1 = \frac{1}{2} \begin{pmatrix} 1 + e^{i\phi} \\ 1 - e^{i\phi} \end{pmatrix}, \tag{2.64}$$

where ϕ is now the phase shift in the interferometer that results from a gravitational wave passing by.

We can calculate the probability of finding a photon in detector D_1 as

[1] You can convert this to Joules by using the famous formula $E = mc^2$, where E is the energy, m the three solar masses in kilograms, and c the speed of light in vacuum in metres per second.

$$p_1 = |\langle D_1|\text{out}\rangle|^2 = \frac{1}{4}\left|\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1+e^{i\phi}\\ 1-e^{i\phi}\end{pmatrix}\right|^2 = \frac{1}{4}\left|1-e^{i\phi}\right|^2$$
$$= \frac{1}{4}(1-e^{i\phi})(1-e^{-i\phi}) = \frac{1}{2}(1-\cos\phi)\,, \tag{2.65}$$

where we used Eq. (2.29). Similarly, we can calculate that

$$p_2 = |\langle D_2|\text{out}\rangle|^2 = \frac{1}{2}(1+\cos\phi)\,, \tag{2.66}$$

and $p_1 + p_2 = 1$. We can estimate these probabilities by repeating the experiment a large number of times (say N times). If N_1 is the number of detections in D_1 and N_2 is the number of detections in D_2, then the probabilities are estimated as

$$p_1 = \frac{N_1}{N} \quad \text{and} \quad p_2 = \frac{N_2}{N}\,. \tag{2.67}$$

Using the fact that

$$p_2 - p_1 = \cos\phi\,, \tag{2.68}$$

the phase ϕ is then measured as

$$\phi = \arccos(p_2 - p_1) = \arccos\left(\frac{N_2 - N_1}{N}\right). \tag{2.69}$$

In the actual LIGO experiment, the vast majority of work was (and still is) dedicated to eliminating external noise sources, such as thermal fluctuations, electronic noise, and nearby logging operations. LIGO also uses fairly high intensity (classical) laser light that contains lots of photons, instead of sending individual photons through the interferometer. The leaders of the collaboration, Rainer Weiss, Kip Thorne and Barry Barish, received the Nobel prize in physics in 2017 for the detection of gravitational waves. While the Michelson–Morley interferometer can also be described with classical light waves, it is an instructive example how we treat a system like this in quantum mechanics.

Exercises

1. Multiply the following matrices:

$$A = \begin{pmatrix}12 & -3\\ 6 & 9\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}2 & 5\\ -1 & 0\end{pmatrix}.$$

   Is AB the same as BA?

2. Express the following complex numbers in polar representation $re^{i\phi}$:

$$z_1 = 3 + 4i\,, \qquad z_2 = 12 - 8i\,, \qquad \text{and} \qquad z_3 = z_1 + z_2^*\,.$$

3. Find the complex solutions to the quadratic equation

$$x^2 - 2x + 10 = 0\,.$$

4. Every matrix has two special properties: the trace and the determinant. The trace is the sum over the diagonal elements (top left to bottom right), while the determinant of a 2×2 matrix is given by

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc\,.$$

Calculate the trace and the determinant of the following matrices:

$$\begin{pmatrix} 3 & 1 \\ 2 & 6 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

5. Normalise the vector $|\psi\rangle = \begin{pmatrix} 3 \\ 4i \end{pmatrix}$.
6. Calculate the scalar product between

$$|\psi\rangle = \begin{pmatrix} 3i \\ -3 \end{pmatrix} \qquad \text{and} \qquad |\phi\rangle = \begin{pmatrix} 5 \\ 7 \end{pmatrix}.$$

Show that $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$.
7. Every month a lottery jackpot is one million pounds. A lottery ticket costs £5, and there are four million people playing the lottery each month. Calculate your average gain at any given month. How long do you have to play the lottery in order to have a 50% chance of winning at least once?
8. A car comes at a fork in the road, and the driver can turn left or right. If we describe the car quantum mechanically (not very realistic!) and the car is twice as likely to turn left as it is to turn right, what is the quantum state of the car after the fork?
9. The state of a photon in an interferometer is given by

$$|\psi\rangle = \frac{3i}{5}|\text{right}\rangle + \frac{4}{5}|\text{down}\rangle\,.$$

What is the probability of getting the measurement outcome "right"?
10. Construct the matrix for a beam splitter with a 70:30 ratio between reflection and transmission. Show how we can achieve perfect destructive interference in a Mach–Zehnder interferometer using two of these beam splitters (see also Exercise 3 of Chap. 1).

11. The unitary transformation of a Mach–Zehnder interferometer is described by

$$U_{\rm MZ} = \frac{1}{4}\begin{pmatrix} 3+\sqrt{3}\,i & 1-\sqrt{3}\,i \\ 1-\sqrt{3}\,i & 3+\sqrt{3}\,i \end{pmatrix}.$$

Calculate the relative phase difference between the arms in the interferometer. You may use Eq. (2.46).

12. A photon in the state

$$|\psi\rangle = \frac{1}{\sqrt{5}}|\text{right}\rangle + \frac{2}{\sqrt{5}}|\text{down}\rangle$$

is sent into a beam splitter. What is the probability of finding the photon in the "down" path?

13. We replace the lower-left mirror of the Mach–Zehnder interferometer in Fig. 1.4 with another 50:50 beam splitter. There are now three input beams and three output beams of the resulting interferometer. Assuming there are no relative path differences, construct the 3×3 transformation matrix describing the interferometer. Calculate the nine probabilities of a photon in any of the three inputs going to any of the three outputs. Compare your results with the 3×3 transformation matrix.
14. Consider again the Mach–Zehnder interferometer in Fig. 1.4. Instead of a single phase shift ϕ in the upper arm, we place two identical phase shifts ϕ, one in the upper arm and one in the lower arm. Calculate the matrix transformation of this interferometer. Sending a photon into the left input, what is the quantum state vector of the photon at the output? Can we measure the phase shift ϕ in both arms based on the probabilities of finding the photon in detectors D_1 and D_2?
15. Consider a quantum state $|\psi\rangle$ that accumulates a global phase $e^{i\phi}$. Show that any probability calculated using this state does not depend on the phase ϕ. The phase is *unobservable*.
16. We record 1000 photons in a gravity wave detector (see Sect. 2.5), and we find $N_1 = 35$ in detector D_1 and $N_2 = 965$ in detector D_2. Calculate the phase shift ϕ.

References

B.P. Abbott et al., GW170104: observation of a 50-solar-mass binary black hole coalescence at redshift 0.2. Phys. Rev. Lett. **118**, 22110 (2015)

M. Born, Zur Quantenmechanik der Stoßvorgänge. Z. Phys. **37**, 863 (1926)

P.A.M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, Oxford, 1988)