# Chapter 12
# Elementary Extensions and Symmetries

**Abstract** In this chapter we will see how one can learn something about a structure by using symmetries of its elementary extensions. We will examine the specific example of the ordering of the natural numbers, and we will prove that the structure $(\mathbb{N}, <)$ is minimal. After so many pages, the reader will probably find it hard to believe that this example was my original motivation to write this book. Initially, it seemed that not much technical preparation was needed.

**Keywords** Minimality · Symmetries of the ordered set of integers · Ehrenfeucht-Mostowski theorem · Ramsey type theorems · Homogeneous sets

## 12.1 Minimality of $(\mathbb{N}, <)$

The structure we will talk about is the ordered set of natural numbers $(\mathbb{N}, <)$. It is easy to visualize. No advanced mathematics is involved. Still, what we will do is not trivial, and we will use much of the power of first-order logic.

To say that we will learn something about the structure of $(\mathbb{N}, <)$ is an exaggeration. $(\mathbb{N}, <)$ is such a simple structure that there is nothing really that we need to learn about it. All one may want to know is already shown in the image of a sequence of points starting at 0, going up or, as the number line is usually drawn, going to the right, and then disappearing into infinity.

$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \cdots$$

We will not learn anything new about this structure but we will learn something nontrivial about its definable sets; hence we will learn something about logic. Our goal is to characterize completely all sets of natural numbers that can be defined in $(\mathbb{N}, <)$. The structure looks simple, but we have a language in which we can express complicated properties, so it is hard to say in advance what can and what cannot be defined.



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What may be unclear about the that simple image of $(\mathbb{N}, <)$ is the role of the three small dots on the right. They indicate that the points go to infinity and they do it in an orderly fashion. But how do they do it? Where does the number line go? It is good to have a picture in mind, especially since the reader will be soon asked to think of new points that will be appended at the right, beyond all those infinitely many points. Another, more concrete representation may help.

Let $E$ be the set of points on the number line corresponding to the numbers

$$1-\frac{1}{2},\ 1-\frac{1}{3},\ 1-\frac{1}{4}, 1-\frac{1}{5},\dots$$

The sequence begins with $1-\frac{1}{2}=\frac{1}{2}$, and then proceeds up on the number line, getting closer and closer to 1, but never reaching it. As ordered sets, $(\mathbb{N}, <)$ and $(E, <)$ are isomorphic. They are two representations of same structure, but while $(\mathbb{N}, <)$ can be viewed as an unbounded subset of the real line, $E$ is bounded, so we will have no problem visualizing how new elements can be added to it. Here is the picture of $E$ with one new point, the number 1, added to it on top:

$$1-\frac{1}{2},\ 1-\frac{1}{3},\ 1-\frac{1}{4}, 1-\frac{1}{5},\dots, 1$$

If we now forget that the elements of listed above are numbers, and we only record how they are ordered, we get a simple picture:

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This is really all we need. The representation using fractions helps to see how an infinite sequence of elements can form an increasing chain, but still be bounded from above.

Now let us take a look at the ordered set $(\mathbb{Z}, <)$. It looks like $(\mathbb{N}, <)$, except that it is unbounded at both ends.

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For a more concrete representation, we can use the ordered set $F$ consisting of the numbers

$$\dots,\ \frac{1}{4},\ \frac{1}{3},\ \frac{1}{2}, 1,\ 2-\frac{1}{2},\ 2-\frac{1}{3},\ 2-\frac{1}{4},\dots$$

In this representation, 1 plays the role of the midpoint from which other points move to the left towards 0, and to the right towards 2. All numbers in $F$ are between 0 and 2, approaching those two numbers arbitrarily closely, but never reaching them.

The structure $(E, <)$ is an isomorphic copy of $(\mathbb{N}, <)$, and $(F, <)$ is an isomorphic copy of $(\mathbb{Z}, <)$. Now we can form a new structure whose domain will be the union of $E$ and $F$, and the ordering relation is the usual ordering on the number line. Here is a picture

$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \cdots \qquad \cdots \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \cdots$$

This new ordered structure is isomorphic to $(\mathbb{N}, <)$ with an isomorphic copy of $(\mathbb{Z}, <)$ added on top. The only reason to consider $E$ and $F$ instead of $\mathbb{N}$ and $\mathbb{Z}$ is to make this extended structure easier to visualize.

$(\mathbb{N}, <)$ has a least element, and $(\mathbb{Z}, <)$ does not. This is the only essential difference, but it has further consequences. As we have seen, in $(\mathbb{N}, <)$ every element is definable.

Recall that a symmetry of a structure is a permutation of its domain that preserves all relations of the structure, and as a consequence it also preserves all first-order properties of tuples of elements of the domain. In particular if $f$ is a symmetry of a structure then, for every element $a$ of the domain, the type of $a$ is the same as the type of its image $f(a)$. Since in $(\mathbb{N}, <)$ every element is definable, each element has its own unique type, and it follows that $(\mathbb{N}, <)$ has no symmetries at all. In a stark contrast, all elements of $\mathbb{Z}$ share the same type. Let us see why.

We will repeat an argument we have already used in Sect. 9.4. Let $a$ and $b$ be integers, and let us assume that $a < b$. Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be the permutation that shifts all elements up by $b - a$, i.e. $f(x) = x + (b - a)$. Then, since $f$ preserves the ordering, it is a symmetry of $(\mathbb{Z}, <)$, and because $f(a) = b$, the type of $a$ is the same as the type of $b$. While in $(\mathbb{N}, <)$ every element has its own unique type, in $(\mathbb{Z}, <)$, any element looks exactly as any other element.

We are now ready for a more involved argument that will show that $(\mathbb{N}, <)$ is minimal.

Suppose that $(\mathbb{N}, <)$ is not minimal. Then, there is a formula $\varphi(x)$ of the first-order language with one binary relation symbol $<$ defining a subset of $\mathbb{N}$ that is neither finite not cofinite. That means that the following holds in $(\mathbb{N}, <)$

$$\forall x \exists y \exists z[(x < y) \wedge (y < z) \wedge \varphi(y) \wedge \neg\varphi(z))]. \qquad (*)$$

We have seen that the relation "$y$ is a successor of $x$" is definable in $(\mathbb{N}, <)$. Let $S(x, y)$ be a formula defining it, i.e. for all natural numbers $m$ and $n$, $S(m, n)$ holds in $(\mathbb{N}, <)$ if and only if $n = m + 1$. Then, there are infinitely many $n$ such $\varphi(n)$ holds and $\varphi(n + 1)$ does not. In symbols:

$$\forall x \exists y \exists z[(x < y) \wedge S(y, z) \wedge \varphi(y) \wedge \neg\varphi(z))]. \qquad (**)$$

By the compactness theorem, $(\mathbb{N}, <)$ has a proper elementary extension $(\mathbb{N}^*, <)$. Since the extension is proper, it has a new element $c$, and this element must be larger than all natural numbers in $\mathbb{N}$. Why? Consider a natural number, for example 50. Could $c$ be smaller than 50? Let us see. The following sentence holds in $(\mathbb{N}, <)$

$$\forall x[x < 50 \Longrightarrow (x = 0 \vee x = 1 \vee x = 2 \vee \cdots \vee x = 49)].$$

The above sentence expresses the fact that every natural number smaller than 50 must be one of the numbers from 0 to 49. Since the extension is elementary, the same sentence holds in $(\mathbb{N}^*, <)$ and since $c$ is a new element, it cannot be smaller than 50.

Since every element of $(\mathbb{N}, <)$ has a successor, the same is true in $(\mathbb{N}^*, <)$. So $c$ has a successor $c + 1$, and $c + 1$ has a successor $c + 2$, and so on. Also, in $\mathbb{N}$, every element, except for 0, has a predecessor. This is expressed formally by

$$\forall x[\neg(x = 0) \Longrightarrow \exists y\ S(y, x)].$$

The same holds in $(\mathbb{N}^*, <)$, and it implies that $c$ has a predecessor, let us call it $c-1$. This $c - 1$ is a also a new element. If $c - 1$ were an old natural number, for example 23, then $c$ would be 24, and we know already that it can't be. So $c - 1$ is new, hence it has a predecessor $c - 2$, which is also new, and so on.

The argument above shows that $\mathbb{N}^*$ has many other new elements, and that every new element, such as the $c$ above, must be a part of a predecessor/successor chain of elements that looks exactly like $(\mathbb{Z}, <)$.

Now comes a crucial point in the proof. Because $(**)$ holds in $(\mathbb{N}^*, <)$, there infinitely many new elements $c$ such that

$$\varphi(c) \wedge \neg\varphi(c + 1) \qquad (***)$$

holds in $(\mathbb{N}^*, <)$.[1] Let us fix such a $c$.

Now we can define a symmetry of $(\mathbb{N}^*, <)$ as follows. For all elements $a$ that are not in the predecessor/successor chain of the $c$ above, we let $f(a) = a$. Those elements don't move. For every element $a$ in the chain of $c$, let $f(a)$ be $a + 1$. In particular, $f(c) = c + 1$. Since for all $a$ and $b$, $a < b$ if and only if $f(a) < f(b)$, $f$ is a symmetry of $(\mathbb{N}^*, <)$. Because $f$ is a symmetry, $c$ and $f(c)$ must have the same type in $(\mathbb{N}^*, <)$, it follows that, $\varphi(c)$ holds in $(\mathbb{N}^*, <)$ if and only $\varphi(c + 1)$ does, but we chose $c$ so that only $\varphi(c)$ holds, so this is a contradiction. We have proved that $(\mathbb{N}, <)$ is minimal.

There are several aspects of the proof we just saw that are worth stressing. First of all, we carried out the whole argument not knowing much about $(\mathbb{N}^*, <)$. The compactness theorem just tells us that it exists, but it tells us little about what it looks like, except that is an elementary extension of $(\mathbb{N}, <)$. Once we know that the extension is elementary, the argument rests on our ability to express relevant properties of $(\mathbb{N}, <)$ in a first-order way, to be able to transfer them and use in the extension $(\mathbb{N}^*, <)$.

The second important aspect is the use of symmetry. We cannot use symmetries directly to argue that certain relations are not definable in $(\mathbb{N}, <)$ because it is a rigid structure, it has no nontrivial symmetries at all. To take advantage of symmetries we

[1] We are using the symbol + here. It is not in the language of $(\mathbb{N}, <)$, but it is allowed as an abbreviation, since the expression $\varphi(c + 1)$ can be written as $\forall z S(c, z) \Longrightarrow \varphi(z)$.

had to move to a larger structure that has them. This may seem as a rather ad hoc trick, but in fact it is a standard method of model theory. It is widely applicable partly due to the fact that every structure with infinite domain has a proper elementary extension that admits nontrivial symmetries. This was proved in 1956 by Andrzej Ehrenfeucht and Andrzej Mostowski. The theorem of Ehrenfeucht and Mostowski is too advanced to be included here with a proof, but in the next section some relevant details are mentioned.

## 12.2 Building Symmetries

Here is a somewhat curious fact of life. At any party that is attended by six or more people, there will always be either at least three mutual acquaintance or at least three mutual strangers. To prove it, let us consider six people $A$, $B$, $C$, $D$, $E$, and $F$. We assume nothing specific about who knows whom, in particular they could all know each other, or all be mutual strangers. In both cases, surely our statement is true. In the first case we do have six mutual acquaintances, in the second six mutual strangers. We could try to check if the same is true for all other cases, but this would be a tedious task. Six people can be paired in 15 ways,[2] and each pair can potentially be a pair of acquaintances or strangers. This gives us $2^{15} = 32{,}768$ possible relations to verify. Instead, we will use a clever argument.

Assume nothing specific about who knows whom, and consider the person $A$. The five remaining people are split into two sets, in the first are the people that $A$ knows, and in the second everybody else. Regardless of whom $A$ knows, one of those sets must include at least three people. If both had less than three members, then the total number of members in both would be less than 5, and that can't be. Simple.

Suppose now that three people, say $B$, $C$ and $D$, are in the first set. The argument is similar if three people are in the second set. If any of the $B$, $C$, or $D$ know each other, that, together with $A$, creates a triangle of three people who know each other. If not, then $B$, $C$, and $D$ form a triangle of strangers, and this proves our statement.

Number six is the smallest number with the property just described. In a group of five people it can happen that neither three people know each other nor there are three mutual strangers.

What we proved about acquaintances and strangers can be formulated in terms or relations as follows.

**Theorem 12.1** *Let $\mathfrak{A}$ be a structure whose domain has at least six elements, with a binary relation $E$ such that*

$$\forall x \forall y\ [(E(x, y) \Longrightarrow E(y, x)) \wedge \forall x\ \neg E(x, x)].$$

[2]This follows from a general fact that the number of pairs in an $n$-element set is $\frac{n(n-1)}{2}$. For small values of $n$ such as 6, one can verify it by listing all possible pairs.

*Then there is a subset $X$ of the domain such $X$ has at least three elements, and either for all distinct a, and b in $X$, $E(a, b)$ holds, or for all distinct a, and b in $X$, $\neg E(a, b)$ holds. Such a set $X$ is called homogenous with respect to $E$.*

Theorem 12.1 is an example of a Ramsey type theorem.[3] All Ramsey type theorems have a common structure, they say that for given $n$ and $k$, and a given property of $k$-tuples, if the domain a structure is large enough, than it has a subset $X$ of size at least $n$, that is homogeneous with respect to the property, i.e. either all $k$-tuples of elements of $X$ have the property, or all of them don't. In other words, all $k$-tuples in $X$ look the same with respect to the property. Notice that this does not mean that all $k$-tuples of $X$ all have the same type. The structure may have other relations, and even if it does not, the elements of $X$ may interact with elements of the structure outside $X$ in different ways, hence they types can be different. Nevertheless, Ramsey type theorems tell us that for any given property, if the domain of a structure is large enough, then it must have many elements that "look alike" with respect to the given property.

There is also a powerful infinitary Ramsey's theorem. Here is a variant of it. Suppose $\mathfrak{N} = (\mathbb{N}, <, \dots)$ is a structure, where $<$ is the usual ordering of the natural numbers, and the dots indicate that the structure may or may not have other relations. Let $\varphi(x_1, x_2, \dots, x_k)$ be a formula of the language of $\mathfrak{N}$. For every infinite set $X$ that is definable in $\mathfrak{N}$, there is an infinite definable $Y$ contained in $X$ that is homogenous for the property defined by $\varphi(x_1, x_2, \dots, x_n)$ in the sense that either for all increasing sequences $a_1 < a_2 < \dots < a_k$ of elements of $Y$, $\varphi(a_1, a_2, \dots, a_k)$ holds, or for all such sequences $\neg\varphi(a_1, a_2, \dots, a_k)$ holds. This means that while $\mathfrak{N}$ may be rigid, for any finite number of first-order properties $\mathfrak{N}$ there is an infinite definable set whose elements all "look alike" with respect to those properties. This theorem and the compactness theorem are the main ingredients of Ehrenfeucht and Mostowski's theorem on existence of elementary extensions with symmetries. A proof of the infinitary version of Ramsey's theorem for pairs ($k = 2$) is given in Appendix A.5.

## Exercises

**Exercise 12.1** *Show that the that there is no formula $\varphi(x, y, z)$ of the language with one binary relation symbol $<$ such that for all natural numbers k, l, m, $\varphi(k, l, m)$ holds in $(\mathbb{N}, <)$ if and only if $k + l = m$. Hint: There are many ways in which this can be shown. For a short argument, think of the set of even numbers, and use minimality of $(\mathbb{N}, <)$.*

**Exercise 12.2** *Use the previous exercise to show that the addition of natural numbers is not definable in the structure $(\mathbb{N}, S)$, where $S$ is the successor relation, i.e. for all natural numbers m and n, $S(m, n)$ holds if and only if $m + 1 = n$.*

[3] After British philosopher, mathematician, and economist Frank Plumpton Ramsey (1903–1930).