Chapter 9
Uncertainty Relations

In this chapter, we show how uncertainty relations arise naturally from quantum theory for position and momentum, time and energy, and so on. We explore the meaning of these relations and apply them to the quantum mechanical description of the pendulum. Finally, we show how we can use entanglement to enhance the precision of measurements.

9.1 Quantum Uncertainty Revisited

We have seen in Chaps. 1, 2, and 3 that a system in quantum mechanics can have a quantum uncertainty; the path of a photon in an interferometer may be uncertain, or an electron spin in the z-direction is fundamentally uncertain about the spin in the x- and y-directions. Moreover, in Chap. 7 we saw that this quantum uncertainty is different from classical uncertainty, in that nature itself does not have the information about the system. It is not a lack of knowledge on our part that makes the path of a photon inside a Mach–Zehnder interferometer uncertain, it is an inherent uncertainty in the path of the photon. When we take away this uncertainty via a QND detector we lose the interference in our photodetectors at the output of the Mach–Zehnder interferometer.

In this chapter, we will study in more detail how these uncertainties come about, and how the uncertainty in one observable of a system is related to the uncertainty of another observable. The famous Heisenberg Uncertainty Principle (Heisenberg 1927) says that it is impossible to measure both the position and the momentum of a particle with arbitrary precision. This principle was historically important in

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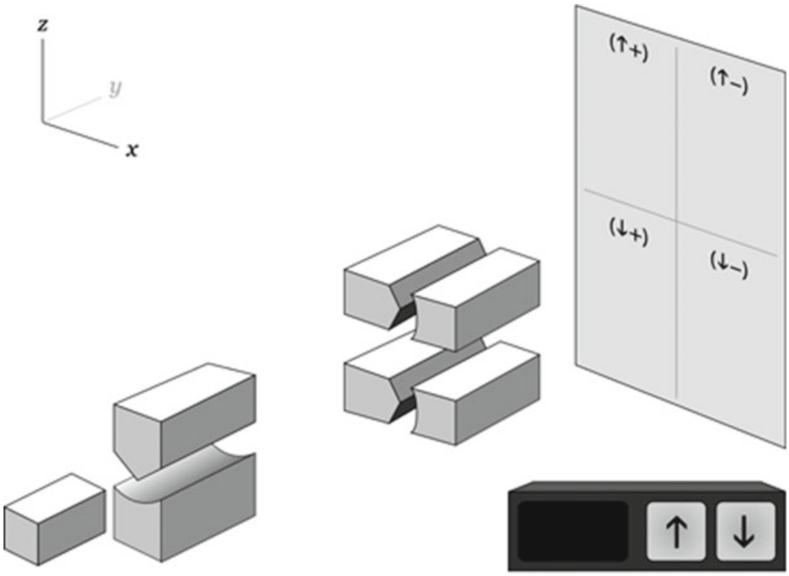

Fig. 9.1 Daisy-chaining two Stern–Gerlach experiments. The interactive figure is available online (see supplementary material 1)

the development of quantum mechanics. Modern treatments of quantum mechanics tend to move away from the uncertainty principle and talk instead of "uncertainty relations", since they follow directly from the rules of quantum mechanics that we identified at the end of Chap. 5. There is no need for a fundamental "Principle" in addition to these rules. A second reason is that there are many ways in which we can define a (quantum) uncertainty. Heisenberg's Uncertainty Principle cannot tell the difference between these different definitions very well, but our modern formulation of the uncertainty relations is perfectly equipped to handle these complications.

To remind ourselves of the issue, consider the following experiment: We prepare an electron in the spin state $|\uparrow\rangle$ in the z-direction and send it into a Stern–Gerlach apparatus with magnets oriented in the z-direction. As a result, the electron will always be deflected upwards. If we create the electron spin in the state $|\downarrow\rangle$, the electron will be deflected downwards. However, instead of hitting a screen, we let the electron continue into a subsequent pair of Stern–Gerlach apparatuses, both oriented in the x-direction (see Fig. 9.1), before being detected on a fluorescent screen. We observe that the electrons indeed always appear either in the upper two quadrants or the lower two quadrants, as expected from the state preparation $|\uparrow\rangle$ or $|\downarrow\rangle$, but the outcome of the spin in the x-direction is completely uncertain. The electrons end up randomly with equal probability in the left or the right quadrant.

Next, we replace the screen in the experiment with two subsequent Stern–Gerlach apparatuses with a third set of Stern–Gerlach apparatuses, this time again oriented in the z-direction. We collect the electrons again on a screen, as shown in Fig. 9.2.

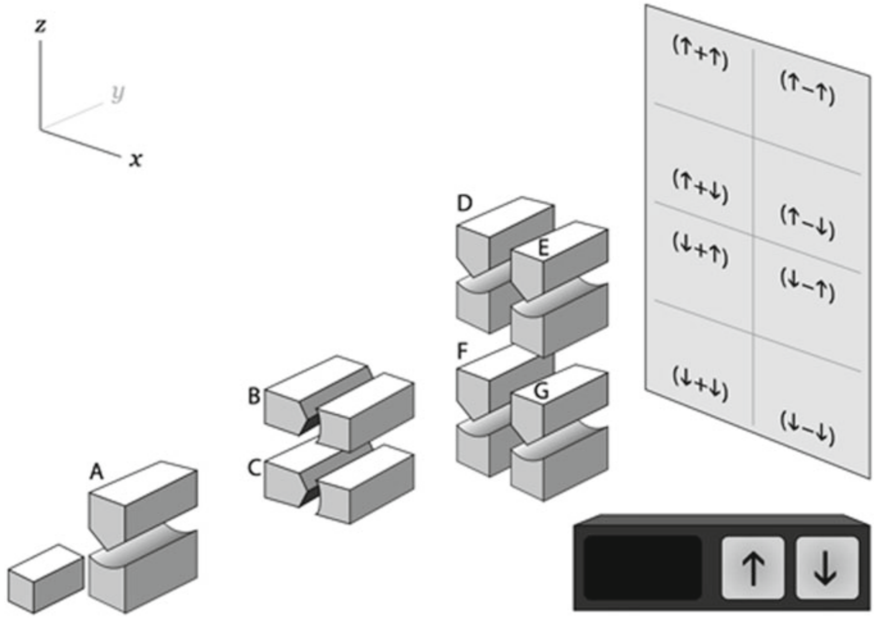

Fig. 9.2 Daisy-chaining three Stern–Gerlach experiments. The interactive figure is available online (see supplementary material 2)

The eight positions on the screen provide enough possibilities that we can infer the original spin in the z-direction, the spin in the x-direction in the second apparatus, and the spin in the z-direction in the last apparatus. When we prepare electrons in the spin states $|\uparrow\rangle$ or $|\downarrow\rangle$ and let them pass through the three Stern–Gerlach apparatuses, we observe that the spin as recorded on the screen is no longer always in the same spin state as the preparation procedure: Even if we prepare the spin state in $|\uparrow\rangle$ initially, there is a 50:50 chance of finding the electron in the final spin state $|\uparrow\rangle$ or $|\downarrow\rangle$. The measurement of the spin in the x-direction has disturbed the spin state in the z-direction.

We can explain the measurement outcomes by calculating the state of the electron spin and the electron position at the screen, and show exactly how the interaction of the spin with the magnets will lead to the uncertainty in the measurement outcomes. First, let's assume that the spin state of the electron is selected as $|\uparrow\rangle$. The electron will always be deflected upwards by the magnets labeled "A". The state after magnet "A" is therefore

$$|\uparrow, \text{up}\rangle \,. \tag{9.1}$$

This is a state of two degrees of freedom for the electron. We treat it in the same way as we did for composite systems in Chap. 6. Similarly, the electron spin $|\downarrow\rangle$ will become

$$|\downarrow, \text{down}\rangle \tag{9.2}$$

after the magnet. Next, in order to predict the behaviour of the electron in magnets “B” and “C”, we write the spin state $|\uparrow\rangle$ in terms of the spin eigenstates in the x-direction, labeled $+$ and $-$:

$$|\uparrow, \text{up}\rangle = \frac{|+, \text{up}\rangle + |-, \text{up}\rangle}{\sqrt{2}} \,. \tag{9.3}$$

After the second set of magnets (“B” and “C”) the component of the quantum state in the spin state $|+\rangle$ is deflected to the left, and the component in the spin state $|-\rangle$ is deflected to the right. When we include this in the description of the quantum state, we obtain

$$\frac{|+, \text{up}, \text{left}\rangle + |-, \text{up}, \text{right}\rangle}{\sqrt{2}} \,. \tag{9.4}$$

Notice that we now have entanglement between the spin state $|\pm\rangle$ and whether the electron is deflected to the left or to the right. The third set of magnets (“D” to “G”) are oriented again in the z-direction, and we need to write the spin states $|\pm\rangle$ in terms of $|\uparrow\rangle$ and $|\downarrow\rangle$. This will determine whether the electron will be deflected upwards or downwards in the magnets. The state becomes

$$\begin{aligned} |\psi\rangle = &\frac{1}{2}|\uparrow, \text{up}, \text{left}, \text{up}\rangle + \frac{1}{2}|\downarrow, \text{up}, \text{left}, \text{down}\rangle \\ &+ \frac{1}{2}|\uparrow, \text{up}, \text{right}, \text{up}\rangle - \frac{1}{2}|\downarrow, \text{up}, \text{right}, \text{down}\rangle \,. \end{aligned} \tag{9.5}$$

Notice that the spin components $|\downarrow, \text{up}, \text{left}, \text{down}\rangle$ and $-|\downarrow, \text{up}, \text{right}, \text{down}\rangle$ do not cancel, because the electron is in a different path in each case (right, instead of left). This lack of cancellation, or interference, means that we will see the electron appear both up and down on the screen in the upper two quadrants. Note also that there is entanglement between the spin and the path of the electron.

We can retrieve the interference in $|\psi\rangle$ when we use magnets to deflect the left and the right electron beams, so that they merge again. This is called “quantum erasure”.

9.2 Uncertainty Relations

It is the eigenstates of the observables S_z and S_x that are the cause of the uncertainty: the states $|\uparrow\rangle$ and $|\downarrow\rangle$ are different from $|+\rangle$ and $|-\rangle$, which is what causes the presence of both the upper and lower paths in Eq. (9.3). So in general, we expect that the uncertainty in a measurement of two observables A and B will be higher the more the eigenstates of A and B differ. Moreover, there should be no uncertainty if the eigenstates are the same, that is, when A and B commute.

Since physics is a quantitative science, we want to know precisely how big this uncertainty is. The so-called *Robertson relation* (Robertson 1929) relates the uncertainty in two observables A and B to the expectation value of their commutator:

$$\Delta A \;\Delta B \geq \frac{1}{2}|\langle[A, B]\rangle| \,. \tag{9.6}$$

This is what people usually mean by the uncertainty relation. It is a mathematical expression that is different from Heisenberg's Uncertainty Principle, which is the qualitative statement given in the previous section. We will now give the proof of the uncertainty relation, but it is not essential to the rest of the chapter, and if you wish you can skip to the next section.

❧ Proof of the Uncertainty Relation

One of the most useful theorems in algebra is the Cauchy–Schwarz inequality for scalar products. We will just state it here without proof:

$$\langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2 \,. \tag{9.7}$$

Intuitively, you can see that this is true for real vectors $\mathbf{u}$ and $\mathbf{v}$ (with lengths u and v) that have a scalar product $\mathbf{u}\cdot\mathbf{v} = uv\cos\theta$, with θ the angle between the two vectors. The Cauchy–Schwarz inequality for these vectors becomes

$$u^2\,v^2 \geq (uv\cos\theta)^2 \qquad \text{or} \qquad \cos^2\theta \leq 1\,. \tag{9.8}$$

The Cauchy–Schwarz inequality tells us that this is true for general complex vectors in arbitrary dimension as well.

The right-hand side of this equation is the modulus square of a complex number $\langle f|g\rangle$, and we can bound this by the imaginary part of $\langle f|g\rangle$:

$$|\langle f|g\rangle|^2 = |\mathrm{Re}\langle f|g\rangle|^2 + |\mathrm{Im}\langle f|g\rangle|^2 \geq |\mathrm{Im}\langle f|g\rangle|^2 = \frac{1}{4}|\langle f|g\rangle| - \langle g|f\rangle|^2 \,, \tag{9.9}$$

where we used that $\mathrm{Im}\,z = (z - z^*)/2i$. The Cauchy–Schwarz inequality then becomes

$$\langle f|f\rangle\langle g|g\rangle \geq \frac{1}{4}|\langle f|g\rangle - \langle g|f\rangle|^2 \,. \tag{9.10}$$

This is the inequality that we will use to derive the Robertson relation.

Let $|f\rangle = (A - \langle A\rangle)|\psi\rangle$ and $|g\rangle = (B - \langle B\rangle)|\psi\rangle$. We calculate the scalar products $\langle f|f\rangle$ and $\langle g|g\rangle$ as

$$
\begin{aligned}
\langle f|f\rangle &= \langle\psi|(A-\langle A\rangle)^2|\psi\rangle = (\Delta A)^2\,, \\
\langle g|g\rangle &= \langle\psi|(B-\langle B\rangle)^2|\psi\rangle = (\Delta B)^2\,.
\end{aligned} \tag{9.11}
$$

The scalar product between $|f\rangle$ and $|g\rangle$ can be calculated as

$$
\langle f|g\rangle = \langle\psi|AB|\psi\rangle - \langle A\rangle\langle B\rangle \quad \Rightarrow \quad \langle g|f\rangle = \langle\psi|BA|\psi\rangle - \langle A\rangle\langle B\rangle. \tag{9.12}
$$

This leads to

$$
|\langle f|g\rangle - \langle g|f\rangle|^2 = |\langle\psi|AB-BA|\psi\rangle|^2. \tag{9.13}
$$

Putting everything together, we obtain

$$
(\Delta A)^2\ (\Delta B)^2 \geq \frac{1}{4}|\langle[A,B]\rangle|^2\,, \tag{9.14}
$$

which gives the Robertson relation in Eq. (9.6) once we recognise that ΔA and ΔB are real positive numbers.

9.3 Position-Momentum Uncertainty

We can use the Robertson formula from the previous section to find the uncertainty relation between position $\hat{x}$ and momentum $\hat{p}$, with

$$
\hat{x} = x \qquad \text{and} \qquad \hat{p} = -i\hbar\frac{d}{dx}\,. \tag{9.15}
$$

The commutator between $\hat{x}$ and $\hat{p}$ can be calculated by remembering that it is an operator acting on a state:

$$
\begin{aligned}
\left[\hat{x},\hat{p}\right]|\psi\rangle &= -i\hbar\left(x\frac{d}{dx} - \frac{d}{dx}x\right)|\psi\rangle = -i\hbar x\frac{d}{dx}|\psi\rangle + i\hbar\frac{d}{dx}\left(x|\psi\rangle\right) \\
&= -i\hbar x\frac{d}{dx}|\psi\rangle + i\hbar|\psi\rangle + i\hbar x\frac{d}{dx}|\psi\rangle = i\hbar|\psi\rangle\,,
\end{aligned} \tag{9.16}
$$

where we used the product rule for differentiation from the second to the third line. Since this is true for any state $|\psi\rangle$, we usually abbreviate this as

$$
[\hat{x},\hat{p}] = i\hbar\,. \tag{9.17}
$$

Substituting this into the Robertson relation in Eq. (9.6) gives the celebrated uncertainty relation between position and momentum:

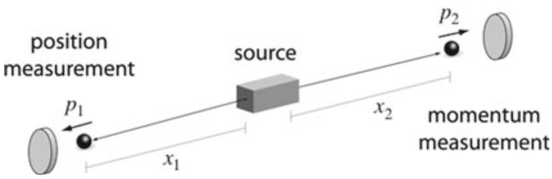

Fig. 9.3 Simultaneous position and momentum measurements on two particles

$$\Delta x \ \Delta p \geq \frac{\hbar}{2} \,. \tag{9.18}$$

It tells us that the uncertainty Δx of the position of a particle means that the uncertainty in the momentum Δp in a subsequent measurement is at least $\hbar/(2\Delta x)$. These uncertainties are a combination of the inherent quantum uncertainty and the uncertainty associated with our classical lack of knowledge about the state of the particle (the uncertainties Δx and Δp are calculated with respect to the state of the system, which may be mixed, as we discovered in Chap. 7). Even when we know all there is to know about the state of a particle, there is an inherent quantum uncertainty in the position and the momentum of the particle due to the fact that the position and momentum operators do not have the same eigenstates, and therefore do not commute.

Albert Einstein was famously unhappy about the inherent quantum uncertainty in the position and the momentum of a particle, and tried to argue that the quantum mechanical description of physical systems is not complete. In 1935 he devised a thought experiment with Boris Podolski and Nathan Rosen that was supposed to prove the incompleteness of quantum mechanics. This has become known as the EPR paradox (Einstein et al. 1935).

The experiment goes as follows: Suppose we construct a source that sends two particles in opposite directions with equal but opposite momentum, so that the total momentum of the two particles is zero (see Fig. 9.3). We can then measure the position of particle 1, and the momentum of particle 2. There is no reason we cannot make each measurement arbitrarily precise: The measurements do not simultaneously measure the non-commuting observables $\hat{x}$ and $\hat{p}$ for each particle. Instead, this is a measurement of the *commuting* observables $\hat{p}_1 - \hat{p}_2$ and $\hat{x}_1 + \hat{x}_2$ for a state that is fully determined by the relations $p_1 + p_2 = 0$ and $x_1 - x_2 = 0$. To prove that the operators $\hat{p}_1 - \hat{p}_2$ and $\hat{x}_1 + \hat{x}_2$ commute, we write

$$\begin{aligned}
[\hat{x}_1 + \hat{x}_2, \hat{p}_1 - \hat{p}_2] &= [\hat{x}_1, \hat{p}_1 - \hat{p}_2] + [\hat{x}_2, \hat{p}_1 - \hat{p}_2] \\
&= [\hat{x}_1, \hat{p}_1] - [\hat{x}_1, \hat{p}_2] + [\hat{x}_2, \hat{p}_1] - [\hat{x}_2, \hat{p}_2] \\
&= [\hat{x}_1, \hat{p}_1] - 0 + 0 - [\hat{x}_2, \hat{p}_2] \\
&= i\hbar - i\hbar = 0 \,,
\end{aligned} \tag{9.19}$$

where in the third line we used that operators on different particles commute.

If the momentum of particle 1 is measured as p, we can infer that the momentum of particle 2 is $-p$ (since $p_1 + p_2 = 0$). Indeed, if we measured the momentum of particle 2 we would always find $-p$. However, instead of the momentum, we measure the position of particle 2 with respect to the position of the source, yielding x. If the source is located at the origin, we know that the distance from the origin of the other particle will also be x (in the opposite direction). Indeed, if we measured the position of the other particle we would find the distance x from the origin every time (that is, with probability 1).

Since the two particles are far away from one another, there is no way a signal from one could travel to the other particle in time to influence the measurement outcome. After all, nothing can travel faster than light. Einstein, Podolski and Rosen then make their profound argument: if you know with certainty what will happen in a measurement, then there should be an element of reality that determines the measurement outcome (having ruled out faster-than-light signalling). So the particles have inherently both a precise position and momentum, contrary to the claims of the uncertainty relation. This is the EPR paradox.

The spatial separation of the particles is crucial for the EPR thought experiment. Without it, we can imagine that the choice of measurement on particle 1 will influence the outcomes of the measurement on particle 2. We can introduce so-called "hidden variables" (the elements of reality of EPR) that are not part of the regular theory of quantum mechanics as it is developed in this book, but that dictate how the particles behave in measurements. The measurement outcome can then be determined by a signal from the other particle, instead of "elements of reality" already present in the particle. By placing the particles outside of each other's causal influence the EPR setup avoids this loophole.

The EPR paradox is one of the highlights of the famous debate between Einstein and Bohr about the meaning of quantum mechanics. To solve this paradox, we note that the two particles are entangled in position and momentum. This means that the position and momentum for each *individual* particle is completely uncertain, since the state of one component of a maximally entangled state is maximally mixed (i.e., maximally uncertain). The certainty of measurement outcomes that the EPR paradox identifies is in the *joint* properties $x_1 - x_2$ and $p_1 + p_2$ of the combined system of the two particles. This is fundamentally different form the properties of the individual particles, and the uncertainty relation for position and momentum as defined for individual particles still stands.

In 1964, John Bell (1964) showed that any theory of quantum mechanics that uses hidden variables to remove the inherent quantum uncertainty will have to involve superluminal signalling between particle 1 and particle 2. For many people this is too big a price to pay for regaining determinism. From a practical point of view, hidden variable theories are more complicated than orthodox quantum mechanics without any computational advantage, and that is why most people don't bother with them. Of course, this does not make the conceptual difficulties go away. We will explore those further in the next chapter.

9.4 Energy-Time Uncertainty

Another uncertainty relation that plays an important role in quantum mechanics is the energy-time uncertainty relation. Contrary to the position-momentum uncertainty relation, this is not a direct manifestation of the Robertson relation, because time is not an operator in quantum mechanics. Nevertheless, it takes a very similar form:

$$\Delta E \, \delta t \geq \frac{\hbar}{2} . \tag{9.20}$$

The question then becomes: what do we mean by ΔE and δt?

For operators such as $\hat{x}$ and $\hat{p}$ the uncertainties $(\Delta x)^2$ and $(\Delta p)^2$ are just the expectation value of the squared value away from the mean:

$$(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle . \tag{9.21}$$

It is a measure of the spread of the measurement outcomes, also known as the variance.

We have an operator for the energy, namely the Hamiltonian, and we can therefore unambiguously define ΔE in the same way as above. However, there is no (universal) time operator in quantum mechanics, so we cannot define the time uncertainty in the same way. That's why we write δt, to distinguish it from an operator uncertainty ΔA.

To find out what exactly we mean by δt, we first give a mathematical derivation. We start with the Robertson relation, where for B we substitute the Hamiltonian H:

$$\Delta A \, \Delta H \geq \frac{1}{2} |\langle [A, H] \rangle| . \tag{9.22}$$

Next, we want to relate the observable A to the time parameter t. For example, A may be the position of the second hand of a clock. Assume that the clock as a whole does not move in time, so the time variation of the expectation value $\langle A \rangle$ is entirely determined by the time evolution of the quantum state $|\psi\rangle$ of the pointer. We can then write, using the product rule, that

$$\begin{aligned} \frac{d\langle A \rangle}{dt} &= \frac{d}{dt} \langle \psi | A | \psi \rangle \\ &= \left(\langle \psi | \frac{\overleftarrow{d}}{dt} \right) A | \psi \rangle + \langle \psi | \left(\frac{dA}{dt} \right) | \psi \rangle + \langle \psi | A \left(\frac{d}{dt} | \psi \rangle \right) \\ &= -\frac{1}{i\hbar} \langle \psi | HA | \psi \rangle \, + \, 0 \, + \, \frac{1}{i\hbar} \langle \psi | AH | \psi \rangle \\ &= \frac{1}{i\hbar} \langle [A, H] \rangle , \end{aligned} \tag{9.23}$$

where we used that the time derivative of A is zero, and we substituted the Schrödinger equation in the third line (twice). We can substitute this expression into the Robertson relation (9.22), which gives us

$$\Delta A \, \Delta H \geq \frac{\hbar}{2} \left| \frac{d\langle A \rangle}{dt} \right| . \tag{9.24}$$

Assuming that the expectation value $\langle A \rangle$ keeps changing over time, so that the right-hand side of Eq. (9.24) is never zero, we can define a time uncertainty as

$$\delta t \equiv \Delta A \left| \frac{d\langle A \rangle}{dt} \right|^{-1} . \tag{9.25}$$

This allows us to write an energy-time uncertainty relation

$$\delta t \; \Delta E \geq \frac{\hbar}{2} , \tag{9.26}$$

where we identified $\Delta E = \Delta H$.

We now have an expression for δt that will help us interpret the energy-time uncertainty relation. The right-hand side of Eq. (9.25) contains only well-defined terms: ΔA is the uncertainty in the measurement outcome of operator A, and $|d\langle A \rangle/dt|$ measures how fast the expectation value of A changes over time. When A is the position of a second hand on a clock, the denominator in Eq. (9.25) is the average (angular) speed of the second hand. The quantum uncertainty in the position of the second hand divided by the average speed is properly interpreted as the time it takes the second hand to move to a new position that is distinguishable from the position a time period δt earlier. This is a proper uncertainty, because it gives us the minimum time resolution of the clock. We can improve the precision of the clock by decreasing ΔA, which according to the uncertainty relation we can achieve only by choosing quantum states with a large uncertainty in energy.

In general, δt is the minimum time it takes for the system in a state $|\psi\rangle$ to evolve to a new state that is distinguishable from $|\psi\rangle$. It is a quantum uncertainty arising from the quantum uncertainty in the observable A. If the state is in an energy eigenstate, such that $\Delta E = 0$, the system merely accrues a global phase over time, and the system will never evolve to a distinguishable state. The time it takes for the system to evolve to a distinguishable state then becomes infinite: $\delta t \to \infty$.

9.5 The Quantum Mechanical Pendulum

How do the uncertainty relations affect real physical situations? To explore this we study one of the simplest and most interesting mechanical systems in physics: the pendulum. It has a regular periodic motion, which makes it an important component

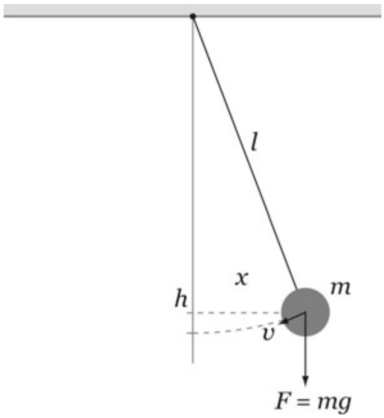

Fig. 9.4 The pendulum. The interactive figure is available online (see supplementary material 3)

of mechanical clocks. It also is the prime example of harmonic motion. The basic features of a classical pendulum are shown in Fig. 9.4.

Let us have a look at the way we describe the motion of a pendulum. First, we assume that all the mass m of the pendulum is concentrated in the bob, which is hanging from the ceiling by a thin wire of length l. Gravity pulls the pendulum towards the equilibrium position with a force $F = mg$. However, in quantum mechanics we prefer to deal with the energy of a system, so we first need to work out the potential energy of the (frictionless) pendulum as a function of the horizontal displacement x.

We choose the energy scale such that in the equilibrium position $x = 0$ the potential energy is zero. This means that for some displacement the potential energy becomes mgh. The period of a pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}} = \frac{2\pi}{\omega}\,, \tag{9.27}$$

where ω is the angular frequency of the pendulum. We can therefore write $g = \omega^2 l$ and use this to eliminate g from the potential energy. Next, we express l in terms of x and h using Pythagoras' theorem:

$$(l-h)^2 + x^2 = l^2\,. \tag{9.28}$$

Solving for l then gives us

$$l = \frac{h}{2}\left(1 + \frac{x^2}{h^2}\right). \tag{9.29}$$

Since for small displacements $x^2 \gg h^2$, we can ignore the term 1 in this equation, and simplify to

$$h \simeq \frac{x^2}{2l}\,. \tag{9.30}$$

Substituting g and h into mgh then gives

$$mgh \simeq \frac{1}{2}m\omega^2 x^2. \tag{9.31}$$

The total kinetic and potential energy of the pendulum is then taken as

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\,, \tag{9.32}$$

where x and p are the *horizontal* position and momentum of the pendulum, respectively. Since x is small, we ignore the vertical movement.

In quantum mechanics, the total energy H becomes the Hamiltonian operator, where we now interpret x and p as operators. Finding the energy eigenstates then amounts to finding the solutions to the equation

$$H|E_n\rangle = E_n|E_n\rangle\,, \tag{9.33}$$

or equivalently solving the differential equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 - E_n\right)\psi_n(x) = 0\,, \tag{9.34}$$

where $\psi_n(x) \equiv \langle x|E_n\rangle$. This differential equation is a bit tricky to solve, so we take a different approach.

We define two new operators, $\hat{a}$, and its Hermitian adjoint $\hat{a}^\dagger$, such that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right) \tag{9.35}$$

and

$$\hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right). \tag{9.36}$$

Now, this is a pretty strange step, and at this point completely unmotivated. However, it is entirely legal, and remember that we added operators before when we constructed the spin operator S_θ in an arbitrary direction in the xz-plane. Right now, I can only promise that this definition will have a grand pay-off in a few pages. You can check that the prefactor of $\hat{x}$ has dimensions of length and that of $\hat{p}$ has dimensions of momentum. This means that $\hat{a}$ and $\hat{a}^\dagger$ are dimensionless. Substituting these operators into Eq. (9.32), we obtain

$$H = \frac{1}{2}\hbar\omega\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) . \tag{9.37}$$

Note that we do not assume that $\hat{a}^\dagger\hat{a} = \hat{a}\hat{a}^\dagger$. This is important. The operators $\hat{x}$ and $\hat{p}$ do not commute: $[\hat{x}, \hat{p}] = i\hbar$, and by imposing this commutation relation on the Eqs. (9.35) and (9.36), we find that $\hat{a}$ and $\hat{a}^\dagger$ must obey the commutation relation

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1 . \tag{9.38}$$

We can use this to rewrite the Hamiltonian in Eq. (9.32) as

$$H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) . \tag{9.39}$$

We will now use this form to extract the energy spectrum for the quantum mechanical pendulum.

The question that surely is going through your mind right now is: what do $\hat{a}$ and $\hat{a}^\dagger$ even mean? To answer this, we will use the fact that we can calculate commutation relations of functions of $\hat{a}$ and $\hat{a}^\dagger$, in particular H. Consider the expression $H\hat{a}|E_n\rangle$. In order to calculate what this is, we can commute $\hat{a}$ with H and apply Eq. (9.39):

$$H\hat{a} = \hat{a}H + \left[H, \hat{a}\right] , \tag{9.40}$$

and

$$\left[H, \hat{a}\right] = -\hbar\omega\,\hat{a} . \tag{9.41}$$

So we find that

$$\begin{aligned} H\hat{a}|E_n\rangle &= \hat{a}H|E_n\rangle \;+\; \left[H, \hat{a}\right]|E_n\rangle = E_n\hat{a}|E_n\rangle - \hbar\omega\,\hat{a}|E_n\rangle \\ &= (E_n - \hbar\omega)\,\hat{a}|E_n\rangle . \end{aligned} \tag{9.42}$$

In other words, the (possibly unnormalised) state $\hat{a}|E_n\rangle$ is again an eigenstate of H with energy $E_n - \hbar\omega$. Similar reasoning leads to the relation

$$H\hat{a}^\dagger|E_n\rangle = (E_n + \hbar\omega)\hat{a}^\dagger|E_n\rangle . \tag{9.43}$$

The operators $\hat{a}$ and $\hat{a}^\dagger$ are therefore "ladder operators" that move up and down the ladder of energy eigenstates (it requires an additional argument to show that there are no energy eigenstates between E_n and $E_n \pm \hbar\omega$; the ladder operators do not skip energy eigenstates).

When the ground state of the quantum pendulum is denoted by $|E_0\rangle$, the lowering operator $\hat{a}$ on this state must return zero:

$$\hat{a}|E_n\rangle = 0 . \tag{9.44}$$

Otherwise it would create a state below the ground state, which is by definition impossible. Secondly, from Eq. (9.39) we see that $\hat{a}^\dagger\hat{a}$ must return the number n of the energy level E_n (the extra term $\hbar\omega/2$ is independent of n and can be subtracted by a redefinition of the energy scale). We therefore conclude that

$$\hat{a}^\dagger\hat{a}|E_n\rangle = n|E_n\rangle\,, \tag{9.45}$$

which we can write as

$$\hat{a}^\dagger\left(\hat{a}|E_n\rangle\right) = \hat{a}^\dagger f_n|E_{n-1}\rangle = f_n g_{n-1}|E_n\rangle = n|E_n\rangle\,. \tag{9.46}$$

Therefore, $f_n\ g_{n-1} = n$, which is satisfied for

$$f_n = \sqrt{n} \qquad \text{and} \qquad g_n = \sqrt{n+1}\,, \tag{9.47}$$

and we find the general relations

$$\hat{a}|E_n\rangle = \sqrt{n}|E_{n-1}\rangle \tag{9.48}$$

and

$$\hat{a}^\dagger|E_n\rangle = \sqrt{n+1}|E_{n+1}\rangle\,. \tag{9.49}$$

Note how this automatically satisfies the requirement that $\hat{a}|E_0\rangle = 0$. The energy spectrum of the quantum mechanical pendulum is therefore

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)\,, \tag{9.50}$$

with energy eigenstates $|E_n\rangle$.

One interesting thing we can calculate is the average position and momentum in the ground state $|E_0\rangle$. For the position we find

$$\begin{aligned}
\langle E_0|\hat{x}|E_0\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\langle E_0|\left(\hat{a}+\hat{a}^\dagger\right)|E_0\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\langle E_0|\hat{a}|E_0\rangle + \langle E_0|\hat{a}^\dagger|E_0\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(0 + \langle E_0|E_1\rangle\right) = 0\,,
\end{aligned} \tag{9.51}$$

since the energy eigenstates $|E_0\rangle$ and $|E_1\rangle$ are orthogonal. Similarly, for the average momentum in the ground state we find

$$\begin{aligned}
\langle E_0|\hat{p}|E_0\rangle &= -i\sqrt{\frac{m\hbar\omega}{2}}\langle E_0|\left(\hat{a}-\hat{a}^\dagger\right)|E_0\rangle \\
&= -i\sqrt{\frac{m\hbar\omega}{2}}\left(\langle E_0|\hat{a}|E_0\rangle - \langle E_0|\hat{a}^\dagger|E_0\rangle\right) \\
&= -i\sqrt{\frac{m\hbar\omega}{2}}\left(0 - \langle E_0|E_1\rangle\right) = 0\,. \qquad (9.52)
\end{aligned}$$

So the average position and momentum for a quantum mechanical pendulum in the ground state is zero, as expected.

However, what about the variances of x and p? We calculate $(\Delta x)^2$ and $(\Delta p)^2$ according to the standard formula:

$$\begin{aligned}
(\Delta x)^2 &= \langle E_0|x^2|E_0\rangle - \langle E_0|x|E_0\rangle^2 \\
&= \frac{\hbar}{2m\omega}\langle E_0|\left(\hat{a}+\hat{a}^\dagger\right)^2|E_0\rangle \\
&= \frac{\hbar}{2m\omega}\langle E_0|\left(\hat{a}\hat{a}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\right)|E_0\rangle \\
&= \frac{\hbar}{2m\omega}\,, \qquad (9.53)
\end{aligned}$$

and for the momentum:

$$\begin{aligned}
(\Delta p)^2 &= \langle E_0|p^2|E_0\rangle - \langle E_0|p|E_0\rangle^2 \\
&= -\frac{m\hbar\omega}{2}\langle E_0|\left(\hat{a}-\hat{a}^\dagger\right)^2|E_0\rangle \\
&= -\frac{m\hbar\omega}{2}\langle E_0|\left(\hat{a}\hat{a}-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\right)|E_0\rangle \\
&= \frac{m\hbar\omega}{2}\,. \qquad (9.54)
\end{aligned}$$

Therefore, the ground state uncertainties obey the relation

$$\Delta x\,\Delta p = \frac{\hbar}{2}\,, \qquad (9.55)$$

and the ground state of the quantum pendulum is a minimum uncertainty state since it satisfies the equality in the uncertainty relation. Even though the position and momentum is zero on average, there are fluctuations of size Δx and Δp, and even in the ground state the pendulum is not completely still. This is fundamental quantum noise, and it cannot be removed from the system. This is a general aspect of quantum mechanics: no system can be completely still. There is always some *quantum noise* that cannot be reduced beyond a certain minimum that is determined by an uncertainty relation. In particle physics this manifests itself as the spontaneous generation and destruction of particles with energy E for a duration shorter than $\hbar/(2E)$.

There is one more thing we should say about the quantum mechanical pendulum. The most characteristic aspect of the pendulum is that it swings back and forth, while we have only determined the energy eigenstates $|E_n\rangle$. These states evolve over time as $e^{-\frac{i}{\hbar}E_n t}|E_n\rangle$, accumulating an unobservable global phase $e^{-\frac{i}{\hbar}E_n t}$. In other words, when a system is in an energy eigenstate, it will not show any "motion". Compare this to our discussion concerning the energy-time uncertainty relation, where we can have a clock timing resolution only if the clock state is not in an energy eigenstate. We need some kind of motion to make a clock, like the pendulum in a grandfather clock.

The state of the pendulum in motion must therefore be a superposition of energy eigenstates. It is not so trivial to find out which state gives rise to the typical pendulum motion, but it turns out to be of the form

$$|\alpha(t)\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n e^{-\frac{i}{\hbar}E_n t}}{\sqrt{n!}} |E_n\rangle \,, \tag{9.56}$$

where α is related to the maximum horizontal displacement (the amplitude) for the pendulum A via

$$A = |\alpha| \sqrt{\frac{2\hbar}{m\omega}} \,. \tag{9.57}$$

Choosing α real, we find that the expectation value of the position operator $\hat{x}$ defined in Eq. (9.35) with respect to $|\alpha(t)\rangle$ can be calculated as

$$\langle x\rangle = \langle \alpha(t)|\hat{x}|\alpha(t)\rangle = A\cos\omega t \,. \tag{9.58}$$

Similarly,

$$\langle p\rangle = \langle \alpha(t)|\hat{p}|\alpha(t)\rangle = -m\omega A \sin\omega t \,, \tag{9.59}$$

as you would expect. You should try to reproduce these expectation values using Eqs. (9.48) and (9.49). The state $|\alpha(t)\rangle$ is called a *coherent state*. What happens when α is complex?

The expectation values of the quantum mechanical operators $\hat{x}$ and $\hat{p}$ follow the classical equations of motion for the pendulum. The probability density function $|\langle x|\alpha(t)\rangle|^2$ of finding pendulum at displacement x at time t is shown as the blue line in Fig. 9.5, and clearly exhibits harmonic motion.

Of course, the larger the amplitude A, the more energy there is in the pendulum. We can calculate this from the expectation value of the Hamiltonian in Eq. (9.39) with respect to the state $|\alpha(t)\rangle$:

$$\langle H\rangle = \hbar\omega\, \langle\alpha(t)| \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) |\alpha(t)\rangle = \hbar\omega|\alpha|^2 = \frac{1}{2}m\omega^2 A^2, \tag{9.60}$$

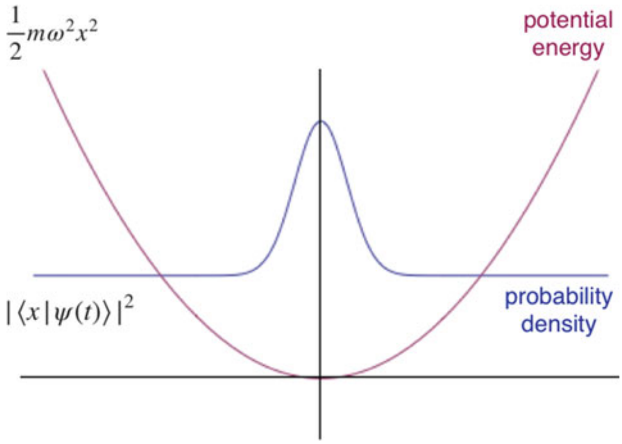

Fig. 9.5 The coherent state for the pendulum. The interactive figure is available online (see supplementary material 4)

which is exactly what we would expect from Eq. (9.32), since classically $p = 0$ at the extreme displacement $x = A$. The uncertainty in the energy for the pendulum in the state of Eq. (9.56) is given by

$$\Delta E = \sqrt{\langle H^2\rangle - \langle H\rangle^2} = \hbar\omega|\alpha| = A\sqrt{\frac{m\hbar\omega^3}{2}}\,. \tag{9.61}$$

From the uncertainty relation between energy and time we then have that

$$\delta t \geq \frac{\hbar}{2\Delta E} = \left(\frac{\hbar^3}{8mA\omega^3}\right)^{\frac{1}{4}}\,. \tag{9.62}$$

This means that if we use the pendulum as the oscillator in a clock, the highest precision we can achieve in principle is given by δt in Eq. (9.62).

You may find it strange that there are energy fluctuations in the swinging motion of the pendulum. What about energy conservation? It turns out that energy is strictly conserved in the quantum mechanical pendulum: potential energy is exactly converted into kinetic energy and vice versa (assuming no friction). However, the total value of the energy is uncertain and subject to quantum fluctuations. Energy conservation applies strictly to the *processes* governing physical systems, not the states themselves.

9.6 ❧ Precision Measurements

Quantum systems can evolve under the action of some interaction Hamiltonian, and we can make repeated measurements of the quantum system to figure out the strength of some of the physical quantities in the interaction Hamiltonian. For example, we can describe an electron spin in a magnetic field with the interaction Hamiltonian derived from Eq. (3.63)

$$H = -\frac{e}{mc}\left(B_x S_x + B_y S_y + B_z S_z\right) , \tag{9.63}$$

where S_x, S_y, and S_z are the spin observables of the electron; B_x, B_y, and B_z are the values of the magnetic field in the direction x, y, and z, respectively; and e and m are the electron charge and mass. We can then prepare the system repeatedly in the same state using the same preparation procedure, and for each preparation measure the electron spin to figure out the strength of the magnetic field.

Let's suppose that we send an electron through a region that has a magnetic field with unknown magnitude B in the z-direction ($B = B_z$), and $B_x = B_y = 0$. The state of the electron is given by

$$|\psi\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} . \tag{9.64}$$

If we know the velocity of the electron and the size of the magnetic region, we can work out the time t the electron experiences the interaction H. The evolution U of the electron state in matrix form is calculated as

$$U = \exp\left(-\frac{i}{\hbar}Ht\right) = \exp\left(\frac{ieBt}{mc\hbar}S_z\right) = \exp\left(\frac{ieBt}{2mc}\sigma_z\right) , \tag{9.65}$$

with

$$\sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} . \tag{9.66}$$

The evolution U then becomes

$$U = \begin{pmatrix}e^{i\omega t/2} & 0\\0 & e^{-i\omega t/2}\end{pmatrix} \quad \text{with} \quad \omega = \frac{eB}{mc} , \tag{9.67}$$

and the state of the electron after interacting with the magnetic field is

$$|\psi(t)\rangle = \frac{e^{i\omega t/2}|\uparrow\rangle + e^{-i\omega t/2}|\downarrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\omega t/2}\\e^{-i\omega t/2}\end{pmatrix} . \tag{9.68}$$

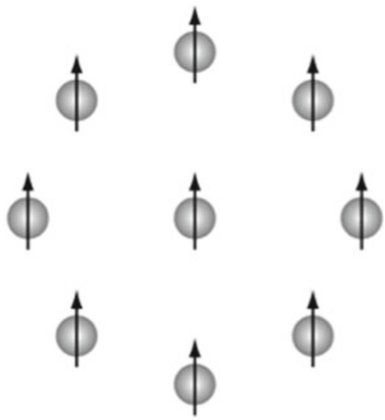

Fig. 9.6 Measuring the spin direction on N separate spins. The interactive figure is available online (see supplementary material 5)

You should be able to verify this using the techniques developed in Chaps. 3 and 4. We can measure the spin of the electron in the x-direction using a Stern–Gerlach apparatus, and the probabilities of the measurement outcomes are

$$\begin{aligned} p_+ &= |\langle +|\psi(t)\rangle|^2 = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{eBt}{mc}\right) , \\ p_- &= |\langle -|\psi(t)\rangle|^2 = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{eBt}{mc}\right) . \end{aligned} \tag{9.69}$$

By repeating this experiment N times we obtain N_+ outcomes "+" and N_- outcomes "−" with $N_+ + N_- = N$. This is shown in Fig. 9.6. When N is large, the relative frequencies of the measurement outcomes approach the probabilities:

$$\frac{N_+}{N} \to p_+ \quad \text{and} \quad \frac{N_-}{N} \to p_- . \tag{9.70}$$

We can then estimate the value B for the magnetic field as

$$B = \frac{mc}{et}\arccos(p_+ - p_-) \approx \frac{mc}{et}\arccos\left(\frac{N_+ - N_-}{N}\right) , \tag{9.71}$$

which becomes exact as $N \to \infty$. Note the similarities with the gravitational wave detector in Chap. 2.

We would like to have some estimate of the error δB in this value of B, since for finite N the value for B will not be exact. We can relate δB to the variance in S_x via the standard formula for errors analogous to Eq. (9.23):

$$\delta B = \left|\frac{d\langle S_x\rangle}{dB}\right|^{-1} \Delta S_x . \tag{9.72}$$

We already calculated $\langle S_x \rangle$:

$$\langle S_x \rangle = \frac{\hbar}{2} p_+ - \frac{\hbar}{2} p_- = \frac{\hbar}{2} \cos\left(\frac{eBt}{mc}\right) , \tag{9.73}$$

and the derivative becomes

$$\frac{d\langle S_x \rangle}{dB} = -\frac{e\hbar t}{2mc} \sin\left(\frac{eBt}{mc}\right) , \tag{9.74}$$

The variance $(\Delta S_x)^2$ is given by

$$(\Delta S_x)^2 = \frac{\hbar^2}{4} \langle \psi(t)|\sigma_x^2|\psi(t)\rangle - \frac{\hbar^2}{4} \langle \psi(t)|\sigma_x|\psi(t)\rangle^2 = \frac{\hbar^2}{4} \sin^2\left(\frac{eBt}{mc}\right) . \tag{9.75}$$

For N measurements, the variance $(\Delta S_x)^2$ adds up to become $N(\Delta S_x)^2$, and the expectation value of S_x is also N times as large, yielding $N\langle S_x \rangle$. When we combine all this we find

$$\delta B = \frac{2mc}{Ne\hbar t} \frac{1}{|\sin \omega t|} \times \frac{\hbar}{2} \sqrt{N} |\sin \omega t| = \frac{mc}{et} \frac{1}{\sqrt{N}} . \tag{9.76}$$

This precision is called the *standard quantum limit*. You see that as $N \to \infty$ the error in B vanishes (assuming no other errors), and the speed with which this happens is given by the square root of the number of measurements N.

This is, however, not the ultimate precision that can be attained in quantum mechanics. Instead of sending the electrons through the magnetic region one by one, we can send them through all together in some suitably chosen quantum state. Suppose that we have again N electrons, with the first one prepared in the same state as before:

$$|\psi\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} . \tag{9.77}$$

The rest of the electrons are prepared in the state $|\uparrow\rangle$. Next, we apply a so-called CNOT operation between the first electron and each one of the others, shown in Fig. 9.7. This will have the following effect on the state:

$$\begin{aligned} &|\uparrow, \uparrow\rangle \underset{\text{CNOT}}{\longrightarrow} |\uparrow, \uparrow\rangle \quad \text{and} \quad |\uparrow, \downarrow\rangle \underset{\text{CNOT}}{\longrightarrow} |\uparrow, \downarrow\rangle , \\ &|\downarrow, \uparrow\rangle \underset{\text{CNOT}}{\longrightarrow} |\downarrow, \downarrow\rangle \quad \text{and} \quad |\downarrow, \downarrow\rangle \underset{\text{CNOT}}{\longrightarrow} |\downarrow, \uparrow\rangle . \end{aligned} \tag{9.78}$$

In other words, the first electron flips the spin of the second if the first is in the state $|\downarrow\rangle$ (see also the XOR operation in Sect. 6.4). After applying $N - 1$ CNOT operations, the spin state of the N electrons becomes

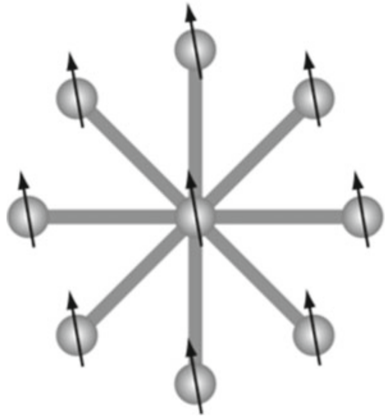

Fig. 9.7 Measuring the spin direction on N entangled spins. The interactive figure is available online (see supplementary material 6)

$$|\psi\rangle = \frac{|\uparrow, \uparrow, \ldots, \uparrow\rangle + |\downarrow, \downarrow, \ldots, \downarrow\rangle}{\sqrt{2}} . \tag{9.79}$$

This is a superposition of all electrons with spin up and all electrons with spin down, and this state has no classical analog. It is a special entangled state, called GHZ state, after Daniel Greenberger, Michael Horne and Anton Zeilinger, who first studied its non-classical properties.

As all electrons travel through the region with the magnetic field of unknown magnitude, each electron picks up the state dependent phase shift $e^{\pm i\omega t}$ according to Eq. (9.67), and the evolved state becomes

$$|\psi(t)\rangle = \frac{e^{iN\omega t/2}|\uparrow, \uparrow, \ldots, \uparrow\rangle + e^{-iN\omega t/2}|\downarrow, \downarrow, \ldots, \downarrow\rangle}{\sqrt{2}} . \tag{9.80}$$

Afterwards, we apply the same set of CNOT operations on the electrons, which turns the state into

$$|\Phi(t)\rangle = \frac{e^{iN\omega t/2}|\uparrow, \uparrow, \ldots, \uparrow\rangle + e^{-iN\omega t/2}|\downarrow, \uparrow, \ldots, \uparrow\rangle}{\sqrt{2}} . \tag{9.81}$$

We can interpret this state as $N - 1$ electrons in the state $|\uparrow\rangle$, and the first electron in the state

$$|\phi(t)\rangle = \frac{|\uparrow\rangle + e^{-iN\omega t}|\downarrow\rangle}{\sqrt{2}} , \tag{9.82}$$

up to a global phase. Notice how the phase shift is now N times larger than before: we have effectively teleported the phase shifts of all the $N - 1$ electrons onto the central electron. A measurement of this electron in the Stern–Gerlach apparatus then yields the probabilities

$$p_+ = |\langle +|\phi(t)\rangle|^2 = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{NeBt}{mc}\right),$$
$$p_- = |\langle -|\phi(t)\rangle|^2 = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{NeBt}{mc}\right). \tag{9.83}$$

Following the same procedure as before, we calculate

$$\delta B = \left|\frac{d\langle S_x\rangle}{dB}\right|^{-1} \Delta S_x\,, \tag{9.84}$$

with

$$\langle S_x\rangle = \frac{\hbar}{2}\cos\left(\frac{NeBt}{mc}\right)\,, \tag{9.85}$$

and

$$\frac{d\langle S_x\rangle}{dB} = -\frac{Ne\hbar t}{2mc}\sin\left(\frac{NeBt}{mc}\right)\,. \tag{9.86}$$

The variance becomes

$$(\Delta S_x)^2 = \frac{\hbar^2}{4}\langle\psi(t)|\sigma_x^2|\psi(t)\rangle - \frac{\hbar^2}{4}\langle\psi(t)|\sigma_x|\psi(t)\rangle^2$$
$$= \frac{\hbar^2}{4}\sin^2\left(\frac{NeBt}{mc}\right)\,. \tag{9.87}$$

Note that we have only one measurement in this situation, so we multiply the variance and expectation value of S_x with 1. This yields for the error in B

$$\delta B = \frac{2mc}{Ne\hbar t}\frac{1}{|\sin N\omega t|} \times \frac{\hbar}{2}|\sin N\omega t| = \frac{mc}{et}\frac{1}{N}\,. \tag{9.88}$$

This is called the *Heisenberg limit* (Holland and Burnett 1993) of the precision in B, and you see it is much more precise than the standard quantum limit for large N. However, there is only a single measurement with a binary outcome $+$ or $-$, and this can never give us the full numerical value of B: there are many bits of information in a numerical value of the form $B = 5.2947 \times 10^{-7}$ T, but there is only one bit of information revealed in a binary measurement outcome. The Heisenberg limit therefore means that we can detect the presence (a "yes/no" question) of a magnetic field of magnitude $B = mc/eNt$. If the magnetic field is switched off (i.e., $B = 0$) the state does not evolve, and the measurement outcome will be $+$ with $p_+ = 1$. For a magnetic field magnitude at the Heisenberg limit the measurement outcome will be $-$ with probability $p_- = 1$. Any magnetic field that has a smaller magnitude will produce uncertainty in the measurement outcome, because $p_- < 1$.

It should also be noted that the CNOT operations are very hard to implement, and currently Heisenberg limited measurements have been demonstrated only for small N.

Exercises

1. Show that the variance of A vanishes when $|\psi\rangle$ is an eigenstate of A.
2. Calculate the maximum time precision for a 1.0 kg pendulum of length 1.0 m and amplitude 10 cm. Is the precision of a grandfather clock restricted by the laws of quantum mechanics?
3. Show that the precision of a clock cannot be improved by increasing the size of the face plate.
4. We can create entanglement by using a quantum erasure protocol. Consider two identical atoms with low lying energy states $|0\rangle$ and $|1\rangle$, and an excited state $|e\rangle$ that is coupled to $|1\rangle$. We can generate a photon if the atoms is in state $|1\rangle$ by exciting the atom to $|e\rangle$ with a laser, followed by the spontaneous emission of a photon.
   (a) both atoms are prepared in the state $(|0\rangle + |1\rangle)/\sqrt{2}$. Write down the state of the two atoms and any emitted photons after we excite both atoms and let them spontaneously emit a photon.
   (b) The modes that may contain photons are mixed on a beam splitter. What is the state of the atoms if we detect exactly one photon in the output modes of the beam splitter? You may assume our detectors are perfect.
   (c) What is the probability that we find exactly one photon after the beam splitter?
5. Prove Eqs. (9.38) and (9.41).
6. Show that the momentum and the total energy can be measured simultaneously only when the potential is constant everywhere. What does a constant potential mean in terms of the dynamics of a particle?
7. A coherent state of a pendulum is defined by Eq. (9.56), and has a complex amplitude α. Calculate the expectation value for the position and momentum of the pendulum. When we write α in polar coordinates, such that $\alpha = re^{i\phi}$, give a physical interpretation of ϕ.
8 Prove that the ladder operators obey the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$.

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